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Output Feedback Variable Structure Control of Uncertain Linear Systems

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Abstract— **The output feedback variable structure control problem is considered for a class of single input single output uncertain linear plants with arbitrary relative degree and with parameter perturbations, unmodeled dynamics and bounded disturbance simultaneously. A novel passivity-based control scheme composed of linear feedback and variable structure control is proposed. If exact differentiations of system output provided, this scheme results in asymptotic stability; if a class of singular perturbation based linear differentiators adopted, it leads to an arbitrarily small regulation/tracking error.**

I. INTRODUCTION

Variable structure control (VSC) has become one of the most popular methods to deal with linear or nonlinear uncertain plants for its remarkable robust properties ([1], [2]). Particularly, the research on static or dynamic output feedback variable structure control (OFVSC) has attracted considerable attentions in the recent decade.

As is well known, relative degree is a crucial factor in a VSC system. For a linear plant with relative degree one, static OFVSC is possible ([3]–[5]), and a general result was achieved in [2]. In contrast, OFVSC of higher relative degree plants is a much more challenging problem, and several complex methods have been introduced. Firstly, the concept *high order sliding mode* (HOSM) may play an important role ([6]–[10]). With the so-called robust exact differentiators ([8], [11]), a number of 2-order sliding mode controllers ([12]) and *universal* HOSM controllers ([7], [13]) have been proposed to deal with nonlinear plants. Secondly, high gain observer (HGO) is another useful tool for output feedback control. For instance, [14], [15] and [16] discussed how to synthesis a VSC system with linear HGOs to realize an arbitrarily small tracking error. Thirdly, variable structure model reference adaptive control (VS-MRAC) schemes are applicable to linear models with unknown parameters and unmodeled dynamics ([17]–[19]). Finally, in a single-relay control system where the relative degree of the continuous part is greater than one, oscillations will inevitably occur. Fortunately, if the continuous system can be divided into a slowly-varying sub-system and a rapidly-varying one, the closed loop behavior can be subtly studied by singular perturbation analysis ([20], [21]).

In this paper, we study a class of single input single output uncertain linear systems. Higher relative degree, bounded external disturbance at the input channel, parameter perturbations and unmodeled dynamics are all considered. We will adopt VSC strategy combined with dynamic output feedback to realize robust stabilization and disturbance attenuation, where the passivity-based linear feedback is introduced to stabilize (passify) the nominal model set and the variable structure control to deal with other uncertainties and disturbances. The proposed scheme guarantees local asymptotic stability if exact differentiations of system output provided, or guarantees an arbitrarily small regulation/tracking error if the exact differentiators are replaced by a class of singular perturbation based linear differentiators.

We summarize the innovations of this paper as follows. First, the passivity idea is introduced to the synthesis of VSC systems, and the robust passification problem is considered and solved for the first time (*Lemma 2*). Second, in comparison with some existing control methods (e.g. [7], [13]–[16] and [17]–[19]), the proposed one seems simpler and more suitable for the stabilization of a general class of well-modeled uncertain *linear* plants (where various kinds of uncertainties are permitted). Finally, this scheme can be viewed as a new analysis framework of a class of relay control systems as well (*Lemma 4*).

II. PROBLEM STATEMENT

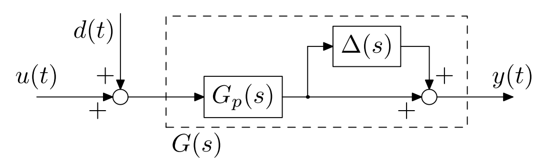

Fig. 1. The model of uncertain linear plant.

Consider a class of single input single output uncertain linear systems as shown in Fig. 1. The plant is described by

$$y(t) = G(s)\left[u(t) + d(t)\right],^{1} \tag{1}$$

where the transfer function is

$$G(s) = G_p(s)\left[1 + \Delta(s)\right] = \frac{n_p(s)}{d_p(s)}\left[1 + \Delta(s)\right]. \tag{2}$$

The nominal model $G_p(s)$, unmodeled dynamics $\Delta(s)$ and external disturbance $d(t)$ satisfy the following assumptions.

Assumption A: The numerator $n_p(s) = n_l s^l + \cdots + n_1 s + n_0$ (of a known degree l) and the denominator $d_p(s) = s^q + d_{q-1}s^{q-1} + \cdots + d_1 s + d_0$ (of a known degree q) are both interval polynomials. System relative degree $r = q - l \geq 2$.[2]

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[1] In the representation $y(t) = G(s)u(t)$, $y(t)$ and $u(t)$ are the output and input signals of linear filter $G(s)$ respectively.

[2] Since the variable structure control problem of systems with $r = 1$ has been well-solved, we will not consider such a simple case. In fact, our conclusions can be shown compatible with the existing results in this case.

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Uncertain coefficients $n_i \in [\underline{n}_i, \bar{n}_i]$ $(i = 1, 2, \cdots, l)$, $d_i \in [\underline{d}_i, \bar{d}_i]$ $(i = 1, \cdots, q-1)$; especially, $\underline{n}_l > 0$. Besides, every possible $n_p(s)$ should be Hurwitz. The set of $n_p(s)$ and $d_p(s)$ are denoted as $\mathcal{F}_n$ and $\mathcal{F}_d$ respectively, and the set of nominal plant $G_p(s)$ is denoted as $\mathcal{G}$, that is:

$$\mathcal{G} := \left\{ G_p(s) \middle| G_p(s) = \frac{n_p(s)}{d_p(s)}, \quad n_p \in \mathcal{F}_n, \ d_p \in \mathcal{F}_d \right\}$$

Assumption B: The unmodeled dynamics $\Delta(s) \in \mathcal{RH}_\infty$ satisfies $\|\Delta(s)\|_1 \le \iota < 1$. The set of $\Delta(s)$ is denoted as $\mathcal{D}$.

Assumption C: The external disturbance $d(t)$ is bounded as $|d(t)| \le \nu(y(t)) + \kappa_0$, where κ_0 is a known constant and $\nu(\cdot)$ is a known continuous non-negative function, $\nu(0) = 0$.

Remark 1:

- Equation (1) describes how the unknown external disturbance $d(t)$ affects the plant, and (2) describes both the unmodeled dynamics and the interval plant model. It is not difficult to transform the transfer function $G_p(s)$ into a state-space representation $\Sigma(A, b, c)$ of a fixed order q, while this is impossible for $G(s)$ because of the presence of unmodeled dynamics $\Delta(s)$.
- *Assumption A* is quite similar to the assumptions of [22] or the assumptions of Theorem 13.1 (simultaneous strong stabilization) in [23]. It is required that every $n_p(s) \in \mathcal{F}_n$ is Hurwitz. This condition can be verified by Kharitonov's Theorem (see e.g. [23]).
- Since $\|\Delta\|_\infty \le \|\Delta\|_1 < 1$, the relative degree of $G(s)$ is the same as the relative degree of $G_p(s)$.
- It should be pointed out that all the coefficients of $G_p(s)$ are defined in bounded closed intervals (compact sets). As a result the set $\mathcal{G}$ is compact; moreover, the range of system norm $\|\cdot\|_\mu : \mathcal{G} \to \mathrm{R}^+$ is a bounded closed interval in R^+, infimum and supremum exist.

In this note, we will first solve the regulation problem. That is, to find a control law under which the system output $y(t)$ will converge to zero or to a small neighborhood of zero. After that, we will revise the proposed control law so that $y(t)$ can track a class of bounded reference signals $y_r(t)$.

III. Exact Differentiation Based OFVSC Control

As implied in [2], strictly positive real (SPR) transfer function may play an important role in variable structure control systems.

Definition 1 (SPR, [24]): A transfer function $\phi(s)$ for a linear single input single output system, with relative degree $m = 1$, is SPR if and only if

(a) $\phi(s)$ is analytic in $\Re\{s\} \ge 0$;
(b) $\Re[\phi(j\omega)] > 0$, $\forall \omega \in (-\infty, \infty)$;
(c) $\lim_{\omega\to\infty} \omega^2 \Re[\phi(j\omega)] > 0$.

Remark 2: If the relative degree $m = 0$, the condition (c) should be modified as: $\phi(\infty) > 0$.

Lemma 1: Consider a closed loop system as shown in Fig. 2, where the strictly proper linear plant $G(s) \in$ SPR, and the time-variant relay satisfies

$$u(t) = -\tilde{K}(t)\,\mathrm{sgn}\,(y(t)), \tag{3}$$

where $0 < \kappa \le \tilde{K}(t) \le \kappa + D$. Then for any initial state x_0, $\lim_{t\to\infty} x(t) \to 0$ and $y(t)$ converges to zero in

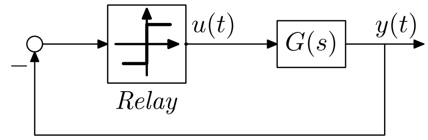


Fig. 2. Closed loop system composed of a SPR linear unit and a time-variant relay.

finite time. Moreover, for any fixed constant $\kappa > 0$, there exists a neighborhood $B(\delta)$ of the equilibrium point such that $\forall x_0 \in B(\delta)$, $y(t)$ converges monotonously.

Proof: Assume $\Sigma(A, b, c)$ is a minimum realization of $G(s)$. Define $V(x) = x^\tau P x$ as a Lyapunov function of the closed loop system, where the positive definite matrix P satisfies the Kalman-Yakubovich-Popov conditions

$$PA + A^T P = -Q < 0, \quad Pb = c^\tau. \tag{4}$$

Then in the non-sliding mode dynamics,

$$\dot{V}(x) = -x^\tau Q x - 2\tilde{K}(t)\,|y(t)| < 0. \tag{5}$$

Since $G(s)$ is minimum phase, the sliding mode dynamics (if exists) is also globally asymptotically stable. Hence the whole VSC system is globally asymptotically stable. After a finite-time transient period, $\|x(t)\| < (cb\kappa - \varepsilon)/\|cA\|$ (where $\varepsilon < cb\kappa$ is a given positive constant). Then

$$\dot{y}(t)\,\mathrm{sgn}\,(y(t)) = cAx\,\mathrm{sgn}\,(y(t)) - cb\tilde{K} < -\varepsilon < 0, \tag{6}$$

which means the existence of sliding mode in Filippov sense and finite-time convergence of $y(t)$. Furthermore, inequality (6) holds in a neighborhood of equilibrium point if κ fixes. This implies the monotonous convergence of $y(t)$. ■

Lemma 1 shows that SPR property is welcome under the VSC framework. Inspired by this result, we will first robustly passify (means rendering a system SPR) the nominal model set $\mathcal{G}$ by a linear dynamic feedback controller $C_1(s)$, a cascade filter $C_2(s)$ and a Hurwitz polynomial $h(s)$ of degree $(r-1)$. Then according to *Lemma 1*, an additional variable structure control scheme may deal with other uncertainties and disturbances. Fig. 3 shows the entire control scheme.

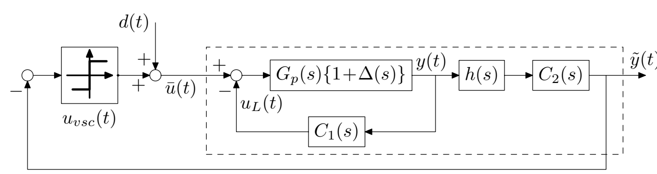


Fig. 3. VSC control scheme with exact differentiations.

Lemma 2 (Robust Passification): There exist two proper, stable and minimum phase transfer functions $C_1(s)$ and $C_2(s)$ and a Hurwitz polynomial $h(s)$ of degree $r-1$, such that $\forall G_p(s) \in \mathcal{G}$, $G(s) \in$ SPR where

$$G(s) := \frac{G_p(s)}{1 + C_1(s)G_p(s)} C_2(s)h(s). \tag{7}$$

Proof: See Appendix. ■

Theorem 3 (exact differentiation based OFVSC scheme): Consider the VSC system shown in Fig. 3, where the plant to be controlled satisfies *Assumptions A*, *B* and *C*. Assume robust exact differentiators can provide 1-order to $(r-1)$-order differentiations of signal $y(t)$, and two linear filters $C_1(s)$, $C_2(s)$ and a Hurwitz polynomial $h(s)$ satisfy *Lemma 1*. Then in a neighborhood of the equilibrium point, the following control law

$$u(t) = u_{vsc}(t) - C_1(s)y(t) \quad (8)$$

$$u_{vsc}(t) = -K_{vsc}\,\mathrm{sgn}(\tilde{y}(t)) \quad (9)$$

$$\tilde{y}(t) = C_2(s)h(s)y(t) \quad (10)$$

(where $K_{vsc} > \kappa_0(1+\iota)/(1-\iota)$ is a positive constant) guarantees asymptotic stability of the closed loop system and the convergence of $y(t)$.

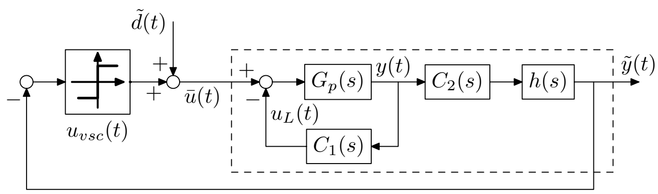


Fig. 4. Equivalent control scheme of Fig. 3.

Proof: Transform the proposed control scheme (Fig. 3) into an equivalent form as shown in Fig. 4, where the uncertain model $G(s)$ is replaced by the nominal model $G_p(s)$ and the disturbance $d(t)$ is modified by $\tilde{d}(t)$ accordingly. The two diagrams are *equivalent* in the sense that their input-output behaviors are the same, that is, for any control input u_{vsc},

$$y = G_p(u_{vsc} + \tilde{d} - C_1 y) = G_p(1+\Delta)(u_{vsc} + d - C_1 y).$$

Then the equivalent disturbance $\tilde{d}(t)$ is

$$\tilde{d}(t) = (1+\Delta)\,d(t) + \Delta(s)u_{vsc}(t) - \Delta(s)\left(C_1 y(t)\right). \quad (11)$$

In a neighborhood of origin, it satisfies

$$\begin{aligned} |\tilde{d}(t)| &\le (1+\iota)[\nu(y)+\kappa_0] + \iota\, K_{vsc} + \iota\, \|C_1\|_1 |y(t)| \\ &< \iota\, K_{vsc} + (1+\iota)\kappa_0 + \varepsilon_0 \end{aligned} \quad (12)$$

where ε_0 is a positive constant depending on the concerned neighborhood.

Substituting (12) into control law (9), we will see the "real" input $\bar{u}(t) = u_{vsc}(t) + \tilde{d}(t)$ implemented on the local closed loop (dashed line area) satisfies

$$\begin{aligned} -\bar{u}(t)\,\mathrm{sgn}(\tilde{y}(t)) &= K_{vsc} + \tilde{d}(t)\,\mathrm{sgn}(\tilde{y}(t)) \\ &> (1-\iota)K_{vsc} - (1+\iota)\kappa_0 - \varepsilon_0 > 0 \end{aligned} \quad (13)$$

if $\varepsilon_0 < [(1-\iota)K_{vsc} - (1+\iota)\kappa_0]$.

Since the transfer function from $\bar{u}(t)$ to $\tilde{y}(t)$ is $\frac{G_p}{1+C_1 G_p}(C_2 h) \in$ SPR, according to *Lemma 1*, the closed loop system is asymptotically stable and the auxiliary variable $\tilde{y}(t)$ converges to 0 in finite time. Therefore system output $y(t) = \frac{1}{C_2(s)h(s)}\tilde{y}(t)$ also converges to 0, although not in finite time. ■

Remark 3:

- Since a transfer functions s^k represents k-order exact differentiation, in (10), $h(s)y(t) = \sum_{k=0}^{r-1} h_k y^{(k)}(t)$ represents a linear combination of the output signal $y(t)$ and its 1-order to $(r-1)$-order differentiations. The construction of *robust exact differentiators* can be found in [11] and [8].
- In *Lemma 1*, only the nominal set $\mathcal{G}$ is considered and passified, despite whether the real plant (2) has been stabilized or not. In another word, single linear dynamic feedback $C_1(s)$ does not guarantee stability.
- *Theorem 3* leads to local asymptotic stability. The larger K_{vsc}, the larger stability region. In a special case that $\mathcal{D}$ is null (no unmodeled dynamics), $\tilde{d}(t) = d(t)$. Let

$$K_{vsc}(t) = \nu(y(t)) + \kappa_0 + \epsilon \quad \epsilon > 0 \quad (14)$$

instead of (9), inequality (13) will hold for all $t \ge 0$. This results in global asymptotic stability.

IV. Linear Differentiation Based OFVSC Control

In comparison with the nonlinear differentiator which is *exact* in Filippov sense, a class of singular perturbation based linear differentiators is more practically used for its simpler structure and less calculation burdens (see e.g. [16]). In this section, we try to answer such a question: with the same control scheme in section III, can we directly replace the nonlinear exact differentiators by linear differentiators? And what is the behavior of the new closed loop system?

Define a Hurwitz polynomial

$$f(s) = a_0 + a_1 s + a_2 s^2 + \cdots + a_{r-1}s^{r-1} \quad (a_0 = 1) \quad (15)$$

whose roots are all negative real, then $F(s) := \frac{1}{f(s)}$ is a low pass filter with $F(0) = 1$. Define

$$f_\mu(s) = f(\mu s) \quad F_\mu(s) = \frac{1}{f_\mu(s)}, \quad (16)$$

then $F_\mu(s)$ may act as a *fast* tracker as $\mu \to 0$. In other words, $\frac{1}{f_\mu(s)}u(t)$ may track $u(t)$, $\frac{s}{f_\mu(s)}u(t)$ may track $u'(t)$, … $\frac{s^{r-2}}{f_\mu(s)}u(t)$ may track $u^{(r-2)}(t)$ respectively. Finally, regarding $\frac{s^{r-1}}{f_\mu(s)}u(t)$ as a tracker of $u^{(r-1)}(t)$, we obtain a group of linear differentiators. Now the linear combination of exact differentiations $h(s)$ may be replaced by $\frac{h(s)}{f_\mu(s)}$, which is a stable, minimum phase and proper linear filter.

Lemma 4: Assume the closed loop system shown in Fig. 5 is composed of a linear plant $G(s) \in$ SPR, a *fast* filter $F_\mu(s)$ defined by (16), and a time-variant relay satisfies (3). Consider a neighborhood of the equilibrium point, for any given error bound ε, there exists a sufficiently small constant μ and a finite time T_0, such that $\forall t > T_0$, $|y(t)| < \varepsilon$.

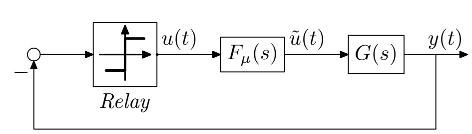


Fig. 5. Closed loop system composed of a SPR linear unit, a *fast* filter and a time-variant relay.

Proof: See Appendix. ∎

Theorem 5 (linear differentiation based OFVSC law): Under the same condition of *Theorem 3*, where the exact differentiation polynomial $h(s)$ is directly replaced by a linear filter $F_\mu(s)h(s)$ ($F_\mu(s)$ is defined by (16) with all poles negative real). In a neighborhood of the origin, for any given constant $\varepsilon > 0$, there exists a sufficiently small constant μ, such that the control law (8), (9) (where $K_{vsc} > \kappa_0 \frac{1+\iota}{1-\iota}$) and (17)

$$\tilde{y}(t) = C_2(s)\left[F_\mu(s)h(s)\right]y(t) \tag{17}$$

guarantees $|y(t)| < \varepsilon$ after a finite-time transient process.

Proof: As have been done in the proof of *Theorem 3*, we first modify the disturbance $d(t)$ by $\tilde{d}(t)$ so that the unmodeled dynamics can be removed. It can be seen that (11) does not vary and (13) still holds in a neighborhood of origin. Dragging the *fast* filter $F_\mu(s)$ to the front of $G_p(s)$, we get an equivalent block diagram (Fig. 6) of the closed loop system described by *Theorem 5*.

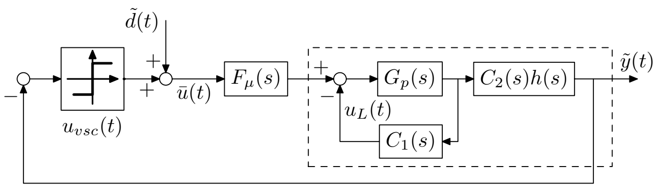


Fig. 6. Equivalent control scheme of *Theorem 5*.

Since $\frac{G_p}{1+C_1G_p}C_2h \in \text{SPR}$, according to *Lemma 5*, if μ is sufficiently small, $\tilde{y}(t)$ will converge to an arbitrarily small neighborhood of zero. Notice that

$$y(t) = \frac{f_\mu(s)}{h(s)}\frac{1}{C_2(s)}\tilde{y}(t) = \left(\sum_{k=0}^{r-1}\frac{\mu^k a_k s^k}{h(s)}\right)\frac{1}{C_2(s)}\tilde{y}(t),$$

$\tilde{y}(t)$ can be arbitrarily small means $y(t)$ can be arbitrarily small as well. Then the theorem conclusion is immediate. ∎

Remark 4:

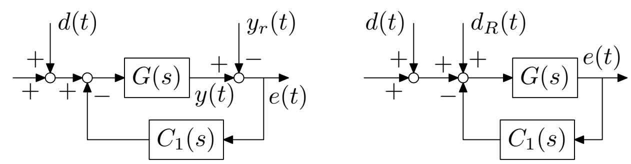
- As singular perturbation based linear filter introduced, there is only one switching function appears in the proposed control law in *Theorem 5*.
- The *peaking phenomenon* aroused by singular perturbation based linear differentiators does not matter, because the auxiliary signal $\tilde{y}(t)$ is not an energy signal and only the sign of it is concerned. But, on the other hand, linear differentiators are sensitive to measurement noise[3], which may be the main defect of this method.
- As $\mu \to 0$ is sufficiently small, the larger K_{vsc}, the larger neighborhood of the origin that can be regulated. Generally speaking, K_{vsc} should be determined according to the system initial states or the neighborhood which is concerned, and μ should be determined according to K_{vsc} and the desired error bound ε.
- This control scheme is different from the proposed ones in [14], [15] and [16], because (a) we introduce signal differentiators rather than state observers; (b) we do not construct any sliding mode dynamics in Filippov sense; in fact, no sliding mode exists at all. Although the closed loop behaviors are similar, our scheme seems simpler and more practical to deal with uncertain linear plants.

[3]In comparison, the exact differentiators are, to some extent, robust ([11]).

V. Output Tracking Problem

Assume that the reference signal $y_r(t)$ satisfies:

Assumption D: Reference signal $y_r(t) = G_R(s)r_0(t)$ is generated by an exo-signal $r_0(t)$ and a reference model $G_R(s)$, where $G_R(s) = n_R(s)/d_R(s)$ is stable, minimum phase and with relative degree r at least, and $r_0(t)$ is uniformly bounded, i.e. $\sup|r_0(t)| \le \vartheta$.[4]



(a) Reference signal. (b) Equivalent *disturbance*.

Fig. 7. Convert a tracking problem into a regulation problem.

The main idea is to regard the signal $y_r(t)$ as a *disturbance*, and discount it to the input channel. (Fig. 7) Define

$$d_R(t) = \frac{y_r(s)}{G(s)} = \frac{G_R(s)}{G_p(s)\left(1+\Delta(s)\right)}r_0(t)$$

and denote

$$\sup_{G_p(s)\in\mathcal{G}}\left\|\frac{G_R(s)}{G_p(s)}\right\|_1 = \sup_{G_p(s)\in\mathcal{G}}\left\|\frac{n_R(s)d_p(s)}{d_R(s)n_p(s)}\right\|_1 := \varsigma,$$

then

$$\sup|d_R(t)| \le \frac{\varsigma}{1-\iota}\vartheta := \psi \tag{18}$$

is uniformly bounded. Thus the tracking problem has been converted into an equivalent regulation problem.

VI. Simulation Example

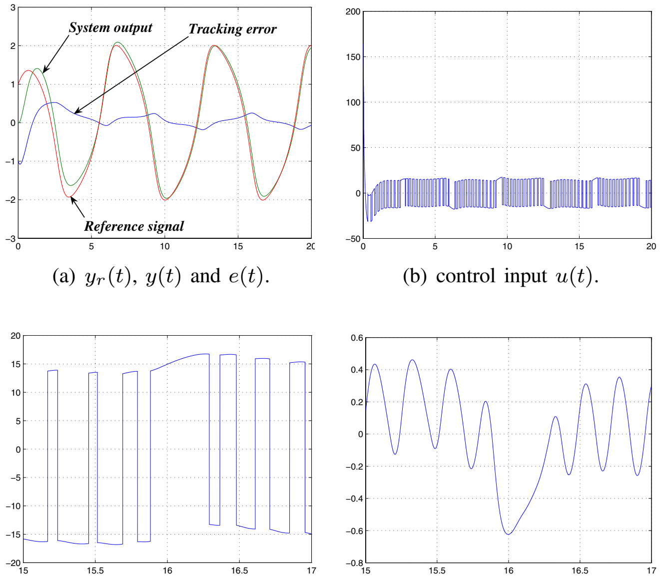
Consider an uncertain linear plant, where the nominal set and unmodeled dynamics are restricted as

$$\mathcal{G} = \left\{\frac{[1, 1.2]}{s^3 + [1,3]s^2 + [-3,0]s + [1,1.5]}\right\}$$

and $\mathcal{D} = \{\|\Delta(s)\|_1 < 0.2\}$; disturbance $|d(t)| \le 2.5$. The system output is expected to track a bounded reference signal $y_r(t)$, which is the solution of Van der Pol equation $\ddot{y}_r + (y_r^2-1)\dot{y}_r + y_r = 0$ with initial states $y_r(0) = 1$ and $\dot{y}_r(0) = 1$. (thus $\max_t|y_r(t)| = 2$.) In simulation, we take $G_p(s) = 1.05/(s^3+3s^2-0.5s+1.2)$ as the "real" model and $\Delta(s) = 0.15\frac{(s+1)(s-2)}{s^2+0.4s+1}$ as the "real" unmodeled dynamics; and $d(t)$ is composed of a periodic pulse signal (amplitude $[0,2]$, period 0.1 and pulse-rate 30%) and a stochastic signal uniformly distributed in $[-0.5, 0.5]$.

It can be verified that $C_1(s) = \frac{150(s+1)^2}{(s+5)(s+10)}$, $C_2(s) = \frac{s+3}{s+10}$ and $h(s) = (s+1)^2$ will guarantee $\frac{G_p}{1+C_1G_p}C_2h \in \text{SPR}$

[4]*Assumption D* is equivalent to such a condition: $y_r(t)$, $y_r'(t)$, $y_r^{(r-1)}(t)$ are all uniformly bounded, so is $d_R(t) := y_r(t)/G_p(s)$.



(a) $y_r(t)$, $y(t)$ and $e(t)$.

(b) control input $u(t)$.

(c) $u(t)$ in period $[15, 17]$.

(d) $\tilde{e}(t)$ in period $[15, 17]$.

Fig. 8. Simulation results.

($\forall G_p \in \mathcal{G}$). (coefficients can be turned-up "off-line" easily, see Appendix A.) The controller is designed according to *Theorem 5*, where $F_\mu(s) = (\mu s + 1)^{-2}$. The large coefficient $K_{vsc} = 15$ and the sufficient small coefficient $\mu = 0.05$ are determined "on-line", according to the "real" model.

Simulation result is shown in Fig. 8. Besides of the tracking ability and convergence, it can be seen (a) the auxiliary signal $\tilde{e}(t)$ oscillates around zero with high switching frequency; (b) the oscillations seem similar to the "chattering" phenomenon in standard VSC systems, although no ideal sliding motion exists at all.

VII. Conclusions

We have proposed a novel output feedback variable structure control scheme based on the passivity idea. In the case that exact differentiations provided, it guarantees asymptotic stability; in the case that singular perturbation based linear differentiators adopted, it guarantees an arbitrarily small regulation/tracking error.

We believe this scheme is practically significant that: (a) it can deal with a wide class of uncertain linear plants; (b) it seems rather simple and the control performance is acceptable; and (c) the coefficients of the proposed controller are physically significative and can be tuned up easily. We also believe the scheme is theoretically significant that it can be viewed as a new analysis framework of a class of relay control systems without sliding mode (*Lemma 4*).

Appendix

A. Sketch of the proof of Lemma 2.

$\forall G_p(s) = n_p(s)/d_p(s) \in \mathcal{G}$, $(s+1)^{r-1} n_p(s)/d_p(s)$ is with relative degree 1. According to the Lemma 3 in [22], a sufficiently large K_1 stabilizes $(s+1)^{r-1} n_p(s)/d_p(s)$, i.e. $n_1(s) := d_p(s) + K_1 n_p(s)(s+1)^{r-1}$ is a Hurwitz polynomial. Moreover, the q roots of $n_1(s)$ approximate the roots of $n_p(s)$, -1 ($r-1$ multiplicity) and $-K_1 n_l$ respectively. That is, (details omitted) $\sup_{\omega\ge 0}\left|\phi\left\{\frac{n_p(j\omega)(j\omega+1)^{r-1}(j\omega+K_1 n_l)}{n_1(j\omega)}\right\}\right|$ can be arbitrarily small if K_1 is large enough.

Next, define $d_1(s) = s d_p(s)$, the relative degree of $n_1(s)/d_1(s)$ is one, and a sufficiently large K_2 (depends on K_1) stabilizes $n_1(s)/d_1(s)$. That is,

$$\begin{aligned} n_2(s) &:= d_1(s) + K_2 n_1(s) \\ &= (s+K_2)d_p(s) + K_1 K_2 n_p(s)(s+1)^{r-1} \end{aligned}$$

is a Hurwitz polynomial. Moreover, the $q+1$ roots of $n_2(s)$ approximate the roots of $n_1(s)$ and $-K_2$ respectively. That is, $\sup_{\omega\ge 0}\left|\phi\left\{\frac{n_1(j\omega)\times(j\omega+K_2)}{n_2(j\omega)}\right\}\right|$ can be arbitrarily small if K_2 is large enough (depends on K_1).

Define $d_2(s) := (s+K_2)d_1(s) = s(s+K_2)d(s)$ and so on, until a sufficiently large constant K_r (depends on $K_1, K_2, \ldots, K_{r-1}$) makes the polynomial $n_r(s) := d_{r-1}(s) + K_r n_{r-1}(s)$ Hurwitz; and moreover, $\sup_{\omega\ge 0}\left|\phi\left\{\frac{n_{r-1}(j\omega)\times(j\omega+K_r)}{n_r(j\omega)}\right\}\right|$ can be arbitrarily small.

It can be verified that we finally obtain a proper, stable and minimum phase controller $C_1(s)$:

$$C_1(s) = \frac{K_1 K_2 \cdots K_r (s+1)^{r-1}}{(s+K_2)(s+K_3)\cdots(s+K_r)} \tag{19}$$

that stabilizes $n_p(s)/d_p(s)$. Define

$$C_{20}(s) = (s + K_1 \bar{n}_l) \quad h(s) = (s+1)^{r-1} \tag{20}$$

the feedback and cascade transfer function will be

$$\begin{aligned} \bar{G}(s) &:= \frac{G_p(s)}{1+G_p(s)C_1(s)} C_{20}(s)h(s) \\ &= \frac{n_p(s)(s+1)^{r-1}(s+K_1\bar{n}_l)\prod_{i=2}^{r}(s+K_i)}{n_r(s)} \end{aligned} \tag{21}$$

which is with relative degree 0 and satisfies

$$\begin{aligned} \sup_{\omega\ge 0}\left|\phi\left\{\bar{G}(j\omega)\right\}\right| \le{}& \sup_{\omega\ge 0}\left|\phi\left(\frac{j\omega+K_1\bar{n}_l}{j\omega+K_1 n_l}\right)\right| \\ &+ \sum_{i=2}^{r}\sup_{\omega\ge 0}\left|\phi\left\{\frac{n_{i-1}(j\omega)(j\omega+K_i)}{n_i(j\omega)}\right\}\right| \\ &+ \sup_{\omega\ge 0}\left|\phi\left\{\frac{n_p(j\omega)(j\omega+1)^{r-1}(j\omega+K_1 n_l)}{n_1(j\omega)}\right\}\right| \end{aligned} \tag{22}$$

Notice that the first item in the righthand of (22) is less than $\frac{\pi}{2}$ and the following two items can be arbitrarily small. This means: properly choosing coefficients K_1, K_2, ..., K_r, we have $\left|\phi\left(\bar{G}(j\omega)\right)\right| < \pi/2$, and thus $\bar{G}(s) \in$ SPR. Since the set of $\bar{G}(s)$ (denoted as $\bar{\mathcal{G}}$) is compact, this inequality is strict. It can be proved that there exists a sufficiently large constant β, such that $\forall \bar{G}(s) \in \bar{\mathcal{G}}$, $G(s) := \bar{G}(s)/(s+\beta) \in$ SPR.

Finally, define a proper, stable and minimum phase transfer function

$$C_2(s) = \frac{C_{20}(s)}{s+\beta} = \frac{s+K_1\bar{n}_l}{s+\beta}, \tag{23}$$

we complete the proof of *Lemma 2* constructively, where the transfer functions $C_1(s)$, $C_2(s)$ and $h(s)$ are defined in (19), (20) and (23). It should be pointed out that it is easy to adjust these coefficients ($K_1, \ldots, K_r, \beta$), because they are merely expected to be *sufficiently large*.

B. Sketch of the proof of Lemma 4.

First, it can be proved that system output $y(t)$ oscillates; that is, if $y(0) \neq 0$, there exists a constant $T < \infty$ such that $y(T) = 0$. When the sign of $y(t)$ changes, the sign of $u(t)$ will change immediately; and after at most $O(\mu)$ time[5], the sign of $\tilde{u}(t)$ will be the same as the sign of $u(t)$ and retain $|\tilde{u}(t)| > \rho\kappa$ $(0 < \rho < 1)$, until $y(t)$ returns to zero and another switching occurs.

Denote t_k as the switching time series of $y(t)$, and denote $t'_k \in (t_k, t_{k+1})$ (if exists) such that $\forall t \in (t'_k, t_{k+1})$, $\tilde{u}(t) > \rho\kappa$ (if $u(t) > 0$) or $\tilde{u}(t) < -\rho\kappa$ (if $u(t) < 0$). Assume $S(x) = x^\tau P x$ is the storage function of $G(s)$, where the positive definite matrix P satisfies conditions (4). Since $G(s) \in$ SPR, $\forall t > 0$, $S(x_t) \leq S(x_0) + \int_0^t \tilde{u}(\tau) y(\tau) d\tau$. Consider the time interval $[t_k, t_{k+1}]$,

$$\begin{aligned} S_{k+1} &\leq S_k + \int_{t_k}^{t'_k} \tilde{u}(\tau) y(\tau) d\tau + \int_{t'_k}^{t_{k+1}} \tilde{u}(\tau) y(\tau) d\tau \\ &< S_k + \beta(t'_k - t_k) - \rho\kappa \int_{t'_k}^{t_{k+1}} |y(\tau)| d\tau \\ &:= S_k + D_k^{(1)} - D_k^{(2)} \end{aligned} \tag{24}$$

where S_k is the abbreviation of $S(x(t_k))$, and $\beta \geq \sup|\tilde{u}(t)| \times \sup|y(t)|$ is a boundary constant (independent of ε and μ). Intuitively, since $(t'_k - t_k) < O(\mu)$, the term $D_k^{(1)}$ can be arbitrarily small as $\mu \to 0$; while the term $D_k^{(2)}$ can not be "too small" because $|\dot{y}(t)|$ is bounded independent of μ. Mathematically speaking,

$$\int_{t'_k}^{t_{k+1}} |y(\tau)| d\tau \geq \frac{1}{2\alpha} \left(\max_{t \in [t'_k, t_{k+1}]} |y(t)| \right)^2 .$$

where $\alpha = \sup|\dot{y}(t)|$ is independent of μ.

Assume $(t'_k - t_k) \leq \lambda\mu$, then

$$D_k^{(1)} + \lambda\beta\mu \geq D_k^{(2)} \tag{25}$$

will hold sooner or later after a finite-time transient process (otherwise the "energy" series $\{S_k\}$ will decrease at a speed greater than $\lambda\beta\mu$ to negative.) From then on, at each interval $[t_k, (t'_k), t_{k+1}]$, the series $\{S_k\}$ can be divided into 3 cases:

(c1) $S_k - S_{k+1} \leq \lambda\beta\mu$;
(c2) $S_k > S_{k+1} + \lambda\beta\mu$ but $S_k \leq S_{k'}$;
(c3) $S_k > S_{k+1} + \lambda\beta\mu$ and $S_k > S_{k'} > S_{k+1}$;

The three cases are complete (if time t'_k exists) and case (c1) is finite-time recurrent.

It can be proved that every time interval $[t_k, t_{k+1}]$ belongs to (c1) or (c2) satisfies

$$\max_{t \in [t_k, t_{k+1}]} |y(t)| < O(\sqrt{\mu}). \tag{26}$$

While case (c3) means "energy" decreasing monotonously and (26) can also be obtained.

Finally, if $|\tilde{u}(t)| < \rho\kappa$ holds in some whole time interval $[t_k, t_{k+1}]$, we also have $\max_{t \in [t_k, t_{k+1}]} < O(\mu)$.

[5]Here the symbol $O(\cdot)$ denotes a non-negative continuous function defined in a neighborhood of zero, satisfying $\lim_{\epsilon \to 0} \frac{O(\epsilon)}{|\epsilon|} < A < \infty$ where A is a positive constant independent of ϵ.

The analysis above indicates that as long as $\mu \to 0$, $y(t)$ will converge to an arbitrarily small neighborhood of zero and will not escape any longer.

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