Proceedings of the
44th IEEE Conference on Decision and Control, and
the European Control Conference 2005
Seville, Spain, December 12-15, 2005

A behavioral approach to the estimation problem and its applications to state-space models

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***Abstract*— The observer design problem is here investigated in the context of linear left shift invariant discrete behaviors, whose trajectories have support on the positive axis. Necessary and sufficient conditions for the existence of a (consistent) dead-beat observer of some relevant variables from some measured ones, in the presence of some unmeasured (and irrelevant) variables, are introduced, and a complete parametrization of all dead-beat observers is given. Finally, several classical problems addressed for state-space models, like state estimation, the design of unknown input observers or the design of fault detectors and identifiers (possibly in the presence of disturbances), are casted in this general framework, and the aforementioned equivalent conditions and parametrizations are tailored to all these special instances.**

***Index Terms*— Behavior, nilpotent autonomous system, reconstructibility, observer, unknown input observer (UIO), fault detector and isolator (FDI).**

I. INTRODUCTION

The original theory of state observers was concerned with the problem of reconstructing (or estimating) the state from the corresponding inputs and outputs. This problem has been later generalized in various ways, and in relatively recent years there has been a great deal of research aiming at state observers in the presence of unknown inputs (disturbances) [5], [8].

Another research issue, which originated in the eighties and flourished in the nineties [2], [5], but still represents a very lively research topic [3] is the fault detection and isolation (FDI) problem. The problem of detecting and identifying the faults affecting the system functioning, possibly in the presence of disturbances, is naturally seen as an estimation problem. Recent years have witnessed a renewed interest for these issues [4], [9], [10].

The aim of this paper is to investigate the observer design problem for linear time-invariant (discrete-time) dynamic systems that are described in behavioral terms by means of a set of difference equations. Recent results on this topic have been presented in [10] for the continuous time case and in [9] for discrete time systems. In this contribution, the observer theory is further explored, by introducing the concept of consistency of a dead-beat observer (DBO), by parametrizing all dead-beat observers and by characterizing those dynamic systems that admit a consistent DBO endowed with a proper rational transfer matrix and hence realizable by means of a state-space model.

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In detail, we will consider a dynamic system $\Sigma = (\mathbb{Z}_+, \mathbb{R}^{\mathtt{w}}, \mathfrak{B})$, whose system variable $\mathbf{w}$ is naturally split into three subvectors, i.e., $\mathbf{w}^T = [\,\mathbf{w}_r^T \quad \mathbf{w}_m^T \quad \mathbf{w}_i^T\,]^T$, where $\mathbf{w}_m$ represents the set of measured variables, $\mathbf{w}_r$ the (unmeasured) variables which are "relevant" for our estimation problem (actually, the target of our estimation problem), and $\mathbf{w}_i$ the set of variables which are unmeasured but "irrelevant", in the sense that we are not interested in evaluating them. Moreover, the behavior trajectories satisfy a difference equation of the following type

$$[\,R_r(\sigma) \quad -R_m(\sigma) \quad -R_i(\sigma)\,] \begin{bmatrix} \mathbf{w}_r(t) \\ \mathbf{w}_m(t) \\ \mathbf{w}_i(t) \end{bmatrix} = 0, \quad t \in \mathbb{Z}_+,$$

for suitable polynomial matrices R_r, R_m, R_i in the left shift operator σ. The natural goal is that of designing an estimator of $\mathbf{w}_r$ based on the knowledge of $\mathbf{w}_m$, such that (s.t.) its estimation error goes to zero in a finite number of steps, independently of $\mathbf{w}_i$. The general results obtained in the first part of the paper will be later applied to state-space models for formalizing, and hence solving, a wide variety of classical estimation problems (state estimation, state estimation in the presence of disturbances, fault detection and isolation, ...).

In the paper we consider polynomial matrices with entries in $\mathbb{R}[z]$. A polynomial matrix $M \in \mathbb{R}[z]^{p\times q}$ is *right prime* if it is of full column rank q and the GCD of its maximal order minors is a unit. M is right prime if and only if it admits a polynomial left inverse or, equivalently, the Bézout equation $X(z)M(z) = I_q$, in the unknown polynomial matrix X, is solvable. An immediate generalization of prime matrices is represented by the class of monomic matrices. A polynomial matrix M is *right monomic* [4] if it is of full column rank and the GCD of its maximal order minors is a monomial. M is right monomic if and only if it admits a Laurent polynomial left inverse or, equivalently, the diophantine equation $X(z)M(z) = z^N I_q$, in the unknown polynomial matrix X, is solvable for some nonnegative integer N. *Left prime* and *left monomic* matrices are similarly defined and characterized.

If M is a $p\times q$ polynomial matrix of rank r, a polynomial matrix H is a *left annihilator* of M if $H(z)M(z) = 0$. A left annihilator H_m of M is a *minimal left annihilator* (MLA, for short) if it is of full row rank and for any other left annihilator H of M we have $H(z) = P(z)H_m(z)$ for some polynomial matrix P. It can be easily proved that an MLA always exists (if M is of full row rank, it coincides with the empty matrix), it is a $(p-r)\times p$ left prime matrix and is uniquely determined modulo a unimodular left factor. *Right annihilators* and *minimal right annihilators* (MRAs) can be similarly defined and enjoy analogous properties.

In the paper, the size of any vector will be denoted by means of the same typewritten letter that is used for denoting the vector itself, e.g. $\mathtt{w}_m := \dim(\mathbf{w}_m), \mathtt{x} := \dim(\mathbf{x})$, etc. In the paper, all trajectories will be assumed defined on the set $\mathbb{Z}_+$ of nonnegative integers. The left (backward) shift operator on $(\mathbb{R}^p)^{\mathbb{Z}_+}$, the set of trajectories defined on $\mathbb{Z}_+$ and taking values in $\mathbb{R}^p$, is defined as

$$\sigma : (\mathbb{R}^p)^{\mathbb{Z}_+} \to (\mathbb{R}^q)^{\mathbb{Z}_+} : (\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, ...) \mapsto (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, ...).$$

If $M(z) = \sum_{i=0}^{L} M_i z^i \in \mathbb{R}[z]^{p\times q}$ is a polynomial matrix, we associate with it the polynomial matrix operator $M(\sigma) = \sum_{i=0}^{L} M_i \sigma^i$.

II. Basic results about behaviors in $(\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}_+}$

A behavior $\mathfrak{B} \subseteq (\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}_+}$ [7], [9] is said to be *linear* if it is a vector subspace (over $\mathbb{R}$) of $(\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}_+}$, and *left shift invariant* if $\sigma\mathfrak{B} \subseteq \mathfrak{B}$. A linear left shift invariant behavior $\mathfrak{B} \subseteq (\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}_+}$ is *complete* if for every sequence $\tilde{\mathbf{w}} \in (\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}_+}$, the condition $\tilde{\mathbf{w}}|_{\mathcal{S}} \in \mathfrak{B}|_{\mathcal{S}}$ for every finite set $\mathcal{S} \subset \mathbb{Z}_+$ implies $\tilde{\mathbf{w}} \in \mathfrak{B}$, where $\tilde{\mathbf{w}}|_{\mathcal{S}}$ denotes the restriction to $\mathcal{S}$ of the trajectory $\tilde{\mathbf{w}}$ and $\mathfrak{B}|_{\mathcal{S}}$ the set of all restrictions to $\mathcal{S}$ of behavior trajectories. Linear left shift invariant complete behaviors (in the following, *behaviors*) are kernels of polynomial matrices in the left shift operator σ, which amounts to saying that the trajectories $\mathbf{w} = \{\mathbf{w}(t)\}_{t\in\mathbb{Z}_+}$ of $\mathfrak{B}$ can be identified with the set of solutions in $(\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}_+}$ of a system of difference equations

$$R_0\mathbf{w}(t)+R_{i1}\mathbf{w}(t+1)+...+R_L\mathbf{w}(t+L) = 0,\ t \in \mathbb{Z}_+, \quad (1)$$

with $R_i \in \mathbb{R}^{p\times \mathtt{w}}$, and hence described by the equation

$$R(\sigma)\mathbf{w} = 0, \quad (2)$$

where $R(z) := \sum_{i=0}^{L} R_i z^i$ belongs to $\mathbb{R}[z]^{p\times\mathtt{w}}$. In the sequel, we will adopt the short-hand notation $\mathfrak{B} = \ker R(\sigma)$. It can be shown that $\ker R_1(\sigma) \subseteq \ker R_2(\sigma)$ if and only if $R_2 = PR_1$ for some polynomial matrix P.

A behavior for which there exists some $N \in \mathbb{N}$ s.t. all its trajectories have (compact) supports included in $[0, N-1]$ is called *nilpotent autonomous* and it is the kernel of a polynomial matrix operator $R(\sigma)$ corresponding to right monomic matrices [9]. In particular, if R is nonsingular square, then $\ker R(\sigma)$ is nilpotent if and only if $\det R = c\cdot z^N$, for some $c \in \mathbb{R}\setminus\{0\}$ and some $N \in \mathbb{Z}_+$. Of course, $\ker R(\sigma)$ is the zero behavior if and only if $\det R = c \neq 0$, namely R is unimodular.

III. Observability and reconstructibility

Consider a dynamic system $\Sigma = (\mathbb{Z}_+, \mathbb{R}^{\mathtt{w}}, \mathfrak{B})$, whose behavior $\mathfrak{B}$ is described as in (2), for some polynomial matrix R. Independently of the physical meaning of the system variables which are grouped together in the vector $\mathbf{w}$, when dealing with any type of estimation problem a first natural distinction is introduced between measured variables, denoted by $\mathbf{w}_m$, and unmeasured variables. These latter, in turn, may be naturally split into the subvector of all system variables which are (unmeasured and) the object of our estimation problem (the "relevant" variables for the specific estimation problem), $\mathbf{w}_r$, and the subvector of all variables which are both unmeasured (for instance because they represent disturbances or modeling errors) and "irrelevant" for our estimation problem. We refer to such a subvector as $\mathbf{w}_i$. As a consequence, $\mathbf{w}^T = [\,\mathbf{w}_r^T \quad \mathbf{w}_m^T \quad \mathbf{w}_i^T\,]^T$. The polynomial matrix R can be accordingly block-partitioned, thus leading to the following description:

$$[\,R_r(\sigma) \quad -R_m(\sigma) \quad -R_i(\sigma)\,]\begin{bmatrix}\mathbf{w}_r(t)\\ \mathbf{w}_m(t)\\ \mathbf{w}_i(t)\end{bmatrix} = 0,\ t \in \mathbb{Z}_+, \quad (3)$$

or, equivalently

$$R_r(\sigma)\mathbf{w}_r(t) = R_m(\sigma)\mathbf{w}_m(t) + R_i(\sigma)\mathbf{w}_i(t),\ t \in \mathbb{Z}_+. \quad (4)$$

Definition 3.1: [9], [10] Given a dynamic system $\Sigma = (\mathbb{Z}_+, \mathbb{R}^{\mathtt{w}}, \mathfrak{B})$ whose behavior $\mathfrak{B}$ is described as in (4), we say that $\mathbf{w}_r$ is reconstructible from $\mathbf{w}_m$, if $(\mathbf{w}_r, \mathbf{w}_m, \mathbf{w}_i), (\bar{\mathbf{w}}_r, \mathbf{w}_m, \bar{\mathbf{w}}_i) \in \mathfrak{B}$ implies that there exists $N \in \mathbb{Z}_+$ s.t. $\mathbf{w}_r(t) - \bar{\mathbf{w}}_r(t) = 0,\ \forall\ t \geq N$. In particular, when $N = 0$, $\mathbf{w}_r$ is said to be observable from $\mathbf{w}_m$. ■

A dead-beat observer (DBO) of $\mathbf{w}_r$ from $\mathbf{w}_m$ is a system that, corresponding to every trajectory $(\mathbf{w}_r, \mathbf{w}_m, \mathbf{w}_i)$ in $\mathfrak{B}$, produces an estimate $\hat{\mathbf{w}}_r$ of the trajectory $\mathbf{w}_r$ (based on the measured variable $\mathbf{w}_m$ alone), that coincides with $\mathbf{w}_r$ except, possibly, for a finite number of time instants.

Definition 3.2: [9] Consider the dynamic system Σ, whose behavior $\mathfrak{B}$ is described as in (4). The system represented by the difference equation

$$Q(\sigma)\hat{\mathbf{w}}_r = P(\sigma)\mathbf{w}_m, \quad (5)$$

with P and Q polynomial matrices of suitable dimensions, is said to be

- a dead-beat observer (DBO) of $\mathbf{w}_r$ from $\mathbf{w}_m$ for Σ if
  - (a) for every $(\mathbf{w}_r, \mathbf{w}_m, \mathbf{w}_i) \in \mathfrak{B}$ there exists $\hat{\mathbf{w}}_r$ s.t. $(\hat{\mathbf{w}}_r, \mathbf{w}_m)$ satisfies (5), and
  - (b) there exists $N \in \mathbb{Z}_+$ s.t. for every $(\mathbf{w}_r, \mathbf{w}_m, \mathbf{w}_i)$ in $\mathfrak{B}$ and $(\hat{\mathbf{w}}_r, \mathbf{w}_m)$ satisfying (5), we have $\mathbf{w}_r(t) - \hat{\mathbf{w}}_r(t) = 0$ for every $t \geq N$;
- a consistent dead-beat observer (cDBO) of $\mathbf{w}_r$ from $\mathbf{w}_m$ for Σ if it is a DBO and for every $(\mathbf{w}_r, \mathbf{w}_m, \mathbf{w}_i)$ in $\mathfrak{B}$ the trajectory $(\mathbf{w}_r, \mathbf{w}_m)$ always satisfies (5). ■

Remarks i) For an observer described by (5), the difference variable $\mathbf{e} := \mathbf{w}_r - \hat{\mathbf{w}}_r$ represents the *estimation error*. So, an observer is dead-beat if the set of its estimation error trajectories constitutes a nilpotent autonomous behavior $\mathfrak{B}_e$.

ii) The concept of consistent DBO may sound somewhat strange and redundant. However, simple examples prove [1] that this is not the case. Nonetheless it can be shown that if a DBO exists then also a cDBO may be found.

Lemma 3.3: [1] Given a dynamic system Σ, whose behavior $\mathfrak{B}$ is described as in (4),

i) if Σ is reconstructible, the polynomial matrix R_r is right monomic;

ii) if Σ admits a DBO (5), the polynomial matrix Q appearing in (5) is right monomic.

Theorem 3.4: Consider a dynamic system, whose behavior $\mathfrak{B}$ is described as in (4), and let H_i denote a minimal left annihilator of R_i. The following facts are equivalent:

ia) there exists a consistent DBO for Σ;

ib) there exists a DBO for Σ;

ii) $\Gamma := H_i R_r$ is right monomic;

iii) R_r is right monomic and there exist $N \in \mathbb{Z}_+$ and a polynomial matrix L satisfying

$$L(z)\,[\,R_r(z) \quad R_i(z)\,] = [\,z^N I \quad 0\,]; \tag{6}$$

iv) $\mathfrak{B}$ is reconstructible.

Proof: ia) $\Rightarrow$ ib) Obvious.

ib) $\Rightarrow$ ii) If there exists a DBO, there exists $N \in \mathbb{Z}_+$ s.t. $\sigma^N \mathbf{w}_r(t) - \sigma^N \hat{\mathbf{w}}_r(t) = \mathbf{w}_r(t+N) - \hat{\mathbf{w}}_r(t+N) = 0, t \in \mathbb{Z}_+$. This easily implies

$$[\,\sigma^N Q(\sigma) \;\; -\sigma^N P(\sigma) \;\; 0\,][\,\mathbf{w}_r(t)^T \;\; \mathbf{w}_m(t)^T \;\; \mathbf{w}_i(t)^T\,]^T = 0.$$

As this condition holds for any $(\mathbf{w}_r(t), \mathbf{w}_m(t), \mathbf{w}_i(t)) \in \mathfrak{B}$, then $\ker[\,R_r(\sigma) \;\; -R_m(\sigma) \;\; -R_i(\sigma)\,] \subseteq \ker[\,\sigma^N Q(\sigma) \;\; -\sigma^N P(\sigma) \;\; 0\,]$. Therefore a polynomial matrix T exists s.t.

$$T(z)[R_r(z) \;\; -R_m(z) \;\; -R_i(z)] = [z^N Q(z) \;\; -z^N P(z) \;\; 0]$$

i.e. $T(z)R_r(z) = z^N Q(z)$, $T(z)R_m(z) = z^N P(z)$, $T(z)R_i(z) = 0$. The last condition implies that T is a left annihilator of R_i and hence $T(z) = S(z)H_i(z)$ for some polynomial matrix S. Consequently $S(z)\,[H_i(z)R_r(z)] = z^N Q(z)$, and since Q is right monomic, by Lemma 3.3, a polynomial matrix V exists s.t. $V(z)Q(z) = z^k I_{\mathtt{w}_r}$ for some $k \in \mathbb{Z}_+$, and hence $V(z)S(z)\Gamma(z) = z^{k+N} I_{\mathtt{w}_r}$, showing that Γ is, in turn, right monomic.

ii) $\Rightarrow$ iii) If Γ is right monomic, there is a polynomial matrix S s.t. $S(z)\Gamma(z) = z^N I_{\mathtt{w}_r}$, for some $N \in \mathbb{Z}_+$. $L(z) := S(z)H_i(z)$ clearly satisfies (6).

iii) $\Rightarrow$ ia) Suppose that there exists a polynomial matrix L satisfying (6), and let $(\mathbf{w}_r(t), \mathbf{w}_m(t), \mathbf{w}_i(t))$ be any trajectory in $\mathfrak{B}$, and hence satisfying (3). Premultiplication by $L(\sigma)$ leads to

$$[\,\sigma^N I_{\mathtt{w}_r} \;\; -L(\sigma)R_m(\sigma) \;\; 0\,][\,\mathbf{w}_r(t)^T \;\; \mathbf{w}_m(t)^T \;\; \mathbf{w}_i(t)^T\,]^T = 0. \tag{7}$$

It is easy to show that $Q(z) = z^N I_{\mathtt{w}_r}$ and $P(z) = L(z)R_m(z)$ define a consistent DBO (5).

iii) $\Rightarrow$ iv) Suppose that there exists a polynomial matrix L satisfying (6) and let $(\mathbf{w}_r(t), \mathbf{w}_m(t), \mathbf{w}_i(t))$ and $(\bar{\mathbf{w}}_r(t), \mathbf{w}_m(t), \bar{\mathbf{w}}_i(t))$ be two trajectories in $\mathfrak{B}$, and hence satisfying (3). Premultiplicating (3) corresponding to $(\mathbf{w}_r(t), \mathbf{w}_m(t), \mathbf{w}_i(t))$ and to $(\bar{\mathbf{w}}_r(t), \mathbf{w}_m(t), \bar{\mathbf{w}}_i(t))$ by $L(\sigma)$, and subtracting one equation to the other, leads to $\sigma^N\,[\mathbf{w}_r(t) - \bar{\mathbf{w}}_r(t)] = 0$, thus ensuring reconstructibility.

iv) $\Rightarrow$ ii) Factorize R_i as $R_i = R_{i1}R_{i2}$, with R_{i1} of size say $p \times k$ and right prime and R_{i2} of full row rank k, and column border R_{i1} up to a unimodular matrix by means of some (right prime) polynomial matrix C. Set $U(z) := [\,R_{i1}(z) \quad C(z)\,]^{-1}$. W.l.o.g. we can assume $U(z) = \begin{bmatrix} S(z) \\ H_i(z) \end{bmatrix}$, so that $\mathfrak{B}$ is equivalently described as

$$\begin{bmatrix} S(\sigma)R_r(\sigma) \\ H_i(\sigma)R_r(\sigma) \end{bmatrix}\mathbf{w}_r = \begin{bmatrix} S(\sigma)R_m(\sigma) \\ H_i(\sigma)R_m(\sigma) \end{bmatrix}\mathbf{w}_m + \begin{bmatrix} R_{i2}(\sigma) \\ 0 \end{bmatrix}\mathbf{w}_i. \tag{8}$$

Consequently, a trajectory $(\bar{\mathbf{w}}_r, 0, \bar{\mathbf{w}}_i)$ belongs to $\mathfrak{B}$ if and only if $\Gamma(\sigma)\bar{\mathbf{w}}_r = 0$ and $S(\sigma)R_r(\sigma)\bar{\mathbf{w}}_r = R_{i2}(\sigma)\bar{\mathbf{w}}_i$. By the full row rank property of R_{i2}, $R_{i2}(\sigma)$ defines a surjective map, and hence for any $\bar{\mathbf{w}}_r \in \ker\Gamma(\sigma)$ there exists $\bar{\mathbf{w}}_i$ which satisfies the second equation. So, if Γ were not right monomic, there would be an infinite support $\bar{\mathbf{w}}_r \in \ker\Gamma(\sigma)$, and a trajectory $\bar{\mathbf{w}}_i$ s.t. $(\bar{\mathbf{w}}_r, 0, \bar{\mathbf{w}}_i)$ belongs to $\mathfrak{B}$. So, both $(0,0,0)$ and $(\bar{\mathbf{w}}_r, 0, \bar{\mathbf{w}}_i)$ would be in $\mathfrak{B}$, thus contradicting reconstructibility. Thus Γ is right monomic. ■

Remark The previous theorem exhibits two limit cases:

• When no irrilevant variables are involved in the behavior description (i.e. R_i is the empty matrix), then H_i reduces to the identity matrix and hence the existence of a DBO is equivalent to the right monomicity of R_r.

• When R_i is of full row rank, then H_i is the empty matrix. When so, Theorem 3.4 can be read in a negative sense, since none of the conditions ia), ib), ii), iii) and iv) can be satisfied.

IV. A PARAMETRIZATION OF ALL CONSISTENT DEAD-BEAT OBSERVERS

Consider, again, the behavior $\mathfrak{B}$, described as in (4), and suppose that it admits a DBO. We aim to provide a complete parametrization of all consistent DBOs for $\mathfrak{B}$. To this end, we first recall the concept of equivalent observers [10] and a useful technical lemma.

Given a DBO, its behavior $\hat{\mathfrak{B}}$ is the set of all solutions $(\hat{\mathbf{w}}_r, \mathbf{w}_m)$ of the difference equation (5). Among all the trajectories of $\hat{\mathfrak{B}}$, however, we are interested only in those that are produced corresponding to the trajectories of $\mathfrak{B}$, namely in the set $\{(\hat{\mathbf{w}}_r, \mathbf{w}_m) \in \hat{\mathfrak{B}} : \mathbf{w}_m \in \mathcal{P}_m\mathfrak{B}\}$, where

$$\mathcal{P}_m\mathfrak{B} := \{\mathbf{w}_m : \exists\; \mathbf{w}_r, \mathbf{w}_i \text{ s.t. } (\mathbf{w}_r, \mathbf{w}_m, \mathbf{w}_i) \in \mathfrak{B}\}.$$

So, by assuming this point of view, it seems reasonable to regard as *equivalent* two observers (5), for the same system Σ, not if their behaviors $\hat{\mathfrak{B}}_1$ and $\hat{\mathfrak{B}}_2$ coincide, but if they satisfy the condition $\{(\hat{\mathbf{w}}_r, \mathbf{w}_m) \in \hat{\mathfrak{B}}_1 : \mathbf{w}_m \in \mathcal{P}_m\mathfrak{B}\} = \{(\hat{\mathbf{w}}_r, \mathbf{w}_m) \in \hat{\mathfrak{B}}_2 : \mathbf{w}_m \in \mathcal{P}_m\mathfrak{B}\}$. We can now introduce the following result about equivalent observers.

Lemma 4.1: [10] If $Q(\sigma)\hat{\mathbf{w}}_r = P(\sigma)\mathbf{w}_m$ is a DBO for Σ, there exists an equivalent DBO $\bar{Q}(\sigma)\hat{\mathbf{w}}_r = \bar{P}(\sigma)\mathbf{w}_m$ with $\bar{Q}$ of full row rank and hence, by Lemma 3.3, nonsingular square and monomic. ■

Due to this lemma, from now on, we will steadily focus on the parametrization of all those observers whose matrix Q is nonsingular square. Aiming at this goal, it is convenient to adopt for the behavior $\mathfrak{B}$ the equivalent description (8), where R_{i2} is of full row rank. $R_{i2}(\sigma)$ defines a surjective map, and hence for any pair $(\mathbf{w}_r, \mathbf{w}_m)$ in

$$\ker [\, H_i(\sigma)R_r(\sigma) \quad - H_i(\sigma)R_m(\sigma)\,] =: \ker [\, \Gamma(\sigma) \quad - \Phi(\sigma)\,]$$

there exists $\mathbf{w}_i$ which satisfies the first equation. So,

$$\begin{aligned}\mathcal{P}_{r,m}\mathfrak{B} \quad &:= \quad \{(\mathbf{w}_r, \mathbf{w}_m) : \exists\ \mathbf{w}_i \text{ s.t. } (\mathbf{w}_r, \mathbf{w}_m, \mathbf{w}_i) \in \mathfrak{B}\} \\ &= \quad \ker [\, \Gamma(\sigma) \quad - \Phi(\sigma)\,]\,.\end{aligned}$$

By adopting a reasoning not too far from those adopted in Theorem 3.4 of [10], we can show the following result.

Theorem 4.2: Consider a system Σ whose behavior $\mathfrak{B}$ is described as in (8), with R_{i2} of full row rank and Γ right monomic. If P and Q are polynomial matrices, with Q nonsingular square, then $Q(\sigma)\hat{\mathbf{w}}_r = P(\sigma)\mathbf{w}_m,$ is a consistent dead-beat observer for Σ if and only if there exists a polynomial matrix Y s.t.

$$[Q(z) \mid -P(z)] := Y(z)\,[\, \Gamma(z) \quad - \Phi(z)\,] \tag{9}$$

with $Y\Gamma$ monomic. ■

Let, now, U be a unimodular matrix s.t. $U(z)\Gamma(z) = \begin{bmatrix} \Delta(z) \\ 0 \end{bmatrix}$, with Δ nonsingular square, and conformably partition $U(z)\Phi(z) = \begin{bmatrix} L_1(z) \\ L_0(z) \end{bmatrix}$. Clearly, Δ is a nonsingular square monomic matrix and hence $\det\Delta = c \cdot z^N, \exists c \in \mathbb{R}, c \neq 0, N \in \mathbb{Z}_+$. $\mathfrak{B}$ can then be equivalently described as follows:

$$\begin{bmatrix} S(\sigma)R_r(\sigma) \\ \Delta(\sigma) \\ 0 \end{bmatrix}\mathbf{w}_r = \begin{bmatrix} S(\sigma)R_m(\sigma) \\ L_1(\sigma) \\ L_0(\sigma) \end{bmatrix}\mathbf{w}_m + \begin{bmatrix} R_{i2}(\sigma) \\ 0 \\ 0 \end{bmatrix}\mathbf{w}_i. \tag{10}$$

By referring to this behavior description, the parametrization given in Theorem 4.2 becomes the following one:

$$[\, Q(z) \quad - P(z)\,] = [\, Y(z) \quad X(z)\,]\begin{bmatrix} \Delta(z) & - L_1(z) \\ 0 & - L_0(z) \end{bmatrix} \tag{11}$$

with Y monomic and X polynomial.

V. Causal dead-beat observers

If we simply mean to obtain a "behavioral approach" to the solution of various types of estimation problems, and a parametric description of all available solutions, the results of the previous sections already provide satisfactory answers. If we aim at applying the previous general results to the specific problems one may address in the state-space setting, however, it is extremely important to investigate the existence of consistent DBOs which admit a state-space realization. This requires the observer transfer matrix $\hat{W}(z) := Q^{-1}(z)P(z)$ to be proper, and since Q is monomic, this implies that $\hat{W}(z)$ has to be a polynomial matrix in the negative powers of z, i.e. $\hat{W}(z) = \bar{W}(z^{-1})$, for some polynomial matrix $\bar{W}(z)$.

The previous parameterization can be fruitfully exploited to investigate this issue. If we refer to the behavior description (8), and hence to the following description of $\mathcal{P}_{r,m}\mathfrak{B}$

$$\Gamma(\sigma)\mathbf{w}_r = \Phi(\sigma)\mathbf{w}_m, \tag{12}$$

it entails no loss of generality assuming that $[\, \Gamma(z) \quad - \Phi(z)\,]$ is a row reduced matrix with row degrees $\mu_1, \mu_2, \ \dots, \ \mu_{p-k}$. Of course, Γ is supposed to be right monomic. So, the general expression of the observer transfer matrix is $\hat{W}(z) = [Y(z)\Gamma(z)]^{-1}\,[Y(z)\Phi(z)]$, with Y a polynomial matrix s.t. $Y\Gamma$ is square monomic.

Theorem 5.1: [1] Consider a dynamic system Σ with behavior $\mathfrak{B}$ described as in (8) and assume that $\mathbf{w}_r$ is reconstructible from $\mathbf{w}_m$. Suppose that $[\, \Gamma(z) \quad - \Phi(z)\,] \in \mathbb{R}[z]^{(p-k)\times(\mathtt{w}_r+\mathtt{w}_m)}$ is row reduced with row degrees $\mu_1, \mu_2, \ \dots, \ \mu_{p-k}$, so that

$$\begin{aligned}[\, \Gamma(z) \quad - \Phi(z)\,] &= \operatorname{diag}\{z^{\mu_1}, z^{\mu_2}, \dots, z^{\mu_{p-k}}\} \\ &\cdot [\, \Gamma_0 + z^{-1} \cdot \bar{\Gamma}(z^{-1}) \quad - \Phi_0 - z^{-1} \cdot \bar{\Phi}(z^{-1})\,],\end{aligned} \tag{13}$$

with Γ_0 and Φ_0 constant matrices, $[\, \Gamma_0 \quad - \Phi_0\,]$ of full row rank, and $\bar{\Gamma}(z^{-1})$ and $\bar{\Phi}(z^{-1})$ polynomial matrices in z^{-1} of suitable sizes. A necessary and sufficient condition for the existence of a consistent DBO endowed with a proper transfer matrix $\hat{W}$ is that Γ_0 is of full column rank.

Remarks. i) Of course, a parametrization of the DBO transfer matrices could be obtained also by referring to the parametrization (11). Indeed, the DBO transfer matrix (upon some simplifications) takes the following form

$$\hat{W}(z) = \Delta^{-1}(z)L_1(z) + \Delta^{-1}(z)Y^{-1}(z)X(z)L_0(z), \tag{14}$$

as Y and X vary over the set of all polynomial matrices of suitable sizes (under the constraint that Y is square monomic).

ii) We may now sketch a simple algorithm for designing a DBO: as a preliminary step, we have to check whether the polynomial matrix Γ is right monomic or not. If not, a DBO does not exist. Otherwise, we may resort to the parametrization of the DBOs given in (11) and to the parametrization of the corresponding transfer matrices given in (14). By performing the rank test on the matrix Γ_0, we may test whether a causal DBO (i.e., endowed with a transfer matrix which is polynomial in z^{-1}) exists. If so, we can realize it by means of a finite memory system:

$$\mathbf{v}(t+1) = F\mathbf{v}(t) + G\mathbf{w}_m(t), \quad \hat{\mathbf{w}}_r(t) = Hv(t) + J\mathbf{w}_m(t).$$

When causal DBOs are not available, we may still resort to state-space realizations. However these systems will provide only "delayed" estimations, in the sense that if $\hat{W}(z) = z^i \cdot \tilde{W}(z^{-1})$, with $\tilde{W}(z^{-1})$ a polynomial matrix in the variable z^{-1}, we can realize $\tilde{W}(z^{-1})$ by means of a state-space model. The corresponding DBO output will be $\hat{\mathbf{w}}_r(t-i)$, instead of $\hat{\mathbf{w}}_r(t)$.

VI. Application to state-space models

In this section we show how the estimation theory developed within the behavioral approach allows to treat in a homogeneous way classical estimation problems for state-space systems. To this end, we consider the most general expression of a state-space model, including not only the state, input and output variables, but also disturbances and additive faults. Once we cast the state-space model in the behavioral framework, by differently choosing the measured, the relevant and the irrelevant variables, we will be able to formalize the following traditional problems:

- the state estimation when neither disturbances nor faults affect the system (as a special case, the Luenberger observer);
- the state estimation when only disturbances affect the system: *unknown input observer* (UIO);
- the fault detection and isolation when no disturbance affects the system (but faults, of course, do) (FDI);
- the fault detection and isolation in the presence of disturbances (dFDI).

A general state-space model is described, for $t \in \mathbb{Z}_+$, by the following equations:

$$\begin{aligned} \mathbf{x}(t+1) &= A\mathbf{x}(t) + B_u\mathbf{u}(t) + B_d\mathbf{d}(t) + B_f\mathbf{f}(t), & (15)\\ \mathbf{y}(t) &= C\mathbf{x}(t) + D_u\mathbf{u}(t) + D_d\mathbf{d}(t) + D_f\mathbf{f}(t), & (16)\end{aligned}$$

where $\mathbf{x}$ denotes the state, $\mathbf{u}$ the controlled input, $\mathbf{y}$ the measured output, $\mathbf{d}$ the disturbance (i.e., the uncontrolled input) and $\mathbf{f}$ the fault. The state-space model (15)-(16) can be rewritten in behavioral form as

$$\begin{bmatrix} \sigma I_{\mathtt{x}} - A & 0 & -B_u & -B_d & -B_f \\ C & -I_{\mathtt{y}} & D_u & D_d & D_f \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \\ \mathbf{u} \\ \mathbf{d} \\ \mathbf{f} \end{bmatrix} = 0, \qquad (17)$$

A. Standard state estimation

If neither faults $\mathbf{f}$ nor disturbances $\mathbf{d}$ affect the system, we are reduced to the case of plain state estimation from the controlled input and the measured output. When so, the relevant variable is $\mathbf{w}_r = \mathbf{x}$, the available measurement is $\mathbf{w}_m = \left[\mathbf{y}^T \ \mathbf{u}^T\right]^T$, and there are no irrelevant variables $\mathbf{w}_i$. The behavioral equation takes the form

$$\left[\begin{array}{c|cc} \sigma I_{\mathtt{x}} - A & 0 & -B_u \\ C & -I_{\mathtt{y}} & D_u \end{array}\right] [\, \mathbf{x}^T \mid \mathbf{y}^T \ \ \mathbf{u}^T \,]^T = 0.$$

As previously remarked, in this case R_i is the empty matrix and hence $H_i(z) = I_{\mathtt{x}+\mathtt{y}}$, while

$$R_r(z) = \begin{bmatrix} zI_{\mathtt{x}} - A \\ C \end{bmatrix} =: \mathcal{O}(z).$$

So, reconstructibility, and hence the existence of a dead-beat state observer, corresponds to the right monomicity of the Hautus observability matrix $\mathcal{O}$, a well-known result [6]. When so, both causal and non-causal DBO can be constructed. Indeed, the polynomial matrix

$$[\, \Gamma(z) \quad -\Phi(z) \,] = \left[\begin{array}{c|cc} zI_{\mathtt{x}} - A & 0 & -B_u \\ C & -I_{\mathtt{y}} & D_u \end{array}\right]$$

is row reduced and the constant matrix $\Gamma_0 = \begin{bmatrix} I_{\mathtt{x}} \\ C \end{bmatrix}$ is of full column rank. Consequently, DBOs endowed with a proper transfer matrix always exist.

B. Unknown input observers (UIOs)

When faults $\mathbf{f}$ are not contemplated, but disturbances $\mathbf{d}$ affect the system dynamics, we are reduced to the problem of designing an UIO: the relevant variable is $\mathbf{w}_r = \mathbf{x}$, while the available measurement is $\mathbf{w}_m = \left[\mathbf{y}^T \ \mathbf{u}^T\right]^T$. The irrelevant variables are of course represented by the disturbances $\mathbf{w}_i = \mathbf{d}$. The behavioral equations can be block-partitioned in the following form

$$\left[\begin{array}{c|cc|c} \sigma I_{\mathtt{x}} - A & 0 & -B_u & -B_d \\ C & -I_{\mathtt{y}} & D_u & D_d \end{array}\right] \begin{bmatrix} \mathbf{x} \\ - \\ \mathbf{y} \\ \mathbf{u} \\ - \\ \mathbf{d} \end{bmatrix} = 0.$$

Upon introducing an MLA of $R_i(z) = \begin{bmatrix} B_d \\ -D_d \end{bmatrix}$, which can always be assumed a constant matrix so that $H_i(z) = [\, H_{iB} \quad H_{iD} \,]$, a dead-beat UIO exists if and only if the polynomial matrix

$$[\, H_{iB} \quad H_{iD} \,] R_r(z) = H_{iB}(zI_{\mathtt{x}} - A) + H_{iD}C =: \Gamma_x(z)$$

is right monomic. This requires, in particular, that the state-space model is reconstructible in the classical sense and that $\mathtt{x} + \mathtt{y} - \mathrm{rank}\,[\, B_d^T \quad -D_d^T \,] \geq \mathtt{x}$, namely $\mathtt{y} \geq \mathrm{rank}\,[\, B_d^T \quad -D_d^T \,]$. In this case

$$[\, \Gamma(z) \quad -\Phi(z) \,] = [\, H_{iB} \quad H_{iD} \,] \left[\begin{array}{c|cc} zI_{\mathtt{x}} - A & 0 & -B_u \\ C & -I_{\mathtt{y}} & D_u \end{array}\right]$$

is not necessarily row reduced. Moreover, causal dead-beat UIOs could not exist, as shown in the following example.

Example. Consider a state-space model (17) with

$$A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, C = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, B_d = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, D_d = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}.$$

This is an observable system devoid of controlled inputs but affected by disturbances. In this case

$$R_r(z) = \begin{bmatrix} zI - A \\ C \end{bmatrix} = \begin{bmatrix} z & 0 \\ -1 & z \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \; R_m(z) = \begin{bmatrix} 0 \\ I \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$R_i(z) = \begin{bmatrix} B_d \\ -D_d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \; H_i(z) = \begin{bmatrix} 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

$\Gamma(z) = \begin{bmatrix} -1 & \sigma \\ 0 & 1 \end{bmatrix} = \Delta(z)$ is unimodular, $L_1(z) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$, while $L_0(z) = \emptyset$. Therefore the DBO transfer matrix is uniquely determined. Moreover $[\Gamma(z) \mid -\Phi(z)]$ is row reduced and Γ_0 is not of full column rank. Therefore there exists a unique non-causal DBO described by $\hat{\mathbf{x}}(t) = \begin{bmatrix} 1 & \sigma \\ 0 & 1 \end{bmatrix} \mathbf{y}(t)$. Real time estimation cannot be performed, since only $\hat{\mathbf{x}}(t-1)$ can be computed at time t. ♠

Another interesting problem, even though less explored in the literature, is that of obtaining estimates both for the state and for the disturbance: in this case the relevant variable is $\mathbf{w}_r = [\mathbf{x}^T \ \mathbf{d}^T]^T$, the measured variable is

$\mathbf{w}_m = \left[\mathbf{y}^T \ \mathbf{u}^T\right]^T$ and no irrelevant variables are involved in the system description:

$$\left[\begin{array}{cc|cc} \sigma I_{\mathrm{x}} - A & -B_d & 0 & -B_u \\ C & D_d & -I_{\mathrm{y}} & D_u \end{array}\right] \left[\begin{array}{c} \mathbf{x} \\ \mathbf{d} \\ - \\ \mathbf{y} \\ \mathbf{u} \end{array}\right] = 0$$

The MLA H_i becomes $I_{\mathrm{x+y}}$ and the reconstructibility property corresponds to the following system matrix

$$R_r(z) = \begin{bmatrix} zI_{\mathrm{x}} - A & -B_d \\ C & D_d \end{bmatrix} =: \Gamma_{x,d}(z)$$

being right monomic.

C. Fault detection and isolation (FDI)

When disturbances $\mathbf{d}$ may be neglected, we may face to two interesting problems.

1) *The design of an observer-based FDI*: this corresponds to assuming as relevant variables both $\mathbf{x}$ and $\mathbf{f}$, i.e. $\mathbf{w}_r = \left[\mathbf{x}^T \ \mathbf{f}^T\right]^T$, while using as measurements $\mathbf{w}_m = \left[\mathbf{y}^T \ \mathbf{u}^T\right]^T$. If so, no irrelevant variables appear in the system description and, again, $H_i(z) = I_{\mathrm{x+y}}$. The behavioral description can be block-partitioned as follows

$$\left[\begin{array}{cc|cc} \sigma I_{\mathrm{x}} - A & -B_f & 0 & -B_u \\ C & D_f & -I_{\mathrm{y}} & D_u \end{array}\right] \left[\begin{array}{c} \mathbf{x} \\ \mathbf{f} \\ - \\ \mathbf{y} \\ \mathbf{u} \end{array}\right] = 0,$$

and a dead-beat FDI exists if and only if

$$R_r(z) = \begin{bmatrix} zI_{\mathrm{x}} - A & -B_f \\ C & D_f \end{bmatrix} =: \Gamma_{x,f}(z)$$

is right monomic.

2) *The design of an FDI which allows to estimate the faults, disregarding the state variable evolution.* In this case $\mathbf{f}$ becomes the only relevant variable $\mathbf{w}_r$, while $\mathbf{x}$ becomes the irrelevant variable $\mathbf{w}_i$. In this case we can write

$$\left[\begin{array}{c|cc|c} -B_f & 0 & -B_u & \sigma I_{\mathrm{x}} - A \\ D_f & -I_{\mathrm{y}} & D_u & C \end{array}\right] \left[\begin{array}{c} \mathbf{f} \\ - \\ \mathbf{u} \\ \mathbf{y} \\ - \\ \mathbf{x} \end{array}\right] = 0.$$

Now R_i is just the Hautus observability matrix and once we select any left coprime matrix fraction description $D_L(z)^{-1}N_L(z)$ of the state to output transfer matrix $C(zI_{\mathrm{x}} - A)^{-1}$, we get $H_i(z) = [\, N_L(z) \quad -D_L(z) \,]$ as an MLA of the Hautus matrix. Consequently, a dead-beat FDI exists if and only if $H_i(z) \begin{bmatrix} B_f \\ -D_f \end{bmatrix} = N_L(z)B_f + D_L(z)D_f =: \Gamma_f(z)$ is right monomic.

D. Fault detection and isolation in presence of disturbances

Similarly to what analyzed in the previous subsection, two different FDI problems in the presence of disturbances may be considered: one may be interested in estimating both $\mathbf{x}$ and $\mathbf{f}$, i.e. $\mathbf{w}_r = \left[\mathbf{x}^T \ \mathbf{f}^T\right]^T$, making use of the measurement $\mathbf{w}_m = \left[\mathbf{y}^T \ \mathbf{u}^T\right]^T$, and disregarding $\mathbf{w}_i = \mathbf{d}$. When so, the behavioral equation takes the form

$$\left[\begin{array}{cc|cc|c} \sigma I_{\mathrm{x}} - A & -B_f & 0 & -B_u & -B_d \\ C & D_f & -I_{\mathrm{y}} & D_u & D_d \end{array}\right] \left[\begin{array}{c} \mathbf{x} \\ \mathbf{f} \\ - \\ \mathbf{y} \\ \mathbf{u} \\ - \\ \mathbf{d} \end{array}\right] = 0$$

Upon denoting by $[\, H_{iB} \quad H_{iD} \,]$ a (constant) MLA of $R_i(z) := \begin{bmatrix} B_d \\ -D_d \end{bmatrix}$, the existence of an observer-based FDI which produces exact estimates of both the state and the fault after a finite number of steps corresponds to the right monomicity of the following matrix

$$[\, H_{iB} \quad H_{iD} \,] \begin{bmatrix} zI_{\mathrm{x}} - A & -B_f \\ C & D_f \end{bmatrix} =: \Gamma_{x,f}(z).$$

The other case corresponds to the problem of estimating the faults, from the input and output measurements, by neglecting the state dynamics and the disturbances. In this case $\mathbf{w}_r = \mathbf{f}$, $\mathbf{w}_m = \left[\mathbf{y}^T \ \mathbf{u}^T\right]^T$ and $\mathbf{w}_i = \left[\mathbf{x}^T \ \mathbf{d}^T\right]^T$. Consequently, we can write

$$\left[\begin{array}{c|cc|cc} -B_f & 0 & -B_u & \sigma I_{\mathrm{x}} - A & -B_d \\ D_f & -I_{\mathrm{y}} & D_u & C & D_d \end{array}\right] \left[\begin{array}{c} \mathbf{f} \\ - \\ \mathbf{y} \\ \mathbf{u} \\ - \\ \mathbf{x} \\ \mathbf{d} \end{array}\right] = 0.$$

The polynomial matrix H_i represents, in this case, an MLA of the system matrix

$$\Gamma_{x,d}(z) = \begin{bmatrix} zI_{\mathrm{x}} - A & -B_d \\ C & D_d \end{bmatrix},$$

and the existence of a (non-observer based) dead-beat FDI in the presence of disturbances is equivalent to the right monomicity of $H_i(z)R_r(z) = H_i(z) \begin{bmatrix} B_f \\ -D_f \end{bmatrix} =: \Gamma_f(z)$.

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