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Zames-Falb multipliers for quadratic programming

W. P. Heath and A. G. Wills

Abstract— **In constrained linear model predictive control a quadratic program must be solved on-line at each control step. If zero is feasible the resultant static nonlinearity is sector bound. We show that the nonlinearity is also monotone nondecreasing and slope restricted; furthermore it may be expressed as the gradient of a convex potential function. Hence we show the existence of Zames-Falb multipliers for such a nonlinearity.**

For completeness, we construct such multipliers both for the general case of multi-input multi-output static nonlinearities and for the particular case where the nonlinearity arises from a quadratic program. We also express the results in terms of integral quadratic constraints. These multipliers may be used in a general and versatile analysis of the robust stability of constrained model predictive control.

I. Introduction

Recently we proposed a new stability test for constrained linear MPC (model predictive control) [1], [2]. We considered the quadratic program that must be solved on-line at each control step as a static nonlinearity. We showed that if zero is feasible then the nonlinearity is sector bound. Hence the multivariable circle criterion gives a sufficient condition for closed-loop stability. The results may be used to demonstrate stability against both structured and unstructured infinity-norm bound model uncertainty [3].

In this paper we show the existence of Zames-Falb multipliers for such a nonlinearity. In particular we show that the nonlinearity is bound, monotone nondecreasing and slope restricted in the sense of [4]. Furthermore it may be expressed as the gradient of a convex potential function; it has been shown [5] that this further condition is necessary for the existence of Zames-Falb multipliers for MIMO (multi-input multi-output) nonlinearities.

For completeness, we construct such multipliers both for the general case and for the case where the nonlinearity arises from a quadratic program. For the general case the construction follows from the results of [4] and [5], but requires some (small) technical details not considered in the original papers. We also express the results in terms of IQCs (integral quadratic constraints).

The multipliers, together with the results in [3], may be used for the analysis of the robust stability of constrained MPC. The analysis is general and versatile, but we should note two caveats. Firstly, as stated above, we require zero to be feasible; this may be overly restrictive for MPC with state (as opposed to input) constraints. Secondly, the Zames-Falb stability criterion does not necessarily guarantee continuity of the input-output map [6].

W. P. Heath is with Control Systems Centre, School of Electrical and Electronic Engineering, The University of Manchester, PO Box 88, Sackville Street, Manchester M60 1QD, UK William.Heath@manchester.ac.uk

A. G. Wills is with the School of Electrical Engineering and Computer Science, University of Newcastle, NSW 2308, Australia Adrian.Wills@newcastle.edu.au

II. Zames-Falb multipliers for MIMO nonlinearities

Here we state and prove the Zames-Falb construction for MIMO (multi-input multi-output) static nonlinearities $\phi : \mathbb{R}^N \to \mathbb{R}^N$. This is in two parts:

ZF1: the first concerns bound and monotone nondecreasing nonlinearities;

ZF2: the second concerns nonlinearities that are also slope-restricted.

Both results are derived for single-valued nonlinearities in [4], where it is commented that they can be "easily" generalized to MIMO nonlinearities.

It is observed in [5] that, in fact, the first result only generalizes under a further condition. Specifically it is shown that a necessary and sufficient further condition for Zames-Falb multipliers (ZF1) to exist is for the line integral $\int_A^B \phi(x)^T dx$ to be independent of path, and this is equivalent to the condition that ϕ is the gradient of some convex potential function. Although the main results of [5] are expressed in terms of skew$(\phi'(x))$, we will wish to apply the result to a non-smooth nonlinearity, where the derivative is not defined everywhere.

We show the second Zames-Falb result (ZF2) is also true provided the conditions of [4] and [5] hold. The contribution is small, but requires some technical details.

Our construction is closely related to (and based on) those of [4], [7], [8]. Note that in [8] a restricted case is considered, and thus more general multipliers are obtained (see also [9]).

We say a MIMO nonlinearity $\phi : \mathbb{R}^N \to \mathbb{R}^N$ is bound (by $c > 0$) and monotone nondecreasing if it satisfies

1) $\phi(0) = 0$
2) $[\phi(x) - \phi(y)]^T (x - y) \geq 0$ for all $x, y \in \mathbb{R}^N$
3) $||\phi(x)|| \leq c||x||$ for all x

We say further that ϕ is slope-restricted to the interval $[a,b]$ if

4. $[\phi(x) - \phi(y) - a(x-y)]^T [\phi(x) - \phi(y) - b(x-y)] \leq 0$ for all x and y.

We first require two lemmas:

Lemma 1: Let $\phi : \mathbb{R}^N \to \mathbb{R}^N$ be bound and monotone nondecreasing with the integral $\int_A^B \phi(x)^T dx$ independent of path. Then for any $x \in L_2^N$ we have

$$\int_{-\infty}^{\infty} x(t+\tau)^T \phi(x(t))dt \leq \int_{-\infty}^{\infty} x(t)^T \phi(x(t))dt \qquad (1)$$

Furthermore, if ϕ is odd then

$$\left|\int_{-\infty}^{\infty} x(t+\tau)^T \phi(x(t))dt\right| \leq \int_{-\infty}^{\infty} x(t)^T \phi(x(t))dt \qquad (2)$$

Proof: See Appendix. □

Lemma 2: Let $\phi : \mathbb{R}^N \to \mathbb{R}^N$ be bound and monotone nondecreasing with the integral $\int_A^B \phi(x)^T dx$ independent of path. Furthermore, let ϕ be slope-restricted to the interval $[\alpha, \beta - \varepsilon]$ for some real α, β, $\varepsilon > 0$ with $\beta - \varepsilon > \alpha$. Let $\tilde{\phi} : \mathbb{R}^N \to \mathbb{R}^N$ be defined as $\tilde{\phi}(x) = \hat{\phi}_1 \circ \hat{\phi}_2^{-1}(x)$ with $\hat{\phi}_1(x) = (\phi - \alpha I)(x)$ and $\hat{\phi}_2(x) = (\beta I - \phi)(x)$. Then $\tilde{\phi}$ is well-defined, $\tilde{\phi}$ is bound and monotone nondecreasing, $\int_A^B \tilde{\phi}(x)^T dx$ is independent of path, and if ϕ is odd then $\tilde{\phi}$ is odd.

Proof: See Appendix. □

Theorem 1 (Zames-Falb multipliers): Let $\phi : \mathbb{R}^N \to \mathbb{R}^N$ be bound and monotone nondecreasing with $\int_A^B \phi(x)^T dx$ independent of path. Let $h \in L_1$ satisfy $\int_{-\infty}^{\infty} |h(t)|dt < 1$ and either let ϕ be odd or let $h(t) \geq 0$ for all t. Then

1) For any $x \in L_2^N$ we have

$$\langle \phi(x), x \rangle \geq \langle \phi(x), h * x \rangle \qquad (3)$$

2) If ϕ is slope-restricted to the interval $[\alpha, \beta - \varepsilon]$ for some real α, β, $\varepsilon > 0$ with $\beta - \varepsilon > \alpha$ then for any $x \in L_2^N$ we have

$$\langle \phi(x) - \alpha x, \beta x - \phi(x) \rangle \geq \langle \phi(x) - \alpha x, h * (\beta x - \phi(x)) \rangle \qquad (4)$$

Proof:

1) Under the assumptions, and exploiting Lemma 1, we find

$$\begin{aligned} \langle \phi(x), h * x \rangle &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau)\phi(x(t))^T x(t-\tau) d\tau dt \\ &\leq \int_{-\infty}^{\infty} |h(\tau)| d\tau \int_{-\infty}^{\infty} \phi(x(t))^T x(t) dt \\ &\leq \int_{-\infty}^{\infty} \phi(x(t))^T x(t) dt \qquad (5) \end{aligned}$$

2) By Lemma 2 we may apply the first part of the Theorem to $\tilde{\phi}(y)$, where $y = \hat{\phi}_2(x)$ for any $x \in L_2^N$. We have

$$\left\langle \tilde{\phi}(y), y \right\rangle \geq \left\langle \tilde{\phi}(y), h * y \right\rangle \qquad (6)$$

and hence

$$\left\langle \hat{\phi}_1(x), \hat{\phi}_2(x) \right\rangle \geq \left\langle \hat{\phi}_1(x), h * \hat{\phi}_2(x) \right\rangle \qquad (7)$$

□

Corollary 1 (Zames-Falb multipliers for discrete systems): Under the same assumptions on ϕ, let $h \in l_1$ satisfy $\sum_{t=-\infty}^{\infty} |h(t)|dt < 1$ and either let ϕ be odd or let $h(t) \geq 0$ for all t. Then

1) For any $x \in l_2^N$ we have

$$\langle \phi(x), x \rangle \geq \langle \phi(x), h * x \rangle \qquad (8)$$

2) If ϕ is slope-restricted to the interval $[\alpha, \beta - \varepsilon]$ for some real α, β, $\varepsilon > 0$ with $\beta - \varepsilon > \alpha$ then for any $x \in L_2^N$ we have

$$\langle \phi(x) - \alpha x, \beta x - \phi(x) \rangle \geq \langle \phi(x) - \alpha x, h * (\beta x - \phi(x)) \rangle \qquad (9)$$

Proof: This follows immediately by setting $\tilde{x} \in L_2^N$, given $x \in l_2^N$, as

$$\tilde{x}(k+t) = x(k) \text{ for } k \text{ an integer and } 0 \leq t < 1 \qquad (10)$$

□

III. Zames-Falb multipliers for quadratic programming

Let the function $f : \mathbb{R}^N \to \mathbb{R}^N$ be given by the quadratic program

$$\begin{aligned} f(x) \quad = \quad & \arg\min_n \frac{1}{2} n^T n - n^T x \\ & \text{subject to } Ln \preceq b \text{ with } b \succeq 0 \\ & \text{and } Mn = 0 \qquad (11) \end{aligned}$$

with "$\preceq$" and "$\succeq$" denoting term-by-term inequality.

In [1] we showed that the nonlinearity that occurs in constrained linear MPC (model predictive control) can be expressed in this manner, provided 0 is feasible. Furthermore f lies in the sector $[0, I]$ in the sense that

$$f(x)^T [f(x) - x] \leq 0 \text{ for all } x \in \mathbb{R}^N \qquad (12)$$

Thus the multivariable circle criterion can be used to ascertain the closed-loop stability of MPC.

Here we show that Zames-Falb multipliers may be constructed for f. In particular, we show in the following two lemmas that f satisfies the conditions of [4] and [5] respectively.

Lemma 3: Let f be given by (11). Then f is bound by 1 and monotone nondecreasing. Furthermore f is slope-restricted to the interval $[0, 1]$.

Proof: See Appendix. □

Lemma 4: Let f be given by (11). Then the line integral $\int_A^B f(x)^T dx$ is independent of path.

Proof: See Appendix. □

Hence our main result:

Theorem 2 (Zames-Falb multipliers for quadratic programming): Let f be given by (11). Let $h \in L_1$ satisfy $\int_{-\infty}^{\infty} |h(t)|dt < 1$ and either let f be odd or let $h(t) \geq 0$ for all t. Then

1) For any $x \in L_2^N$ we have

$$\langle f(x), x \rangle \geq \langle f(x), h * x \rangle \qquad (13)$$

2) For any $x \in L_2^N$ and for any $\varepsilon > 0$ we have

$$\langle f(x), (1+\varepsilon)x - f(x) \rangle \geq \langle f(x), h * [(1+\varepsilon)x - f(x)] \rangle \qquad (14)$$

Proof:

1) This follows immediately from applying the first result of the Zames-Falb theorem to f.
2) This follows from applying the second result of the Zames-Falb theorem to $f(x)+\alpha x$ with $\beta = 1+\alpha+\varepsilon$.

□

The equivalent result for discrete time systems follows immediately (c.f. Corollary 1):

Corollary 2 (Zames-Falb multipliers for discrete systems with quadratic programming): Let f be given by (11). Let $h \in l_1$ satisfy $\sum_{t=-\infty}^{\infty} |h(t)| < 1$ and either let f be odd or let $h(t) \geq 0$ for all t. Then

1) For any $x \in l_2^N$ we have

$$\langle f(x), x\rangle \geq \langle f(x), h*x\rangle \tag{15}$$

2) For any $x \in l_2^N$ and for any $\varepsilon > 0$ we have

$$\langle f(x), (1+\varepsilon)x - f(x)\rangle \geq \langle f(x), h*[(1+\varepsilon)x - f(x)]\rangle \tag{16}$$

□

The results can be extended to quadratic programs with more general positive definite Hessians via straightforward substitution. Let the function $\tilde{f} : \mathbb{R}^N \to \mathbb{R}^N$ be given by the quadratic program

$$\begin{aligned} \tilde{f}(\tilde{x}) \quad = \quad & \arg\min_{\tilde{n}} \frac{1}{2}\tilde{n}^T G \tilde{n} - \tilde{n}^T \tilde{x} \\ & \text{subject to } \tilde{L}\tilde{n} \preceq \tilde{b} \text{ with } \tilde{b} \succeq 0 \\ & \text{and } \tilde{M}\tilde{n} = 0 \end{aligned} \tag{17}$$

with $G = G^T > 0$. This can be transformed to the form of (11) by the substitutions

$$\begin{aligned} f &= G_r \tilde{f} \\ x &= G_r^{-T} \tilde{x} \end{aligned} \tag{18}$$

where $G_r^T G_r = G$. Hence we may say:

Corollary 3 (Zames-Falb multipliers for quadratic programming with generalized Hessian): Let $\tilde{f}$ be given by (17). Let $h \in L_1$ satisfy $\int_{-\infty}^{\infty} |h(t)|dt < 1$ and either let $\tilde{f}$ be odd or let $h(t) \geq 0$ for all t. Then

1) For any $\tilde{x} \in L_2^N$ we have

$$\left\langle \tilde{f}(\tilde{x}), \tilde{x}\right\rangle \geq \left\langle \tilde{f}(\tilde{x}), h*\tilde{x}\right\rangle \tag{19}$$

2) For any $\tilde{x} \in L_2^N$ and for any $\varepsilon > 0$ we have

$$\begin{aligned} &\left\langle \tilde{f}(\tilde{x}), (1+\varepsilon)\tilde{x} - G\tilde{f}(\tilde{x})\right\rangle \\ &\quad \geq \left\langle \tilde{f}(\tilde{x}), h*[(1+\varepsilon)\tilde{x} - G\tilde{f}(\tilde{x})]\right\rangle \end{aligned} \tag{20}$$

□

A similar result for the discrete case also follows immediately.

IV. Results expressed as IQCs

In [10] a unified approach to robustness analysis via IQCs (integral quadratic constraints) was introduced. Such techniques may be used to analyze the robustness of MPC [3]. Here we express the various inequalities of the theorems in standard form. In particular, given $\phi : L_2^N \to L_2^N$, we adopt the notation of [11] and say

$$\phi \in \text{IQC}(\Pi) \tag{21}$$

when

$$\int_{-\infty}^{\infty} \begin{bmatrix} x(j\omega) \\ \phi(x)(j\omega) \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} x(j\omega) \\ \phi(x)(j\omega) \end{bmatrix} d\omega \geq 0 \tag{22}$$

for all $x \in L_2^N$. Similarly if $\phi : l_2^N \to l_2^N$ we say

$$\phi \in \text{IQC}(\Pi) \tag{23}$$

when

$$\int_{-\pi}^{\pi} \begin{bmatrix} x(e^{j\omega}) \\ \phi(x)(e^{j\omega}) \end{bmatrix}^* \Pi(e^{j\omega}) \begin{bmatrix} x(e^{j\omega}) \\ \phi(x)(e^{j\omega}) \end{bmatrix} d\omega \geq 0 \tag{24}$$

for all $x \in l_2^N$.

Let H be the continuous (or discrete) Fourier transform of h. Then following [8] we may say:

1) The inequality

$$\langle \phi(x), x\rangle \geq \langle \phi(x), h*x\rangle \tag{25}$$

may be expressed as $\phi \in \text{IQC}(\Pi)$ with

$$\Pi = \begin{bmatrix} 0 & (1-H^*)I \\ (1-H)I & 0 \end{bmatrix} \tag{26}$$

2) The inequality

$$\langle \phi(x) - \alpha x, \beta x - \phi(x)\rangle \geq \langle \phi(x) - \alpha x, h*(\beta x - \phi(x))\rangle \tag{27}$$

may be expressed as $\phi \in \text{IQC}(\Pi)$ with

$$\Pi = \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{12}^T & \Pi_{22} \end{bmatrix} \tag{28}$$

where

$$\begin{aligned} \Pi_{11} &= -\alpha\beta(2-H-H^*)I \\ \Pi_{12} &= \alpha(1-H)I + \beta(1-H^*)I \\ \Pi_{22} &= (-2+H+H^*)I \end{aligned} \tag{29}$$

3) The inequality

$$\langle f(x), (1+\varepsilon)x - f(x)\rangle \geq \langle f(x), h*[(1+\varepsilon)x - f(x)]\rangle \tag{30}$$

may be expressed as $f \in \text{IQC}(\Pi)$ with

$$\Pi = \begin{bmatrix} 0 & (1+\varepsilon)(1-H^*)I \\ (1+\varepsilon)(1-H)I & (-2+H+H^*)I \end{bmatrix} \tag{31}$$

4) The inequality

$$\begin{aligned} &\left\langle \tilde{f}(\tilde{x}), (1+\varepsilon)\tilde{x} - G\tilde{f}(\tilde{x})\right\rangle \\ &\quad \geq \left\langle \tilde{f}(\tilde{x}), h*[(1+\varepsilon)\tilde{x} - G\tilde{f}(\tilde{x})]\right\rangle \end{aligned} \tag{32}$$

may be expressed as $\tilde{f} \in \text{IQC}(\Pi)$ with

$$\Pi = \begin{bmatrix} 0 & (1+\varepsilon)(1-H^*)I \\ (1+\varepsilon)(1-H)I & (-2+H+H^*)G \end{bmatrix} \tag{33}$$

V. CONCLUSION

We have shown that Zames-Falb multipliers may be applied to the nonlinearity that arises in constrained linear MPC, provided 0 is feasible. The result maybe incorporated into a robustness analysis of MPC based on IQCs [3].

APPENDIX: PROOF OF THE LEMMAS

Proof of Lemma 1: The potential function $P:\mathbb{R}^N \to \mathbb{R}$ that satisfies $\nabla P(x) = \phi(x)$, $P(0) = 0$ is well-defined. Since ϕ is monotone nondecreasing,

$$P(b) - P(a) \leq (b-a)^T \phi(b) \tag{34}$$

We have $\phi(x) \in L_2^N$ and $P(x) \in L_1$ for all $x \in L_2^N$. Given x, put $a = x(t+\tau)$, $b = x(t)$ and integrate to give

$$\int_{-\infty}^{\infty} x(t+\tau)^T \phi(x(t))dt \leq \int_{-\infty}^{\infty} x(t)^T \phi(x(t))dt \tag{35}$$

Similarly, if ϕ is odd, putting $b = x(t)$, $a = -x(t+\tau)$ and integrating gives

$$-\int_{-\infty}^{\infty} x(t+\tau)^T \phi(x(t))dt \leq \int_{-\infty}^{\infty} x(t)^T \phi(x(t))dt \tag{36}$$

and hence

$$\left|\int_{-\infty}^{\infty} x(t+\tau)^T \phi(x(t))dt\right| \leq \int_{-\infty}^{\infty} x(t)^T \phi(x(t))dt \tag{37}$$

□

Before we prove Lemma 2 we require the further lemma:

Lemma A1: Let $\phi:\mathbb{R}^N \to \mathbb{R}^N$ be slope restricted to the interval $[a,b]$ for some $0 < a < b$. Then:

1) For any $x, y \in \mathbb{R}^N$

$$\begin{aligned}[\phi(x)-\phi(y)-a(x-y)]^T(x-y) &\geq 0\\ [\phi(x)-\phi(y)-b(x-y)]^T(x-y) &\leq 0\end{aligned} \tag{38}$$

2) Furthermore for any $x \in \mathbb{R}^N$

$$||\phi(x)||^2 \leq b^2 ||x||^2 \tag{39}$$

Proof of Lemma A1:

1) For a given $x, y \in \mathbb{R}^N$ define the scalar $\lambda = \lambda(x,y)$ as

$$\lambda = \frac{[\phi(x)-\phi(y)]^T[x-y]}{||x-y||^2} \tag{40}$$

We have the relation $[\phi(x)-\phi(y)-\lambda(x-y)]^T(x-y) = 0$. Thus

$$\begin{aligned}&[\phi(x)-\phi(y)-a(x-y)]^T[\phi(x)-\phi(y)-b(x-y)]\\ &= [\phi(x)-\phi(y)-\lambda(x-y)-(a-\lambda)(x-y)]^T\\ &\quad\times[\phi(x)-\phi(y)-\lambda(x-y)-(b-\lambda)(x-y)]\\ &= ||\phi(x)-\phi(y)-\lambda(x-y)||^2\\ &\quad+(a-\lambda)(b-\lambda)||x-y||^2\end{aligned} \tag{41}$$

So we must have

$$(a-\lambda)(b-\lambda) \leq 0 \tag{42}$$

and hence

$$a \leq \lambda \leq b \tag{43}$$

Hence for any x and y we have

$$\begin{aligned}&[\phi(x)-\phi(y)-a(x-y)]^T(x-y)\\ &= [\phi(x)-\phi(y)-\lambda(x-y)-(a-\lambda)(x-y)]^T\\ &\quad\times(x-y)\\ &= -(a-\lambda)||x-y||^2\\ &\geq 0\end{aligned} \tag{44}$$

Similarly

$$\begin{aligned}&[\phi(x)-\phi(y)-b(x-y)]^T(x-y)\\ &= [\phi(x)-\phi(y)-\lambda(x-y)-(b-\lambda)(x-y)]^T\\ &\quad\times(x-y)\\ &= -(b-\lambda)||x-y||^2\\ &\leq 0\end{aligned} \tag{45}$$

2) Putting $y = 0$ we have

$$[\phi(x)-ax]^T[\phi(x)-bx] \leq 0 \tag{46}$$

and

$$[\phi(x)-bx]^T x \leq 0 \tag{47}$$

Hence

$$\begin{aligned}||\phi(x)||^2 &\leq (a+b)\phi(x)^T x - ab||x||^2\\ &\leq (a+b)b||x||^2 - ab||x||^2\\ &= b^2||x||^2\end{aligned} \tag{48}$$

□

Proof of Lemma 2: From Lemma A1 we have for any x and y

$$\begin{aligned}&[\hat{\phi}_2(x)-\hat{\phi}_2(y)]^T(x-y)\\ &= -[\phi(x)-\phi(y)-\beta(x-y)]^T(x-y)\\ &= -[\phi(x)-\phi(y)-(\beta-\varepsilon)(x-y)]^T(x-y)+\varepsilon||x-y||^2\\ &\geq \varepsilon||x-y||^2\end{aligned} \tag{49}$$

Hence $\hat{\phi}_2$ is monotone increasing which is sufficient for $\hat{\phi}_2^{-1}$ to exist. Thus:

1) Since $\hat{\phi}_2^{-1}$ is well-defined then $\tilde{\phi} = \hat{\phi}_1 \circ \hat{\phi}_2^{-1}$ is also well-defined.
2) We have:
   (1) $\tilde{\phi}(0) = \hat{\phi}_1 \circ \hat{\phi}_2^{-1}(0) = \hat{\phi}_1(0) = 0$
   (2) Given x and y, put $\hat{x} = \hat{\phi}_2^{-1}(x)$ and $\hat{y} = \hat{\phi}_2^{-1}(y)$

$$\begin{aligned}&[\tilde{\phi}(x)-\tilde{\phi}(y)]^T(x-y)\\ &= [\hat{\phi}_1(\hat{x})-\hat{\phi}_1(\hat{y})]^T[\hat{\phi}_2(\hat{x})-\hat{\phi}_2(\hat{y})]\\ &= -[\phi(\hat{x})-\phi(\hat{y})-\alpha(\hat{x}-\hat{y})]^T\\ &\quad\times[\phi(\hat{x})-\phi(\hat{y})-\beta(\hat{x}-\hat{y})]\\ &= -[\phi(\hat{x})-\phi(\hat{y})-\alpha(\hat{x}-\hat{y})]^T\\ &\quad\times[\phi(\hat{x})-\phi(\hat{y})-(\beta-\varepsilon)(\hat{x}-\hat{y})]\\ &\quad+\varepsilon[\phi(\hat{x})-\phi(\hat{y})-\alpha(\hat{x}-\hat{y})]^T(\hat{x}-\hat{y})\\ &\geq 0\end{aligned} \tag{50}$$

(3) We will show that $\tilde{\phi}$ is slope restricted to the sector $[0,k]$ for some k and hence by Lemma A1 $||\phi(x)||^2 \leq k^2||x||^2$. With $\hat{x}$ and $\hat{y}$ defined as before we find

$$\begin{aligned}
&[\tilde{\phi}(x)-\tilde{\phi}(y)]^T[\tilde{\phi}(x)-\tilde{\phi}(y)-k(x-y)]\\
&= [\hat{\phi}_1(\hat{x})-\hat{\phi}_1(\hat{y})]^T\\
&\quad \times[\hat{\phi}_1(\hat{x})-\hat{\phi}_1(\hat{y})-k\{\hat{\phi}_2(\hat{x})-\hat{\phi}_2(\hat{y})\}]\\
&= (1+k)[\phi(\hat{x})-\phi(\hat{y})-\alpha(\hat{x}-\hat{y})]^T\\
&\quad \times\left[\phi(\hat{x})-\phi(\hat{y})-\frac{\alpha+k\beta}{1+k}(\hat{x}-\hat{y})\right]\\
&= (1+k)[\phi(\hat{x})-\phi(\hat{y})-\alpha(\hat{x}-\hat{y})]^T\\
&\quad \times\Big[\phi(\hat{x})-\phi(\hat{y})-(\beta-\varepsilon)(\hat{x}-\hat{y})\\
&\qquad +\left(\beta-\varepsilon-\frac{\alpha+k\beta}{1+k}\right)(\hat{x}-\hat{y})\Big]\\
&\leq [(1+k)(\beta-\varepsilon)-(\alpha+k\beta)]\\
&\quad \times[\phi(\hat{x})-\phi(\hat{y})-\alpha(\hat{x}-\hat{y})]^T[\hat{x}-\hat{y})]\\
&= (\beta-\varepsilon-k\varepsilon-\alpha)\\
&\quad \times[\phi(\hat{x})-\phi(\hat{y})-\alpha(\hat{x}-\hat{y})]^T[\hat{x}-\hat{y}]\\
&\leq 0
\end{aligned} \tag{51}$$

provided $(\beta-\varepsilon-k\varepsilon-\alpha)\leq 0$. So we require

$$k \geq \frac{\beta-\varepsilon-\alpha}{\varepsilon} \tag{52}$$

Hence $\tilde{\phi}$ is bound and monotone nondecreasing.

3) We know [5] that ϕ is the gradient of a convex potential function. Hence $\hat{\phi}_2$ is a one-to-one mapping and the gradient of a convex function. It follows from Theorem 26.6 of [12] that $\hat{\phi}_2^{-1}$ is also the gradient of a convex function. We may express $\tilde{\phi}$ as

$$\begin{aligned}
\tilde{\phi}(.) &= (\phi-\alpha I)\circ(\beta I-\phi)^{-1}(.)\\
&= (\beta-\alpha)\hat{\phi}_2^{-1}(.)-I(.)
\end{aligned} \tag{53}$$

Hence $\tilde{\phi}$ is the gradient of a potential function, and since $\tilde{\phi}$ is non-decreasing it is the gradient of a convex potential: see [5]. Thus $\int_A^B \tilde{\phi}(x)^T dx$ is independent of path.

4) If ϕ is odd then both $\hat{\phi}_1$ and $\hat{\phi}_2$ are odd. It follows that $\hat{\phi}_2^{-1}$ is odd and hence $\tilde{\phi}$ is odd.

□

Proof of Lemma 3:

1) Since $b \succeq 0$, zero is feasible and condition 1 follows trivially.

2) The Karoush Kuhn Tucker (KKT) conditions [13] for f give

$$\begin{aligned}
f(x)+M^T\zeta(x)+L^T\lambda(x)-x &= 0\\
Mf(x) &= 0\\
Lf(x)+s(x) &= b\\
s(x)^T\lambda(x) &= 0
\end{aligned} \tag{54}$$

with $s(x)\succeq 0$ and $\lambda(x)\succeq 0$. Substituting for x (and y) from the first condition gives:

$$\begin{aligned}
&[f(x)-f(y)]^T(x-y)\\
&= [f(x)-f(y)]^T\\
&\quad \times[f(x)+M^T\zeta(x)+L^T\lambda(x)\\
&\qquad -f(y)-M^T\zeta(y)-L^T\lambda(y)]\\
&= ||f(x)-f(y)||^2\\
&\quad +[f(x)-f(y)]^T M^T[\zeta(x)-\zeta(y)]\\
&\quad +[(f(x)-f(y)]^T L^T[\lambda(x)-\lambda(y)]\\
&= ||f(x)-f(y)||^2+0\\
&\quad +[s(y)-s(x)]^T[\lambda(x)-\lambda(y)]\\
&= ||f(x)-f(y)||^2+s(y)^T\lambda(x)+s(x)^T\lambda(y)\\
&\geq 0
\end{aligned} \tag{55}$$

3) Given x, suppose f_x is feasible with $||f_x||^2>||x||^2$ and hence $||f_x||^2>f_x^T x$. Let $\tilde{f}_x=(1-\varepsilon)f_x$ with $\varepsilon>0$. By convexity $\tilde{f}_x$ is also feasible. Put $J_x(f)=\frac{1}{2}f^T f-f^T x$. Then

$$\begin{aligned}
J_x(\tilde{f}_x) &= \frac{1}{2}(1-\varepsilon)^2 f_x^T f_x-(1-\varepsilon)f_x^T x\\
&= J_x(f_x)-\varepsilon(f_x^T f_x-f_x^T x)+\frac{1}{2}\varepsilon^2 f_x^T f_x\\
&< J_x(f_x) \text{ for } \varepsilon \text{ sufficiently small}
\end{aligned} \tag{56}$$

Hence f_x cannot be equal to $f(x)$ and so we must have $||f(x)||^2\leq||x||^2$.

4) Given x, y suppose f_x and f_y are feasible with $(f_x-f_y)^T(x-y)<||f_x-f_y||^2$. Let $\tilde{f}_x=f_x+\varepsilon(f_y-f_x)$ and $\tilde{f}_y=f_y+\varepsilon(f_x-f_y)$ for some $\varepsilon>0$. By convexity $\tilde{f}_x$ and $\tilde{f}_y$ are also feasible. Furthermore

$$\begin{aligned}
&J_x(\tilde{f}_x)+J_y(\tilde{f}_y)\\
&= J_x(f_x)+\varepsilon f_x^T(f_y-f_x)+\frac{1}{2}\varepsilon^2||f_x-f_y||^2\\
&\quad -\varepsilon(f_y-f_x)^T x+J_y(f_y)+\varepsilon f_y^T(f_x-f_y)\\
&\quad +\frac{1}{2}\varepsilon^2||f_y-f_x||^2-\varepsilon(f_x-f_y)^T y\\
&= J_x(f_x)+J_y(f_y)-\varepsilon(1-\varepsilon)||f_x-f_y)||^2\\
&\quad +\varepsilon(f_x-f_y)^T(x-y)\\
&< J_x(f_x)+J_y(f_y) \text{ for } \varepsilon \text{ sufficiently small}
\end{aligned} \tag{57}$$

Hence f_x and f_y cannot be concurrently equal to $f(x)$ and $f(y)$, so we must have $[f(x)-f(y)]^T(x-y)\geq||f(x)-f(y)||^2$, and hence $[f(x)-f(y)]^T[f(x)-f(y)-(x-y)]\leq 0$.

□

Proof of Lemma 4: It is well-known (e.g. [14]) that f is both piecewise affine [15] and continuous [16]. Within each affine region R_i, the KKT conditions give (c.f. [15], [13])

$$\begin{bmatrix} I & L_i^T & M^T\\ L_i & 0 & 0\\ M & 0 & 0\end{bmatrix}\begin{bmatrix} f_i(x)\\ \lambda_i(x)\\ \zeta(x)\end{bmatrix}=\begin{bmatrix} x\\ b_i\\ 0\end{bmatrix} \tag{58}$$

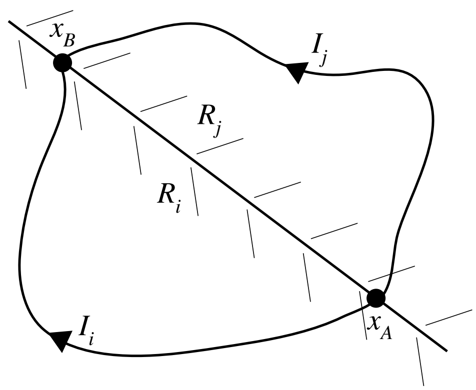

Fig. 1. This illustrates the final part of the proof of Lemma 4. Given x_A and x_B on the border of adjoining regions R_i and R_j, the integral from x_A to x_B is independent of choice of path.

where $f = f_i$ within region R_i, where L_i and b_i correspond to the constraints which are active in the region (i.e. L_i is made up of rows of L, and b_i is made up of the corresponding entries in b, and we have the equality $f_i(x) = b_i$ within R_i), and where $\lambda_i(x)$ are the corresponding Lagrange multipliers. Thus the piecewise affine control law is

$$f_i(x) = F_i x + a_i \tag{59}$$

with

$$\begin{aligned} F_i &= I - \begin{bmatrix} L_i \\ M \end{bmatrix}^T \left(\begin{bmatrix} L_i \\ M \end{bmatrix} \begin{bmatrix} L_i \\ M \end{bmatrix}^T \right)^{-1} \begin{bmatrix} L_i \\ M \end{bmatrix} \\ a_i &= \begin{bmatrix} L_i \\ M \end{bmatrix}^T \left(\begin{bmatrix} L_i \\ M \end{bmatrix} \begin{bmatrix} L_i \\ M \end{bmatrix}^T \right)^{-1} \begin{bmatrix} b_i \\ 0 \end{bmatrix} \end{aligned} \tag{60}$$

In particular $F_i = F_i^T \geq 0$ and within each region R_i we can form the potential function

$$P_i(x) = \frac{1}{2} x^T F_i x + x^T a_i \tag{61}$$

such that

$$f_i(x) = \nabla P_i(x) \tag{62}$$

This is sufficient to show the integral is independent of path within each region [5].

So it suffices to show that if x_A and x_B lie on the border of adjoining regions R_i and R_j, then the integral from A to B is the same via region R_i or region R_j—see Fig 1.

Let I_i be the integral via region R_i, and R_j be the integral via region R_j. Thus

$$I_i = \int_A^B (F_i x + a_i)^T dx = \frac{1}{2} x_B^T F_i x_B + x_B^T a_i - \frac{1}{2} x_A^T F_i x_A - x_A^T a_i \tag{63}$$

and similarly

$$I_j = \int_A^B (F_j x + a_j)^T dx = \frac{1}{2} x_B^T F_j x_B + x_B^T a_j - \frac{1}{2} x_A^T F_j x_A - x_A^T a_j \tag{64}$$

Since x_A and x_B lie on the boundary of region R_i and R_2 we have the relations

$$\begin{aligned} F_i x_A + a_i &= F_j x_A + a_j \\ F_i x_B + a_i &= F_j x_B + a_j \end{aligned} \tag{65}$$

In particular

$$a_i - a_j = (F_j - F_i) x_A = (F_j - F_i) x_B \tag{66}$$

Thus

$$\begin{aligned} I_j - I_i &= \frac{1}{2} x_B^T F_j x_B + x_B^T a_j - \frac{1}{2} x_A^T F_j x_A - x_A^T a_j \\ &\quad - \frac{1}{2} x_B^T F_i x_B - x_B^T a_i + \frac{1}{2} x_A^T F_i x_A + x_A^T a_i \\ &= \frac{1}{2} x_B^T (F_j - F_i) x_B + \frac{1}{2} x_A^T (F_i - F_j) x_A \\ &\quad + (x_A - x_B)^T (a_i - a_j) \\ &= \frac{1}{2} (x_A - x_B)^T (a_i - a_j) \\ &= \frac{1}{2} (x_A - x_B)^T (F_j - F_i) x_A \\ &= 0 \end{aligned} \tag{67}$$

as required. □

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