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Production control of a manufacturing system subject to deterioration

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Abstract— **In this paper we consider a control problem for a manufacturing system comprising a single machine subject to a deterioration process. The system produces a single part type and the model is fluid. The objective is to minimize a long term average cost index which penalizes both inventory surplus and backlog. The machine deterioration rate depends on the production rate and a maintenance operation is performed when the deterioration reaches a specified alarm level. In this paper it is shown that the optimal control operates the machine alternating the maximum production capacity with idle or conservative periods (policy μ-d-μ) if the machine deterioration rate is a concave function of the production rate while it is a continuous feedback function of the buffer level if the deterioration rate is convex. This confirms the results obtained in the past for Markov failure prone systems.**

I. INTRODUCTION

The problem of failure prone manufacturing systems has been investigated since a long time. Analytical results are known in some particular cases, e.g. if the failure process is Markov, the control minimizing a long term average expected cost including inventory surplus and backlog penalties is the *hedging point policy* if and only if the dependence of the failure rate on the production rate is affine [3]. The result in [3] includes the well known constant failure rate case, investigated among others by [1], where the analytical expression of the safety stock was also derived. In the general, still Markov, case, according to [3], it seems optimal to decrease the production rate as the buffer level approaches a hedging level, to gain in reliability of the machine. Actually, in [5], using a numerical approach, it was observed a major difference between the case where the failure rate is a concave function of the production rate and the case where such a dependence is convex. In particular, it was observed that the hedging point policy is optimal (among stationary feedback policies) if the dependence of the failure rate on the production rate is a concave function. This seems to contrast the result of [3] where the optimality of the hedging point policy is stated to hold if and *only if* such a dependence is affine. To explain this, in [5] it was remarked that in the concave non affine case the hedging point policy is optimal only among *stationary feedback* policies and is optimal among *all* policies only in the affine case. So in the concave non-affine case, it was conjectured that the optimal policy is a *non stationary feedback* policy which, to maintain the buffer at the safety stock level, switches infinitely fast the production rate among 0 and the maximum capacity of the machine. In the convex case, the hedging point policy is not optimal and numerical findings [5] confirm that the production is continuously reduced as the buffer level approaches a safety stock. These results are also confirmed by the analysis in [7] and [8]. In order to gain insight into this problem, a slightly different version of it has been considered in this paper, where the down time of the machine is deterministic and the uptime ends when the deterioration state of the machine (which increases through a deterministic function of the production rate) reaches an alarm level. This deterministic formulation can be successfully approached using the maximum principle. The results in this paper confirm the general behavior described above for the Markov system, in the sense that also here a major difference arises among the convex and the concave case. In particular, it seems that if the deterioration rate is a concave function of the production rate, the optimal policy either works at maximum rate either at the demand rate. Moreover, also in this deterministic scenario, in the concave non affine case, a production rate equivalent to the demand rate is obtained through a fast switch among 0 and the maximum capacity production. Finally, in the convex case, the production rate changes in a continuous fashion as the buffer level approaches certain optimal levels of the buffer. The problem considered in this paper is not an optimal maintenance planning problem, investigated by a large body of literature in the past (see e.g. [2]). This mainly depends on the fact that, as stated above, this research has been performed to better understand the interesting switching phenomenon observed in the Markovian case. So the point of view is quite different. However, it is important to remark that the interesting behavior observed and analyzed in this paper is not simply a theoretical subject but may provide useful applications to real world systems, as briefly discussed at the end of Section IV-C1.

II. NOTATION AND PROBLEM FORMULATION

According to a standard notation, let $x(t)$ denote the buffer content at time t, with $x(t) > 0$ representing an inventory surplus and $x(t) < 0$ a backlog of $-x(t)$. Let d be the constant demand rate to be met. Then the buffer level $x(t)$ at time t satisfies the dynamical equation $\dot{x}(t) = u(t) - d$ where the production rate $u(t) \equiv 0$ if at time t the machine is in the down state (also referred to as state 0), and $u(t) \in [0, \mu]$ if at time t the machine is in the up state (also referred to as state 1). The down time T_g of the machine is constant and deterministic. The uptime is also a deterministic quantity but is not constant: it depends on the production history of the machine since the last repair time t_{0f}. In particular, we introduce a deterioration function

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$z(t) = \int_{t_{0f}}^{t} \left(au^{\beta}(\tau) + b\right) d\tau$ where a, b and β are non-negative constants. The machine is stopped at time t_{ff} where t_{ff} is such that $z(t_{ff}) = 1$. After the repair the machine is good as new. We consider an instantaneous quadratic cost $g\left[x(t)\right] = cx^2(t)$, $c > 0$, which penalizes equivalently the backlog and the inventory surplus. The problem considered in this paper is then the determination of the optimal control $u(\cdot)$ which minimizes the following cost index:

$$J = \lim_{t\to+\infty} \frac{1}{t} \int_{t_0}^{t} cx^2(\tau)d\tau \tag{1}$$

In the following, an equivalent formulation is given which allows to apply directly the maximum principle.

A. An equivalent formulation

Since the problem is deterministic and stationary, if an optimal control exists (this will be discussed below), the solution at steady state will be periodic. We will call *cycle* each period and T its time duration. Let:

$$J_T = \frac{1}{T} \int_{0}^{T} cx^2(t)dt \tag{2}$$

Once the problem of minimizing (2) has been solved, the original problem of minimizing (1) is simply solved by considering any policy which brings the buffer level to an optimal initial level X_0 and then, from that time on, applies the control minimizing (2) every T time units. So, from now on, we will consider the problem of the determination of the production control which minimizes (2). Let T_f be the uptime of the machine (T_f is a function of the production control law applied since the last failure, i.e. since 0 for the new problem). Since the machine is down in $[T_f, T]$, the problem of finding the optimal $u(\cdot)$ is restricted to $[0, T_f]$. Clearly $T = T_f + T_g$ is also a function of the control applied in $[0, T_f]$. Let X_0 and X_1 be respectively the buffer level at the beginning of the cycle and at the end. Then, considering the down interval $[T_f, T]$, it trivially follows $X_0 = X_1 - dT_g$ Then, the index in (2) can be written taking into account the contribution in $[0, T_f]$ and the contribution in $[T_f, T]$. The first quantity depends on the production control, the second one, once the X_1 level has been specified, can be univocally determined. We have $\frac{1}{T}\int_{T_f}^{T} cx^2(\tau)d\tau = \frac{cT_g\left(d^2T_g^2 - 3X_1 dT_g + 3X_1^2\right)}{3(T_f+T_g)}$. To apply the maximum principle to the considered problem, we have performed some transformations on the original formulation. In particular, we introduce an auxiliary variable $y(t) := \int_0^t c\,x^2(\tau)d\tau$ and we will denote $y_f = y(T_f)$. The problem can be now formulated as follows.

Problem 1: Determine the policy $u(\cdot)$ solving the following constrained optimization problem:

$$\min_{u(\cdot)} \frac{y_f}{(T_f + T_g)} + \frac{cT_g\left(d^2T_g^2 - 3X_1dT_g + 3X_1^2\right)}{3(T_f + T_g)} \tag{3}$$

subject to

$$\left[\dot{x}(t),\ \dot{z}(t),\ \dot{y}(t)\right]^T = \left[u(t) - d,\ au^{\beta}(t) + b,\ cx^2(t)\right]^T \tag{4}$$

$u(t) \in [0, \mu]$, $0 \le t \le T_f$, with the following boundary initial conditions:

$$\left[x(0),\ z(0),\ y(0)\right]^T = \left[X_1 - dT_g,\ 0,\ 0\right]^T \tag{5}$$

and final conditions:

$$\left[x(T_f),\ z(T_f),\ y(T_f)\right]^T = \left[X_1,\ 1,\ y_f\right]^T \tag{6}$$

We will denote as J^* the minimum value of (3) (hence of (2) and also of (1) if the machine has enough capacity).

B. Feasibility analysis

A finite solution for Problem 1 exists if the machine has enough capacity to meet the demand. In such a case there exists a function $u(\cdot)$ which gives a finite cost, i.e. the minimum value of J in (1) is finite. If such a control exists, as mentioned above, the control law (and the optimal trajectory) is periodic, hence it satisfies the following equations:

$$\begin{cases} \int_0^{T_f} \left(au^{\beta}(\tau) + b\right) d\tau = 1 \\ \int_0^{T_f} (u(\tau) - d)d\tau = dT_g \\ u(t) \in [0, \mu] \qquad 0 \le t \le T_f \end{cases} \tag{7}$$

To understand if (7) can be solved we introduce the following feasibility function:

$$S(u) := \frac{u - d}{a\,u^{\beta} + b} - d\,T_g \tag{8}$$

Equation (8) represents the buffer level reached at the end of a cycle if the buffer level at time 0 starts at 0 and the machine (during the uptime) is operated at a constant production rate u. The following theorem allows to analyze the feasibility problem in a straightforward manner.

Theorem 1: It is possible to find a solution to (7) (hence the system is feasible) if and only if there exists at least a $\bar{u} \in [0, \mu]$ such that $S(\bar{u}) \ge 0$.

Proof. As for the sufficiency, assume a $\bar{u}$ such that $S(\bar{u}) \ge 0$ exists. Then, if it is possible to find a $\bar{u}$ such that $S(\bar{u}) = 0$, $u(t) = \bar{u}$ for all $t \in [0, T_f]$ is solution of (7). If $S(\bar{u}) > 0$, just apply $u(t) = \bar{u}$ until the buffer level $d\,T_g$ is reached (if starting the cycle at 0) and keep then $u(t) = d$ (hence the buffer level constant) until failure. This control satisfies (7). As for the necessity, assume that for all $u \in [0, \mu]$, $S(u) < 0$ and consider the following optimization problem:

$$X_{MAX}(T_f) = \max_{u(.)} \int_0^{T_f} (u(\tau) - d)d\tau \tag{9}$$

with

$$\begin{cases} \dot{z}(t) = au^{\beta}(t) + b \\ z(0) = 0 \\ z(T_f) = 1 \\ u(t) \in [0, \mu] \qquad 0 \le t \le T_f \end{cases} \tag{10}$$

where $X_{MAX}(T_f)$ denotes the maximum buffer level that can be reached in an uptime interval starting from 0. Introducing the Hamiltonian

$$H_{es}(u(t), t, \lambda(t)) = -(u(t) - d) + \lambda(t)\left(au^{\beta}(t) + b\right), \tag{11}$$

the auxiliary system is described by the following dynamic:

$$\frac{d\lambda(t)}{dt} = -\frac{\partial H_{es}(u(t), t, \lambda(t))}{\partial z} = 0 \tag{12}$$

From this it trivially follows that $\lambda(t) = \lambda$ is constant, hence H_{es} is stationary and the optimal $u(\cdot)$ is constant. Now, $S(u) + dT_g$ represents the value of the buffer level that can be reached in an uptime interval starting from 0 and working at constant rate u. Since $S(u) < 0$ for all $u \in [0, \mu]$, and the maximum buffer level is reached working at a constant rate, we have $X_{MAX}(T_f) < dT_g$ for all control policy $u(\cdot)$, hence no solution can be found to (7).

Remark 1: Observe that, while the existence of a u such that $S(u) = 0$ is enough to maintain the steady state periodic trajectory (i.e. to find a solution to (7)), if the buffer level at the beginning is below the optimal initial level X_0, we can only obtain a bounded J^* whose value depends on initial conditions. In other words it is not always possible to bring the buffer to 0 (or in general to any desired level) if we don't have $S(u) > 0$ for some $u \in [0, \mu]$.

Based on Theorem 1, the feasibility of the system can simply be checked by verifying if $S(u)$ is positive for some $u \in [0, \mu]$. Now, if $\beta < 1$, there always exists a U_1 such that $S(u) > 0$ for all $u > U_1$. If $\beta = 1$, it is possible to find a U_1 as for the $\beta < 1$ case only if $1/a > dT_g$. Otherwise $S(u) < 0$ for all u. A solution to Problem 1 exists in these cases only if $\mu \geq U_1$. If $\beta > 1$, $S(u)$ is bounded from above and goes to $-dT_g$ as $u \to \infty$. If the maximum of $S(u)$ is non negative, there exists an interval $[U_1, U_2]$ where $S(u) \geq 0$. If such an interval exists, a solution to Problem 1 can be found if $\mu \geq U_1$. We will denote by $\beta^* > 1$ the value of β such that the two solutions of $S(u) = 0$ are coincident, i.e. $U_1 \equiv U_2$ (so, if $\beta = \beta^*$, $S(u) < 0$ for all $u \neq U_1 \equiv U_2$).

III. THE $\beta = 0$ CASE

This very simple case is reported to introduce the general problem considered in the sequel of the paper. In this case, the uptime is independent of the production and is given by $T_f = \frac{1}{a+b}$. With an approach similar to the one used in [4], it is possible to show that, in this case, the optimal policy in each cycle spends the maximum time allowed with the buffer empty, going from 0 to the safety stock X_1 at the last possible time (hence working at maximum rate μ from 0 to X_1) and reaching as soon as possible from X_0, at the beginning of the cycle, the 0 level (hence working at maximum rate μ if $X_0 < 0$ and not working at all if $X_0 > 0$). The cost associated with such a policy can be evaluated in a straightforward manner: it only depends on the safety stock X_1. Minimizing with respect to this quantity, gives

$$X_1 = -X_0 = \frac{dT_g}{2} \tag{13}$$

which agrees with intuition since the instantaneous cost function is symmetric. The optimal policy, which due to its behavior will be called in the following μ-d-μ policy, during the active part of each cycle (i.e. $[0, T_f]$), works according to the following equation (defined if the system is feasible):

$$\begin{cases} u(t) = \mu & 0 \leq t < \frac{dT_g - X_1}{\mu - d} \\ u(t) = d & \frac{dT_g - X_1}{\mu - d} \leq t < \frac{X_1(a+b)+d-\mu}{(a+b)(d-\mu)} \\ u(t) = \mu & \frac{X_1(a+b)+d-\mu}{(a+b)(d-\mu)} \leq t < \frac{1}{a+b} \end{cases} \tag{14}$$

IV. THE GENERAL CASE

Problem 1 is approached through the maximum principle [6]. First of all, we define the following Hamiltonian function, where $\mathbf{x}(t) = [x(t),\, z(t),\, y(t)]^T$ and $\lambda(t) \in \mathbb{R}^3$:

$$H(\mathbf{x}(t), u(t), t, \lambda(t)) = \lambda_1(t)(u(t) - d) + \lambda_2(t)\left(au^\beta(t) + b\right) + \lambda_3(t)cx^2(t) \tag{15}$$

The auxiliary system dynamics are given by:

$$\frac{d\lambda_1(t)}{dt} = -\frac{\partial H}{\partial x} = -2\lambda_3(t)cx(t) \tag{16}$$

$$\frac{d\lambda_2(t)}{dt} = -\frac{\partial H}{\partial z} = 0 \tag{17}$$

$$\frac{d\lambda_3(t)}{dt} = -\frac{\partial H}{\partial y} = 0 \tag{18}$$

Based on (17) and (18), we have that $\lambda_2(t)$ and $\lambda_3(t)$ are constant (λ_2 and λ_3 in the following). According to the maximum principle, the optimal $\left(\mathbf{x}^*(\cdot), u^*(\cdot), T_f^*, \lambda^*(\cdot)\right)$ must satisfy the following equations (necessary conditions):

1) Minimum condition:

$$H\left(\mathbf{x}^*(t), u^*(t), T_f^*, \lambda^*(t)\right) \leq H\left(\mathbf{x}^*(t), u, T_f^*, \lambda^*(t)\right) \quad t \in [0, T_f^*], \forall u \in [0, \mu] \tag{19}$$

2) Transversality condition:

$$\lambda_1^*(T_f^*)\left(u^*(T_f^*) - d\right) + \lambda_2^*\left(a(u^*(T_f^*))^\beta + b\right) + \lambda_3^* c X_1^{*2} - \frac{y_f^*}{(T_f^* + T_g)^2} - \frac{cT_g\left(d^2T_g^2 - 3X_1^* dT_g + 3X_1^{*2}\right)}{3(T_f^* + T_g)^2} = 0 \tag{20}$$

3) Orthogonality conditions:

$$\lambda_1^*(T_f^*) - \frac{cT_g\left(6X_1^* - 3dT_g\right)}{3(T_f^* + T_g)} = \theta_1 \tag{21}$$

$$\lambda_2^* = \theta_2 \tag{22}$$

$$\lambda_3^* - \frac{1}{T_f^* + T_g} = \theta_3 \tag{23}$$

Since θ_1, θ_2 and θ_3 are free constants, the transversality conditions are always trivially satisfied. Let, as above, X_0 the buffer level at the beginning of the cycle. The maximum principle only provides necessary conditions, so the optimality of the considered policies can not be established. However, the continuity of the deformation of the optimal trajectory as β increases and the knowledge of the optimal policy if $\beta = 0$ seem to indicate that the considered policies are actually the optimal policies. The main assumption we will use in the following to derive these candidate policies is that, since the instantaneous cost function is symmetric with respect to the 0 level of the buffer, a symmetry property characterizes the optimal trajectory around the point $x(T_f/2) = 0$. In particular, (13) is satisfied. This will imply a symmetry property around time $T_f/2$ also of other variables.

A. $\beta = 0$. In Section III, it was observed that if $\beta = 0$ the optimal policy is the μ-d-μ policy. This policy operates the machine at maximum capacity μ from X_0 to 0, then works at rate d keeping the buffer at 0 for a certain amount of time, finally operates the machine at maximum rate μ until the buffer level X_1 which is reached at time T_f, where the machine goes down (see Fig. 2). It is straightforward to check that the μ-d-μ policy satisfies the maximum principle. The Hamiltonian becomes in this case a line with slope $\lambda_1(t)$. If $\lambda_1(t) > 0$, the minimum is achieved by $u(t) = 0$, if $\lambda_1(t) < 0$, the minimum is achieved by $u(t) = \mu$. Finally, if $\lambda_1(t) = 0$, any value of $u(t)$ in $[0, \mu]$ minimizes the Hamiltonian. Since no constraint applies on λ_2 and λ_3, the transversality condition is always satisfied. Selecting:

$$\lambda_{10} := \lambda_1(0) = -\frac{\lambda_3^2 c d^2 T_g^2}{4(\mu - d)} \tag{24}$$

and $\lambda_3 > 0$, the integration of (16) is such that, with a μ-d-μ policy, $\lambda_1(t)$ is negative ($\Rightarrow u(t) = \mu$) until $x(t)$ is negative. When both $x(t)$ and $\lambda_1(t)$ become 0, we set $u(t) = d$, which maintains both $x(t)$ and $\lambda_1(t)$ at 0. Finally, when the control becomes μ, $x(t)$ increases while $\lambda_1(t)$ decreases, according to the minimum condition.

B. $\beta = 1$. This case is similar to the previous one, the μ-d-μ is still optimal. The Hamiltonian is again a line but the slope is now $\lambda_1(t) + a\lambda_2$. Following a procedure similar to the one of the $\beta = 0$ case, the slope of this line must be negative in such a way that starting at the beginning with a production rate $u(t) = \mu$ satisfies the maximum principle. It is necessary to select λ_{10} in such a way that at the time instant where $x(t)$ reaches 0, one has $\lambda_1(t) = -a\lambda_2$, which makes admissible, according to the maximum principle, the choice $u(t) = d$ (with $\lambda_1(t)$ remaining constant) and subsequently $u(t) = \mu$ (with $\lambda_1(t)$ which returns below $-a\lambda_2$).

C. $\beta \neq 1$, $\beta \neq 0$. In this case the optimal policy depends on the convexity properties of the Hamiltonian. If the Hamiltonian is convex with respect to $u(t)$ and its local minimum is in the interval $[0, \mu]$, the production control satisfying the maximum principle will be given by the value of u for which the derivative of the Hamiltonian is 0. If such a minimum does not belong to the interval $[0, \mu]$, the solution will be either 0 or μ. If the Hamiltonian is concave, the minimum will be either 0 or μ. To discriminate if the Hamiltonian is convex or concave, we will compute its second derivative with respect to u:

$$\left.\frac{\partial^2 H(\mathbf{x}(t), u(t), T_f, \lambda(t))}{\partial u^2}\right|_{u(t)=\tilde{u}(t)} = a\beta(\beta - 1)\lambda_2 \tilde{u}^{\beta-2}(t) \tag{25}$$

Since a, β and $u(t)$ are all positive, (25) is positive (and the Hamiltonian is convex) if and only if $(\beta - 1)\lambda_2 > 0$. Otherwise the Hamiltonian is concave. In the following we will study the optimal production control in the two cases (convex and concave). Clearly the sign of λ_2 is not known and the analysis below would appear not applicable. However, we will present at the end of this section a procedure to estimate the sign of λ_2.

C1. $(\beta - 1)\lambda_2 < 0$: *concave Hamiltonian*. In this case, the minimum is either at $u = 0$ or μ. We define:

$$\Delta H := H(\mu) - H(0) = \mu \left(\lambda_1(t) + a\lambda_2 u^{\beta-1}\right) \tag{26}$$

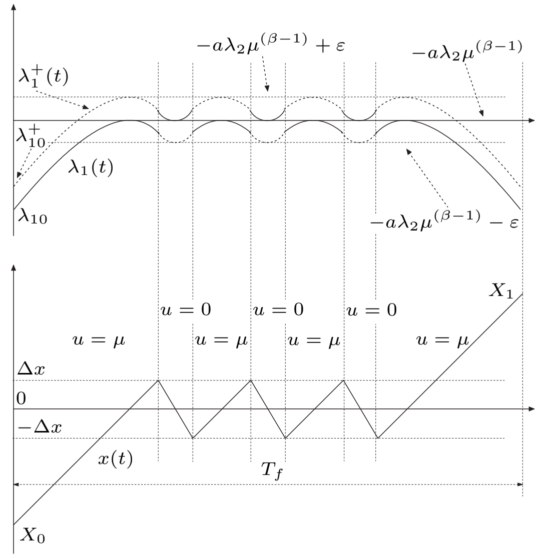

Fig. 1. λ_1, λ_1^+ and x as a function of time for the concave case

When $\lambda_1(t) < -a\lambda_2 u^{\beta-1}$, we will produce at rate $u(t) = \mu$, otherwise at rate 0. However, it is not straightforward in this case to find a policy satisfying the maximum principle as in the previous cases. Suppose for instance, as in the previous cases, to apply a μ-d-μ policy and to select the initial λ_{10} in such a way that, through the integration of (16), $\lambda_1(t) = -a\lambda_2 u^{\beta-1}$ when $x(t) = 0$. Once $x(t)$ has reached 0, however, it is not possible to work at rate d as in the μ-d-μ policy, since now the rate d does not minimize the Hamiltonian. On the other hand, working at rate μ, $\lambda_1(t)$ decreases and $x(t)$ increases arriving at X_1 too early, i.e. with the machine not completely deteriorated. Similarly, if we set $u = 0$ once the buffer has reached 0, $\lambda_1(t)$ would increase and the buffer level decrease, never reaching the X_1 level. If we consider a policy that at the beginning and at the end of the uptime works at maximum capacity ($u(t) = \mu$), the only way to satisfy the maximum principle once the buffer has reached the 0 level, is to work with a fast switching among $u(t) = 0$ and $u(t) = \mu$ with an equivalent average production rate equal to d. It is possible to see (see below) that this kind of policy, if the number of switches goes to infinity, meets all the constraints of the maximum principle. The buffer trajectory $x(t)$ on a cycle and the corresponding $\lambda_1(t)$ for the switching policy just described (with a finite number of switches) are reported in Fig. 1. This policy does not satisfy the maximum principle if the number of switches is finite, as it will be discussed below, but, as the number of switches goes to infinity, the obtained trajectories all satisfy the constraints of the maximum principle. To see how this can happen, we introduce a small quantity $\Delta x > 0$ and consider the aforementioned switching policy, which works at maximum rate μ from X_0 to 0 at the beginning of the cycle and also from 0 to X_1 at the end of the uptime. It remains around 0 (between $-\Delta x$ and Δx) for a certain amount of time with a switching production which alternates maximum and 0 production, each switching occurring when the trajectory reaches $-\Delta x$ (switch from 0 to μ) or Δx (switch from μ to 0). So, in this intermediate phase, the average resulting production rate is $\bar{u} = d$. For this policy, it is not possible to find a $\lambda_1(t)$ to satisfy the maximum principle. So we will proceed as indicated in Fig. 1. At first, we consider as above a function $\lambda_1(t)$ with initial condition λ_{10} selected in such a way that, through the integration of (16), $\lambda_1(t)$ reaches $-a\lambda_2 u^{\beta-1}$ when $x(t)$ reaches 0. Then an additional auxiliary function $\lambda_1^+(t)$ with different initial condition $\lambda_{10} + \varepsilon$ is considered. The value of $\lambda_{10} + \varepsilon$ is selected as the minimum initial condition for which $\lambda_1^+(t)$, obtained by integrating (16), is always not smaller than $-a\lambda_2 u^{\beta-1}$ during the switching phase (see figure). Clearly ε depends on Δx and goes to 0 as $\Delta x \to 0$. During the switching phase, we will switch among $\lambda_1(t)$ and $\lambda_1^+(t)$, considering active the auxiliary function which at the current moment, according to the minimum condition (19), allows to apply the production control giving the desired switching of $x(t)$ between $-\Delta x$ and Δx (the selected $\lambda_1(t)$ is the solid curve in Fig. 1, while the two auxiliary functions λ_1 and λ_1^+ are the dotted curves). Now, if $\Delta x \to 0$, also $\varepsilon \to 0$, the switching rate increases to infinity and λ_1^+ and λ_1 collapse in a unique, differentiable function with which the considered policy meets the maximum principle constraints.

From a practical point of view, a fast switching policy does not seem realistic. In real applications, we expect that, for a given plant, there exists a maximum finite switching rate which can be applied. Comparing the cost of this finite switching rate policy with the cost of a μ-d-μ policy, will allow to select the optimal production control. We want to remark that it is always possible to compute the number of switches such that the corresponding finite switching rate policy has the same cost of the μ-d-μ policy.

C2. $(\beta - 1)\lambda_2 > 0$, *convex Hamiltonian*. In this case, the Hamiltonian is minimized by

$$u(t) = \left(-\frac{\lambda_1(t)}{a\beta\lambda_2}\right)^{\frac{1}{\beta-1}} \quad (27)$$

A unique positive $u(t)$ satisfying (27) may exist, and it does exist if $\lambda_1(t)$ and λ_2 have different sign and $\lambda_2 \neq 0$. Otherwise, if $\lambda_1(t)$ and λ_2 have the same sign, the solution exists only for some particular values of β. Integrating (16) and substituting in (27) gives:

$$u(t) = \left(\frac{2\lambda_3 c \int_0^t x(\tau)d\tau - \lambda_{10}}{a\beta\lambda_2}\right)^{\frac{1}{\beta-1}}, \quad (28)$$

which is a feedback control. At this point, λ_{10}, λ_2 and λ_3 should be derived such that (7) are met. This problem will be solved numerically by introducing two constants C_1 and C_2 and writing (28) as

$$u(t) = \left(C_1 + C_2 \int_0^t x(\tau)d\tau\right)^{\frac{1}{\beta-1}} \quad (29)$$

Given C_1, a unique C_2 may exist satisfying (7) with a $x(t)$ symmetric around $x(T_f/2) = 0$. So, the set of admissible values for C_1 and C_2 is actually a curve in the (C_1, C_2) plane and we can consider, in this case, the cost J in (1) as a function of C_1, i.e. $J = J(C_1)$. Then, the optimal policy may be derived by numerically finding the value of C_1 which minimizes $J(C_1)$. Observe that the set of C_1 for which a C_2 can be found to satisfy (7) with a $x(t)$ symmetric around $x(T_f/2) = 0$ is a bounded interval. Let $[C_1^{min}, C_1^{max}]$ such an interval. A numeric search of the optimal C_1 will be then performed in $[C_1^{min}, C_1^{max}]$. In the remaining of the paper, this numeric search will be referred to as C_1-*search*.

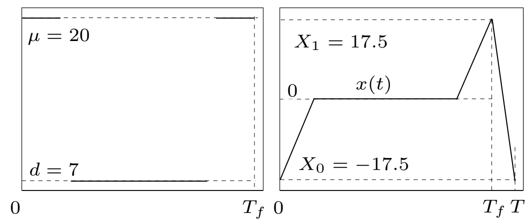

Fig. 2. The μ-d-μ policy optimal for the $\beta = 0$ case of Section V: the production control u (left) and the corresponding trajectory x (right) as a function of time

C3. *The procedure for the case* $\beta \neq 1$, $\beta \neq 0$. As mentioned above, the method described so far can be applied only if the sign of λ_2 is known. How can we determine it without knowing the solution? The answer to this question is based on the following procedure. We first consider as candidate optimal policy the switching policy described for the concave case. This would allow to derive λ_2, using the orthogonality conditions, since $u(T_f) = \mu$ (for the kind of policy considered) and $\lambda_1(T_F) = \lambda_{10}$ (for the symmetry properties). Then, λ_{10} can be computed considering that $\lambda_1(t) = -a\lambda_2 u^{\beta-1}$ when $x(t) = 0$. This is because the integration of (16) is the integration of a line and it is easy to compute $\lambda_1(t)$ as a function of λ_{10} and of λ_2. Once the sign of λ_2 is known, based on the value of β, it is possible to understand if we are in the convex or in the concave case above, and, consequently, if the considered switching policy can be optimal. If the answer is positive the problem has been solved. On the contrary, it is necessary to apply the C_1-search method described in the convex case, which allows to determine the optimal C_1 and C_2, from which the correct λ_{10} and λ_2 can be derived. Now, in all the examples considered, λ_2 was actually a positive quantity. If one is able to show that this holds in general, it would be possible to say that the switching policy is optimal for $0 < \beta < 1$, while, for $\beta > 1$, the optimal policy would be the one given by the C_1-search algorithm described above.

V. A NUMERICAL CASE STUDY

In this section we report a case study for a system with the following parameters: $a = 0.0008$, $b = 0.0002$, $c = 1$, $d = 7$, $T_g = 5$, $\mu = 20$. The value of β will be specified in the following. Time durations are expressed in hours. Based on the results of Section II-B, it is possible to see that the considered system is feasible if $0 \leq \beta \leq \beta^* \simeq 2.093$. In addition, (13) gives $X_1 = -X_0 = 17.5$. In the following, we will consider different values for β, which will allow to

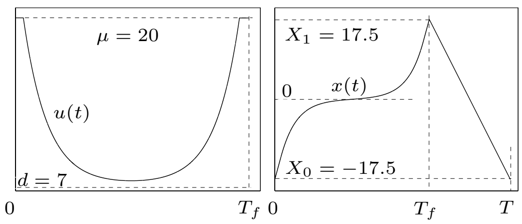

Fig. 3. The policy optimal for the $\beta = 2$ case of Section V: the production control u (left) and the corresponding trajectory x (right) as a function of time

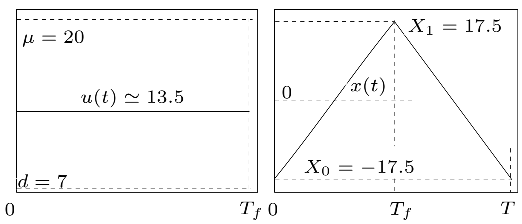

Fig. 4. The policy optimal for the $\beta = 2.093$ case of Section V: the production control u (left) and the corresponding trajectory x (right) as a function of time

explore all the cases analyzed in this paper.

A. $\beta = 0$. The μ-d-μ policy, which is optimal in this case, gives $J = .7813$ and $T_f = 1000$. The control and the state trajectory of the optimal policy are reported in Fig. 2. Observe that the state trajectory on the right is reported for all the cycle (i.e. also during the repair interval $[T_f, T]$), while the control law is reported on the left only in the uptime portion of the cycle (in the remaining part of the cycle the production control is trivially 0).

B. $\beta = 0.5$. If we try to apply a switching policy, we get $\lambda_2 = 2.86$, hence $(\beta - 1)\lambda_2 < 0$, which allows to establish that the computed switching policy is indeed optimal. Applying this policy, we get $J = 1.14$ and $T_f = 684$. The control law and the state trajectory look macroscopically undistinguishable from the previous case. So they are still represented in Fig. 2. However, we want to remark that now, during the phase where the average production is d, actually an infinite number of switches is performed between μ and 0. It is interesting to observe that if the μ-d-μ (non switching) policy was applied to this case, a much larger cost $J = 1.81$ and a much shorter final uptime $T_f = 430$ would be obtained. Another interesting observation is that it is possible to compute the minimum number of switches such that a switching policy performs better than the μ-d-μ policy: we get 1091 switches, with a switching time of 0.62 hours.

C. $\beta = 1$. As for the $\beta = 0$ case, the μ-d-μ policy is optimal and gives a cost $J = 4.55$ and an uptime of duration $T_f = 168$. The control law and the state trajectory are always given in Fig. 2.

D. $\beta = 2$. If we try to apply a switching policy in this case, we get $\lambda_2 = 55.86$, with $(\beta-1)\lambda_2 > 0$. So, actually, we are in the convex case, and the considered switching policy is not optimal. The C_1-search is then applied and the optimal production control computed is reported in Fig. 3 (left), with $J = 57.56$ and $T_f = 9.53$. The corresponding state trajectory is reported in Fig. 3 (right). If the μ-d-μ policy was applied in this case, a significantly larger cost $J = 70.16$ and a shorter uptime $T_f = 6.19$ would have been obtained.

E. $\beta = \beta^* = 2.093$. In this case $S(u) \geq 0$ for a unique value of u (which is $u \simeq 13.5$). Actually, applying the procedure for the convex case above, we get $C_2 = 0$. Hence the optimal policy is constant and it is given by $u \simeq 13.5$. The solution for this case is reported in Fig. 4. In this case $J = 102.73$ and $T_f = 5.15$.

VI. Conclusions

The solution of the problem considered in this paper, established through the maximum principle, shows that when the deterioration is an affine function of the production rate, the optimal policy, called μ-d-μ, operates the machine at maximum rate toward the origin of the buffer space, and remains there (with $u = d$) until the last possible time. Subsequently, the production control is set again to its maximum value until the machine is halted for maintenance. In the concave case, the optimal production control is macroscopically equivalent to the affine case, but the production control actually obtains the d rate with an infinite number of switches among the maximum and the minimum production rate. In the convex case the optimal production control continuously changes as the buffer level increases.

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