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Approximate input-output linearization using L_2-optimal bilinearization

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***Abstract*— In this contribution a numerical method for deriving an approximate input-output linearizing controller for minimum phase systems is presented. The approach is based on the derivation of a bilinear approximation model of higher order, which approximates the considered system on a prespecified multivariable state-space interval. By means of the exact input-output linearizing state feedback of the bilinear model an approximate input-output linearizing controller for the original system with polynomial numerator and denominator is determined. Local stability of the closed-loop system can be assured by subsequently applying a suitable adjustment procedure. The results of the paper are demonstrated by means of a simple example.**

I. INTRODUCTION

Exact input-output linearization (see [1]) is one of the most commonly used methods of nonlinear control theory in practical applications. Applying this approach a nonlinear state feedback can be derived which achieves a linear input-output behavior of the closed-loop system with easily designable dynamics. However, apart from the restriction that the system under consideration must have a stable zero dynamics the derivation of the nonlinear controller usually requires symbolic calculations, which often turn out to be rather sophisticated (see e.g. [2]). Another drawback is that the structure of the resulting input-output linearizing feedback may be very complicated and therefore unsuitable with respect to implementation.

As remedy for these problems several methods have been developed based on approximate and/or numerical calculations. One possibility is the computation of the exact linearizing control law by automatic differentiation (see [3]). This approach computes the values of the control input generated by the input-output linearizing controller numerically. Kang [4] showed how an approximate input-output linearizing controller can be obtained using a Taylor expansion of the model nonlinearities. Recently, Deutscher [5] proposed another numerical approach applying multivariable Legendre polynomials to achieve an exact algebraic expression for the exact linearizing feedback. In general, however, state space embedding of nonpolynomial nonlinearites must be accomplished beforehand, which is only feasible by means of symbolic evaluation.

In this paper an alternative way of using multivariable Legendre polynomials is described, which from the beginning allows the efficient use of numerical software tools for deriving an approximation of the input-output linearizing controller for minimum phase SISO systems. To this end, the ideas of [6] are used to derive a bilinear approximation model of higher order, which approximates the behavior of the original system on a prespecified multivariable state-space interval. Based on the exact input-output linearizing feedback law for the bilinear system, which is given in an explicit form, an approximation for the corresponding linearizing controller of the original model with polynomial numerator and denominator is determined. Local stability of the closed-loop system can be guaranteed by adjusting the resulting control law such that its linearization coincides with a suitable linear stabilizing reference controller. Note that the quality of the derived approximation is nearly equally good on the entire predefined state-space interval. This represents a significant advantage compared to Kang's [4] approach, which is only suitable for small deviations of the operating point due to the local character of the Taylor series expansion.

After briefly introducing multivariable Legendre polynomials the derivation of the bilinear approximation model and the computation of its relative degree are explained in Section II. Section III shows the calculation of the approximate input-output linearizing feedback followed by a systematic adjustment procedure, which guarantees local stability. Finally, the proposed method is applied to a simple example in Section IV.

II. L_2-OPTIMAL BILINEARIZATION

A. *Multivariable Legendre polynomials*

The proposed bilinearization method is based on the use of multivariable Legendre polynomials, which are briefly introduced in the following (for further details see e.g. [6]). It is well known that the Legendre polynomials $\varphi^1(x_\nu)$ in one variable x_ν represent an orthogonal set on the space $L_2([-1,1])$ of Lebesgue-measurable, square-integrable functions. The set of multivariable Legendre polynomials in n variables x_ν, $\nu = 1, 2, \ldots, n$, can be introduced according to

$$\varphi_{k_1 \cdots k_n}(x) = \varphi^1_{k_1}(x_1) \cdot \ldots \cdot \varphi^1_{k_n}(x_n) \tag{1}$$

where the degree of $\varphi_{k_1 \cdots k_n}(x)$ is defined as $k = \sum_{\nu=1}^{n} k_\nu$ and $k_\nu = \deg\left(\varphi^1_{k_\nu}(x_\nu)\right)$. As shown in [6] the polynomials (1) fulfill the orthogonality relation

$$\langle \varphi_{j_1\cdots j_n}(x), \varphi_{k_1\cdots k_n}(x)\rangle =$$
$$= \int\limits_{I_{pol}} \varphi^1_{j_1}(x_1)\varphi^1_{k_1}(x_1)\cdot\ \ldots\ \cdot\varphi^1_{j_n}(x_n)\varphi^1_{k_n}(x_n)dx$$
$$= \begin{cases} \prod\limits_{\nu=1}^{n} \frac{2}{2k_\nu+1}\ :\ j_\nu = k_\nu \\ \quad 0\ :\ j_\nu \neq k_\nu \end{cases} \quad \forall\ \nu = 1,2,\ldots,n \tag{2}$$

on $L_2^n(I_{pol})$ with $I_{pol} = [-1,1]^n$. In order to achieve a compact notation for the subsequent calculations all multivariable Legendre polynomials are comprised in the n_Φ-dimensional vector

$$\Phi(x) = \begin{bmatrix}\Phi_1(x) & \Phi_2(x) & \ldots & \Phi_{n_\Phi}(x)\end{bmatrix}^T \tag{3}$$

of all polynomials up to degree N_{deg}, where $n_\Phi = \binom{n+N_{deg}}{n}$ and $\Phi_1(x) = 1$. Using (3) as a basis within the set of polynomials up to degree N_{deg} another important property of multivariable Legendre polynomials can be specified, namely that by introducing the operational matrices for multiplication M_μ, $\mu = 1,2,\ldots,n_\Phi$, and for differentiation D_ν, $\nu = 1,2,\ldots,n$, according to

$$\Phi_\mu(x)\Phi(x) \approx M_\mu\Phi(x) \tag{4}$$
$$\frac{\partial\Phi(x)}{\partial x_\nu} = D_\nu\Phi(x) \tag{5}$$

any differentiation and multiplication operation can be accomplished by simple matrix manipulations. However, since the multiplication $\Phi_\mu(x)\Phi(x)$ with $\mu > 1$ results in a vector containing polynomials of degrees higher than N_{deg} the relationship (4) only means the best feasible approximation with polynomials up to degree N_{deg} in an L_2-error norm sense.

Remark 1: In the sequel the notation "$\approx$" always denotes an approximation in the minimal L_2-error norm sense.

B. Derivation of the bilinear approximation model

Consider the nth order nonlinear SISO system

$$\dot{x} = f(x) + g(x)u \tag{6}$$
$$y = h(x) \tag{7}$$

It is assumed that $f(x)$ and $g(x)$ are elements of the space $L_2^n(I_{aprx})$ with

$$I_{aprx} = [x_{1,min}, x_{1,max}] \times \ldots \times [x_{n,min}, x_{n,max}] \tag{8}$$

and $h(x)$ is an element of the space $L_2(I_{aprx})$. In the following an approximation procedure for the L_2-approximation of the system (6)–(7) on a prespecified interval I_{aprx} in the state space is outlined. For further details the reader is referred to [6].

Step 1 (interval transformation):: Before the multivariable Legendre polynomials can be applied, the approximation interval I_{aprx} (see (8)) of the nonlinear system (6)–(7) must be adjusted to the interval $I_{pol} = [-1,1]^n$ on which the multivariable Legendre polynomials (1) are defined. This can be achieved by the linear affine state transformation

$$x_\nu = \frac{x_{\nu,max} - x_{\nu,min}}{2}\tilde{x}_\nu + \frac{x_{\nu,max} + x_{\nu,min}}{2} \tag{9}$$

$\nu = 1,2,\ldots,n$, which results in the state space model

$$\dot{\tilde{x}} = \tilde{f}(\tilde{x}) + \tilde{g}(\tilde{x})u \tag{10}$$
$$y = \tilde{h}(\tilde{x}) \tag{11}$$

with the associated approximation interval $I_{pol} = [-1,1]^n$.

Remark 2: This simple transformation represents the only operation which requires symbolic evaluation techniques. All succeeding steps of the proposed controller design process can be performed by purely numerical calculations.

Step 2 (L_2-approximation of the nonlinearities):: Next, the nonlinearities on the right hand side of (10)–(11) are approximated by multivariable Legendre polynomials in an L_2-optimal way. For lack of space this is only shown for the vector function $\tilde{f}(\tilde{x})$ in the following (see (10)). After choosing a suitable approximation degree N_{deg}, the vector field $\tilde{f}(\tilde{x})$ is approximated by the polynomial vector function

$$\tilde{f}(\tilde{x}) \approx F\Phi(\tilde{x}) \tag{12}$$

where the (n, n_Φ) coefficient matrix F has to be determined such that the L_2-norm error

$$e = \|\tilde{f}(\tilde{x}) - F\Phi(\tilde{x})\|_2$$
$$= \left\langle \tilde{f}(\tilde{x}) - F\Phi(\tilde{x}), \tilde{f}(\tilde{x}) - F\Phi(\tilde{x})\right\rangle^{\frac{1}{2}} \tag{13}$$

is minimized. As described in [6] the solution to the stated problem is given by

$$F = \left\langle \tilde{f}(\tilde{x}), \Phi^T(\tilde{x})\right\rangle \left\langle \Phi(\tilde{x}), \Phi^T(\tilde{x})\right\rangle^{-1} \tag{14}$$

where $\left\langle \tilde{f}(\tilde{x}), \Phi^T(\tilde{x})\right\rangle$ and $\left\langle \Phi(\tilde{x}), \Phi^T(\tilde{x})\right\rangle$ denote the associated Gram matrices, which can be evaluated by numerical integration. Applying the same procedure to compute the coefficient matrix G and the coefficient vector h^T with respect to $\tilde{g}(\tilde{x})$ and $\tilde{h}(\tilde{x})$ in (10)–(11) an approximate system description of the form

$$\dot{\tilde{x}} \approx \hat{f}(\tilde{x}) + \hat{g}(\tilde{x})u \tag{15}$$
$$y \approx \hat{h}(\tilde{x}) \tag{16}$$

with the solely polynomial nonlinearities $\hat{f}(\tilde{x}) = F\Phi(\tilde{x})$, $\hat{g}(\tilde{x}) = G\Phi(\tilde{x})$ and $\hat{h}(\tilde{x}) = h^T\Phi(\tilde{x})$ is obtained.

Step 3 (bilinear approximation model):: Finally, based on the system representation (15)–(16) a bilinear approximation model of higher order can be derived by state space embedding of the Legendre polynomials up to degree N_{deg}. Introducing the $(n_\Phi - 1)$-dimensional state vector z of the bilinear system according to

$$\begin{bmatrix}1\\ z\end{bmatrix} = \Phi(\tilde{x}) \tag{17}$$

and differentiating (17) with respect to time lead to the expressions

$$\frac{d}{dt}\begin{bmatrix}1\\z\end{bmatrix} \approx \frac{\partial\Phi(\tilde{x})}{\partial\tilde{x}}\left(F\Phi(\tilde{x}) + G\Phi(\tilde{x})u\right) \tag{18}$$

$$y \approx h^T\begin{bmatrix}1\\z\end{bmatrix} \tag{19}$$

with the Jacobian matrix $\frac{\partial\Phi(\tilde{x})}{\partial\tilde{x}}$. Applying the relations (4)–(5) to (18) and performing some basic manipulations yield

$$\frac{d}{dt}\begin{bmatrix}1\\z\end{bmatrix} \approx A_\Phi\Phi(\tilde{x}) + N_\Phi\Phi(\tilde{x})u \tag{20}$$

with the matrices

$$A_\Phi = \left(D_1 \otimes e_1^T F + \ldots + D_n \otimes e_n^T F\right) M \tag{21}$$

$$N_\Phi = \left(D_1 \otimes e_1^T G + \ldots + D_n \otimes e_n^T G\right) M \tag{22}$$

where "$\otimes$" denotes the Kronecker tensor product (see e.g. [7]) and e_ν means the νth unit vector. The matrices $M = [M_1^T \ldots M_{n_\Phi}^T]^T$ and D_ν are given by (4) and (5). Defining the partitions

$$A_\Phi = \begin{bmatrix}0 & 0^T\\ a_0 & A\end{bmatrix},\ N_\Phi = \begin{bmatrix}0 & 0^T\\ b & N\end{bmatrix},\ h^T = \begin{bmatrix}c_0 & c^T\end{bmatrix} \tag{23}$$

in (21)–(22), omitting the trivial first equation in (20) and setting "$=$" instead of "$\approx$" finally leads to the affine bilinear approximation model

$$\dot{z} = f_b(z) + g_b(z)u \tag{24}$$

$$y = h_b(z) \tag{25}$$

(i.e. with right hand sides affine in z) where $f_b(z) = Az + a_0$, $g_b(z) = b + Nz$ and $h_b(z) = c^T z + c_0$.

C. Relative degree of the approximation model

In this section a simple algorithm for computing the relative degree of the bilinear approximation model (24)–(25) at a point z_0 in the state space is presented and its connection to the relative degree of the original system (10)–(11) is established.

The relative degree r_b of (24)–(25) at z_0 is determined by

$$L_{g_b}L_{f_b}^i h_b(z) = c^T A^i(b + Nz) = 0 \quad \forall z \in U_{z_0}, i = 0, 1, \ldots, r_b - 2 \tag{26}$$

$$L_{g_b}L_{f_b}^{r_b-1} h_b(z)\big|_{z=z_0} = c^T A^{r_b-1}(b + Nz_0) \neq 0 \tag{27}$$

(see [1]) where U_{z_0} means a neighborhood of z_0 and L_{g_b}, L_{f_b} denote the Lie derivatives along g_b and f_b, respectively. Obviously, in the case of $c^T A^i N \neq 0^T$ (26) describes an $(n_\Phi - 2)$-dimensional hyperplane in the state space meaning that a neighborhood U_{z_0} of solutions to (26) does not exist. Consequently, (26) can only be fulfilled in U_{z_0} if

$$\left.\begin{array}{rcl} c^T A^i b & = & 0 \\ c^T A^i N & = & 0^T \end{array}\right\} \ \forall i = 0, 1, \ldots, r_b - 2 \tag{28}$$

Thus, the relative degree r_b of the bilinear approximation model at a point z_0 can easily be calculated by finding the smallest i for which one of the equations in (28) is violated and subsequently checking the condition (27) for $r_b = i + 1$. If the latter is not fulfilled, the relative degree at z_0 is not well-defined. This result is also valid for the underlying nonlinear system (10)–(11), if its input-output behavior is described satisfactorily by the associated bilinear approximation. This can be assumed for an adequate approximation order N_{deg} and a sufficiently small approximation interval I_{aprx}. On that condition the relative degree r of (10)–(11) at a point $\tilde{x}_0$ can directly be related to r_b at the corresponding point

$$z_0 = [\Phi_2(\tilde{x}_0) \ldots \Phi_{n_\phi}(\tilde{x}_0)]^T \tag{29}$$

(see (17)). Actually, it can be shown that a system with polynomial nonlinearities of the form (15)–(16) and its corresponding bilinear approximation model share the same relative degree $\hat{r} = r_b$ if the approximation order N_{deg} for the state space embedding (see step 3 in Section II-B) satisfies

$$N_{deg} \geq \deg(\hat{g}) + l_{\hat{r}-1} - 1 \tag{30}$$

where

$$l_i = \deg(\hat{h}) + i\deg(\hat{f}) - i \tag{31}$$

and the operator $\deg(\cdot)$ applied to a vector function means the greatest degree of all elements. This is briefly verified by evaluating the general conditions

$$L_{\hat{g}}L_{\hat{f}}^i\hat{h}(\tilde{x}) = 0 \ \ \forall\, \tilde{x} \in U_{\tilde{x}_0},\ i = 0, 1, \ldots, \hat{r} - 2 \tag{32}$$

$$L_{\hat{g}}L_{\hat{f}}^{\hat{r}-1}\hat{h}(\tilde{x})\big|_{\tilde{x}=\tilde{x}_0} \neq 0 \tag{33}$$

for the relative degree $\hat{r}$ of (15)–(16) at $\tilde{x}_0$. At first, consider the Lie derivative

$$L_{\hat{f}}\hat{h}(\tilde{x}) = h^T\frac{\partial\Phi(\tilde{x})}{\partial\tilde{x}}F\Phi(\tilde{x}) \tag{34}$$

Comparing (34) to (18) and (20) leads to the expression

$$L_{\hat{f}}\hat{h}(\tilde{x}) = h^T A_\Phi \Phi(\tilde{x}) \tag{35}$$

which is exact as long as the approximation degree $N_{deg} \geq \deg(\hat{h}) + \deg(\hat{f}) - 1$ ensures that no error is involved by applying the operational matrices M_μ for multiplication (4). Proceeding to Lie derivatives of higher order the relationship

$$L_{\hat{g}}L_{\hat{f}}^i\hat{h}(\tilde{x}) = h^T A_\Phi^i N_\Phi \Phi(\tilde{x}) \tag{36}$$

can be derived, which is valid for

$$N_{deg} \geq \deg(\hat{g}) + l_i - 1 \tag{37}$$

with l_i given by (31). Finally, replacing A_Φ, N_Φ and h^T by (23) yields

$$L_{\hat{g}}L_{\hat{f}}^i\hat{h}(\tilde{x}) = [c^T A^i b \ \ c^T A^i N]\Phi(\tilde{x}) \tag{38}$$

After substituting (38) into (32)–(33) and replacing $\Phi(\tilde{x})$ by (17) the equivalence of the conditions (32)–(33) and the corresponding relations (26)–(27) for the associated bilinear approximation model with suitable approximation degree (30) becomes obvious.

III. L_2-OPTIMAL I/O-LINEARIZATION

A. Approximate I/O-linearizing feedback

On the basis of the bilinear approximation model an approximate expression for the exact input-output linearizing feedback of the original system (10)–(11) in a neighborhood of an operating point $(\tilde{x}_0, u_0)$ can be determined provided that the relative degree r of the original system is well-defined at $\tilde{x}_0$ and the corresponding bilinear approximation model shares the same relative degree $r_b = r$ at the associated point z_0 (see (29)). To this end, consider the exact input-output linearizing feedback

$$u = \frac{1}{L_{g_b} L_{f_b}^{r_b-1} h_b(z)} \left(-\sum_{i=0}^{r_b} \alpha_i L_{f_b}^i h_b(z) + \alpha_0 w \right) \tag{39}$$

for the bilinear system (24)–(25) with $\alpha_{r_b} = 1$ and the new input w (see [1]). Applying the feedback law (39) to (24)–(25) results in the linear closed-loop input-output dynamics

$$y^{(r_b)} + \alpha_{r_b-1} y^{(r_b-1)} + \ldots + \alpha_0 y = \alpha_0 w \tag{40}$$

which can easily be assigned by means of the constant parameters α_i. Substituting the nonlinearities f_b, g_b and h_b according to (24)–(25) into (39) and successively evaluating the Lie derivatives (27) and

$$L_{f_b}^i h_b(z) = c^T A^{i-1} a_0 + c^T A^i z \tag{41}$$

$i = 1, 2, \ldots, r_b$, yield the explicit formula

$$u = \frac{-\alpha_0 c_0 - c^T \sum\limits_{i=0}^{r_b-1} \alpha_{i+1} A^i a_0 - c^T \sum\limits_{i=0}^{r_b} \alpha_i A^i z + \alpha_0 w}{c^T A^{r_b-1}(b + Nz)} \tag{42}$$

for the exact input-output linearizing feedback of the bilinear approximation model. For applying (42) to the original system (10)–(11) $[1 \ z^T]^T$ is replaced by (17) which leads to the controller

$$u = \frac{1}{q^T \Phi(\tilde{x})} \left(p^T \Phi(\tilde{x}) + \alpha_0 w \right) \tag{43}$$

with polynomial numerator and denominator determined by the n_Φ-dimensional coefficient vectors

$$p^T = \left[-\alpha_0 c_0 - c^T \sum_{i=0}^{r_b-1} \alpha_{i+1} A^i a_0 \quad - c^T \sum_{i=0}^{r_b} \alpha_i A^i \right] \tag{44}$$

$$q^T = \left[c^T A^{r_b-1} b \quad c^T A^{r_b-1} N \right] \tag{45}$$

For implementation the feedback law (43) must be expressed in terms of the original coordinates x by applying the inverse of the state transformation (9), since x is supposed to be measured.

Again a precise statement about the accuracy of the approximate controller (43) is feasible for state space models of the type (15)–(16) with polynomial nonlinearites. In that case the input-output linearizing law (43) is exact if the approximation degree N_{deg} for the state space embedding process (see Section II-B, step 3) is chosen according to

$$N_{deg} \geq \max(\deg(\hat{g}) + l_{\hat{r}-1} - 1, l_{\hat{r}}) \tag{46}$$

with l_i given by (31) (see [5]).

B. Assuring local stability

A drawback of L_2-optimal approximation methods is the fact that in general the behavior of the original system is not met exactly at the operating point. In the following, a remedy for this problem is presented based on an adjustment of the approximate control law (43). To this end, consider the linearization of the exact input-output linearizing feedback for (10)–(11)

$$\Delta u_{lin} = k^T \Delta\tilde{x} + m \Delta w \tag{47}$$

with $\Delta\tilde{x} = \tilde{x} - \tilde{x}_0$ and $\Delta w = w - \tilde{h}(\tilde{x}_0)$, which can easily be calculated by means of the Jacobian linearization of (10)–(11) about $(\tilde{x}_0, u_0)$. In the sequel, the feedback (47) is taken as a kind of local reference controller. Setting $w = w_0 = \tilde{h}(\tilde{x}_0)$ (i.e. $\Delta w = 0$) and linearizing the nonlinear feedback (43) about $(\tilde{x}_0, u_0)$ lead to

$$\begin{aligned} \Delta u_{nl} &= \frac{\partial u(\tilde{x}, w_0)}{\partial \tilde{x}}\Big|_{\tilde{x}=\tilde{x}_0} \cdot \Delta\tilde{x} + u(\tilde{x}_0, w_0) - u_0 \\ &= \left(p^T P + s^T \right) \Delta\tilde{x} + v \end{aligned} \tag{48}$$

with $\Delta u_{nl} = u - u_0$, the constant offset error $v = u(\tilde{x}_0, w_0) - u_0$, the vector

$$s^T = \frac{\alpha_0 w_0}{[q^T \Phi(\tilde{x}_0)]^2} q^T \frac{\partial \Phi(\tilde{x})}{\partial \tilde{x}}\Big|_{\tilde{x}=\tilde{x}_0} \tag{49}$$

and the (n_Φ, n) matrix

$$P = \frac{\left([q^T \Phi(\tilde{x}_0)] I_{n_\Phi} - \Phi(\tilde{x}_0) q^T \right) \frac{\partial \Phi(\tilde{x})}{\partial \tilde{x}}\Big|_{\tilde{x}=\tilde{x}_0}}{[q^T \Phi(\tilde{x}_0)]^2} \tag{50}$$

where $\frac{\partial \Phi(\tilde{x})}{\partial \tilde{x}}\Big|_{\tilde{x}=\tilde{x}_0} = [D_1 \Phi(\tilde{x}_0) \ \ldots \ D_n \Phi(\tilde{x}_0)]$ (see (5)), p^T and q^T are given by (44)–(45) and I_{n_Φ} denotes the (n_Φ, n_Φ) identity matrix. Comparison between (47) and (48) for $\Delta w = 0$ shows that for local exactness in a neighborhood of $(\tilde{x}_0, u_0)$ the nonlinear controller (43) must satisfy the conditions

$$v = 0 \tag{51}$$

$$p^T P = k^T - s^T \tag{52}$$

However, if (51)–(52) are not fulfilled the control (43) may be replaced by

$$\hat{u} = \frac{1}{q^T \Phi(\tilde{x})} \left(\hat{p}^T \Phi(\tilde{x}) + \alpha_0 w \right) + \hat{v} \tag{53}$$

with the adjusted numerator coefficient vector $\hat{p}^T$ and the constant offset correction $\hat{v}$. Local exactness of (53) in terms of (51)–(52) can be guaranteed with minor changes compared to the control law (43) by determining $\hat{p}^T$ and $\hat{v}$ as follows. After replacing p^T by $\hat{p}^T$ and assuming fixed denominator coefficients q^T the relation (52) can be regarded as an under-determined linear system of equations, which is always solvable since it can be verified that $\text{rank}(P) = n$. The general solution to (52) reads

$$\hat{p}^T = \lambda^T B^T + \kappa^T \tag{54}$$

with $\kappa^T = (k^T - s^T)(P^T P)^{-1} P^T$ where all remaining degrees of freedom are comprised in the $(n_\Phi - n)$-dimensional vector λ^T and the columns of the $(n_\Phi, n_\Phi - n)$ matrix B represent a basis of the null space of P^T. Next, the parameter vector λ^T is chosen such that the numerator polynomial functions of (43) and (53) match in the minimum L_2-error norm sense on $I_{pol} = [-1, 1]^n$, i.e.

$$\begin{aligned}&\|\hat{p}^T\Phi(\tilde{x}) - p^T\Phi(\tilde{x})\|_2 = \\ &\|\lambda^T B^T\Phi(\tilde{x}) - (p^T - \kappa^T)\Phi(\tilde{x})\|_2 \stackrel{!}{=} \min \end{aligned} \tag{55}$$

Using the results of [6] the solution to this problem is given by

$$\lambda^T = (p^T - \kappa^T)\big\langle\Phi(\tilde{x}), \Phi^T(\tilde{x})\big\rangle B\big(B^T\big\langle\Phi(\tilde{x}), \Phi^T(\tilde{x})\big\rangle B\big)^{-1} \tag{56}$$

with the Gram matrix $\big\langle\Phi(\tilde{x}), \Phi^T(\tilde{x})\big\rangle$, which has already been computed during the derivation of the bilinear approximation model (see (14)).

Remark 3: The inverse on the right hand side of (56) always exists since the positive definiteness of $\big\langle\Phi(\tilde{x}), \Phi^T(\tilde{x})\big\rangle$ and $\mathrm{rank}(B) = n_\Phi - n$ imply the positive definiteness of the matrix $B^T\big\langle\Phi(\tilde{x}), \Phi^T(\tilde{x})\big\rangle B$.

Finally, the condition (51), which with respect to (53) is equivalent to

$$\hat{u}(\tilde{x}_0, w_0) \stackrel{!}{=} u_0 \tag{57}$$

is satisfied by setting

$$\hat{v} = u_0 - \frac{1}{q^T\Phi(\tilde{x}_0)}\big(\hat{p}^T\Phi(\tilde{x}_0) + \alpha_0 w_0\big) \tag{58}$$

If the linear approximation of the zero dynamics of (10)–(11) is asymptotically stable, then the linear feedback (47) stabilizes the operating point $(\tilde{x}_0, u_0)$ of (10)–(11) (see [1]). Thus, applying the adjusted controller (53) to (10)–(11) also yields a closed-loop system with asymptotically stable operating point $(\tilde{x}_0, u_0)$, since the linear approximation of (53) coincides with (47).

IV. Example

Consider the 2nd order nonlinear system

$$\begin{aligned}\dot{x}_1 &= -x_1 + \tfrac{1}{x_2+2} + u \\ \dot{x}_2 &= -x_2 + (x_1^2 + 1)u \end{aligned} \tag{59}$$

$$y = x_1^2 + x_2 \tag{60}$$

defined on the approximation interval $x \in I_{aprx} = [-1, 1]^2$. The operating point is given by $x_0 = [\frac{1}{2}, 0]^T$ and $u_0 = 0$. The bilinear approximation model associated with (59)–(60) can be derived for the approximation degree $N_{deg} = 2$ by proceeding as proposed in Section II-B. However, since $I_{aprx} = I_{pol}$ no interval transformation is necessary and the first step can be omitted. Before the second step is accomplished, the basis

$$\Phi(x) = \begin{bmatrix} 1 & x_1 & x_2 & \frac{1}{2}(3x_1^2 - 1) & x_1x_2 & \frac{1}{2}(3x_2^2 - 1) \end{bmatrix}^T \tag{61}$$

(see (3)) within the set of Legendre polynomials in two variables up to degree two must be established. Using (61) the L_2-optimal polynomial approximations of the nonlinearities on the right hand side of (59)–(60) are determined by evaluating (14) as well as the corresponding expressions for $\tilde{g}$ and $\tilde{h}$. Applying the results of step 3 in Section II-B yields the matrices and vectors

$$\begin{aligned}
A &= \begin{bmatrix} -1 & -0.296 & 0 & 0 & 0.106 \\ 0 & -1 & 0 & 0 & 0 \\ 1.648 & 0 & -2 & -0.888 & 0 \\ 0 & 0.592 & 0 & -2 & -0.197 \\ 0 & 0 & 0 & 0 & -2 \end{bmatrix} \\
b &= \begin{bmatrix} 1 & 1.333 & 0 & 0 & 0 \end{bmatrix}^T \\
N &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.667 & 0 & 0 \\ 3 & 0 & 0 & 0 & 0 \\ 1.6 & 1 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 \end{bmatrix} \\
a_0 &= \begin{bmatrix} 0.549 & 0 & -1 & 0.099 & -1 \end{bmatrix}^T \\
c^T &= \begin{bmatrix} 0 & 1 & 0.667 & 0 & 0 \end{bmatrix} \\
c_0 &= 0.333
\end{aligned} \tag{62}$$

of the 5th order bilinear approximation model (24)–(25) associated with (59)–(60). Based on that an approximate input-output linearizing controller is calculated. First, it is verified by means of (27) that the bilinear model has the relative degree $r_b = 1$ at the point $z_0 = \begin{bmatrix}0.5 & 0 & -0.125 & 0 & -0.5\end{bmatrix}^T$ corresponding to x_0 in view of (29). Assuming that the linear closed-loop input-output behavior

$$\dot{y} + \alpha_0 y = \alpha_0 w \tag{63}$$

(see (40)) with $\alpha_0 = 1$ is required the coefficient vectors

$$\begin{aligned}
p^T &= \begin{bmatrix} -\alpha_0 c_0 - c^T a_0 & -\alpha_0 c^T - c^T A \end{bmatrix} \\
&= \begin{bmatrix} 0.333 & -1.099 & 0 & 0.667 & 0.592 & 0 \end{bmatrix} \qquad (64) \\
q^T &= \begin{bmatrix} c^T b & c^T N \end{bmatrix} \\
&= \begin{bmatrix} 1.333 & 2 & 0 & 0.667 & 0 & 0 \end{bmatrix} \qquad (65)
\end{aligned}$$

determining the approximate input-output linearizing feedback (43) are calculated by applying (44) and (45). It is readily seen that the linearization of the calculated control law for $w_0 = \frac{1}{4}$ (see (48)) violates the conditions for local exactness (51)–(52). Thus, the linearization of the nonlinear feedback must be matched to the linear reference controller (47) with $k^T = [0 \; 0.111]$ designed on basis of the Jacobian linearization of (59)–(60). This leads to a linear system of two equations for the adjusted numerator coefficient vector $\hat{p}^T$ (see (52)) with the general solution of the form (54). By means of (56) the special solution

$$\hat{p}^T = \begin{bmatrix} 0.331 & -1.097 & -0.026 & 0.682 & 0.552 & 0.006 \end{bmatrix} \tag{66}$$

is determined which ensures the smallest possible change with respect to the L_2-optimal nonlinear controller (43) in terms of (55). Finally, after evaluating the offset correction $\hat{v} = 0.025$ according to (58) and replacing $\Phi(x)$ by (61) the

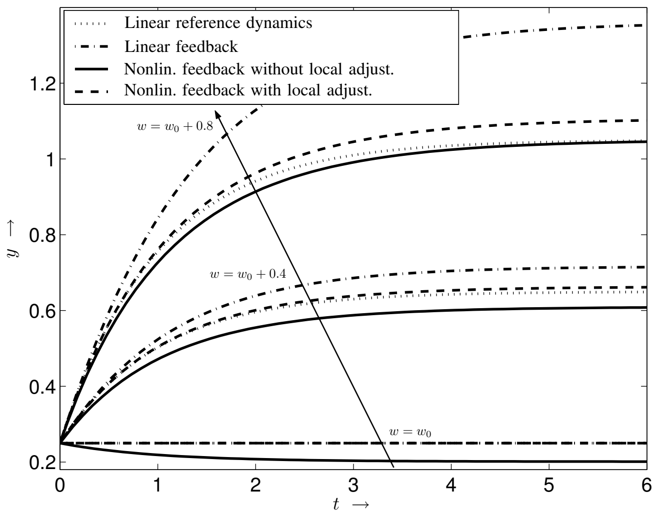

Fig. 1. Step responses of the closed-loop system

adjusted feedback law (53) reads

$$u = \frac{1.023x_1^2 + 0.552x_1x_2 + 0.009x_2^2 - 1.097x_1 - 0.026x_2 - 0.013}{(x_1+1)^2} + \frac{1}{(x_1+1)^2}w, \; x_1 \neq -1 \tag{67}$$

In Figure 1 step responses of the closed-loop system applying the approximate input-output linearizing controllers with and without local adjustment are compared to the required linear dynamics (63) and to the input-output characteristics obtained by employing the linear state feedback (47). Obviously, the application of both nonlinear feedback laws results in a suitable performance with small steady state errors. As expected, ideal behavior around the operating point is achieved using the proposed local adjustment algorithm. Furthermore, in contrast to the linear controller the system responses do not worsen significantly when far-off inputs with respect to the nominal value $w_0 = \frac{1}{4}$ are considered. It can also be shown that the performance of both nonlinear controllers will improve, if an higher approximation degree N_{deg} is chosen. Actually, the corresponding step responses for $N_{deg} = 4$ with and without local adjustment coincide with the required theoretical step responses in terms of plotting accuracy. Finally, it should be mentioned that all necessary calculations for deriving the approximate input-output linearizing feedbacks could be accomplished numerically using MATLAB.

V. Conclusions

In this paper an efficient procedure for the computation of an approximate input-output linearizing feedback for nonlinear SISO systems was presented, which can be completely accomplished using numerical software tools. Moreover, local stability of the resulting closed-loop system could be guaranteed by means of a systematic adjustment of the resulting nonlinear controller.

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