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A new method to design discontinuous stabilizing controllers for cascade systems

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***Abstract*— In this paper, we propose a new method to design discontinuous stabilizing controllers for a class of cascade systems including various kind of nonholonomic systems such as chained systems and symmetric affine systems. The cascade system consists of two systems: One system is a linear system with constant coefficient matrices, and another system is a linear system with matrices depending on the variables of the first system. We replace the independent variable of the second system with the variable that corresponds to the slowest mode of the first system. The result is a differential equation with regular singularity at the origin. If this system is not "controllable", it can be transformed into a "controllable" system by "similar transformation". This transformation results in a discontinuous state feedback. Our approach can be seen as a generalization of the method based on σ process that is developed by Astolfi and Laiou.**

I. Introduction

In this paper, we deal with the stabilization problem of cascade systems of the form

$$\Sigma_1 \quad : \quad \dot{x} = A(\xi)x + b(\xi)u, \tag{1}$$
$$\Sigma_2 \quad : \quad \dot{\xi} = C\xi + Dv, \tag{2}$$

which include various kind of nonholonomic systems such as chained systems and symmetric affine systems. The control problem of this type of systems have been addressed in [7][8].

It is known that chained systems and nonholonomic symmetric affine systems cannot be stabilized to a point by time-invariant continuous state feedback controllers, though they are controllable. Many control strategies have been developed for such systems. They can be classified as time-varying smooth state feedback[15][3][13][16][4][10] and time-invariant discontinuous state feedback[2][18][1][9][14][12].

If $A(0), b(0)$ satisfies the condition

$$\operatorname{rank}\begin{pmatrix} b(0) & A(0)b(0) & \cdots & A(0)^{n-1}b(0)\end{pmatrix} = n, \tag{3}$$

(1) can be stabilized by a smooth state feedback. In this paper, we focus on the stabilization of (1) in the case that the condition (3) are not satisfied. Assume that Σ_2 is controllable and v is chosen so that the closed loop system of Σ_2 is stable. Suppose that z is the variable which corresponds to the eigenvalue ν which has the smallest absolute value of the real part among all eigenvalues of Σ_2.

By replacing the independent variable t in Σ_1 with z, we have

$$z\frac{dx}{dz} = \nu^{-1}(A(z)x + b(z)u), \tag{4}$$

This is a differential equation with regular singularity at $z = 0$.

Then, by the variable transformation $x = P(z)y$, (4)is transformed into

$$z\frac{dy}{dz} = \hat{A}(z)y + \hat{b}(z)u. \tag{5}$$

In some cases, there exists $P(z)$ such that

$$\operatorname{rank}\begin{pmatrix} \hat{b}(0) & \hat{A}(0)\hat{b}(0) & \cdots & \hat{A}(0)^{n-1}\hat{b}(0)\end{pmatrix} = n, \tag{6}$$

holds. In this case, we can choose F_0 so that all eigenvalues of $\hat{A}(0) + \hat{b}(0)F_0$ are in the right half plane of the complex plane. Therefore, a discontinuous state feedback control $u = F(z)P(z)^{-1}x$, where $F(0) = F_0$, stabilizes Σ_1.

In this paper, we show a sufficient condition of the existence of such a $P(z)$.

In [1][6], the variable transformation $x = P(z)y$ is called σ process. Since $P(z)$ is limited to diagonal matrices in [1][6], our method is considered as a generalization of the method of [1][6]. Moreover, we provide the systematic way to construct $P(z)$, while one should find $P(z)$ by try and error when using the method in [1][6].

The organization of this paper is as follows: In the section II, we deal with the stabilization problem of the cascade systems Σ_1, Σ_2 in the case that Σ_2 is a scalar system. In this section, we show a sufficient condition of the existence of the variable transformation $x = P(z)y$ such that (6) holds. An example is also shown in this section. This is the example that there exists no variable transformation $x = P(z)y$ with $P(z)$ which is diagonal such that (6) holds. In other words, σ process in [1][6] fails to satisfy (6). In the section III, we deal with the stabilization problem of the cascade systems Σ_1, Σ_2 in the case that Σ_2 is a multiple variable system. In this section, as an example a stabilizing state feedback law is derived for a higher order chained system.

II. Cascade system of a nonlinear system and a scalar linear system

A. *transformation of independent variable*

We first consider the following system

$$\frac{dx}{dt} = A'(t)x, \tag{7}$$

where $t \in \mathbb{R}$ is the independent variable, $x \in \mathbb{R}^n$ is the state vector, and $A'(t)$ is an $n \times n$ real matrix valued function of t.

Lemma 1: Let $A_\infty = \lim_{t\to\infty} A'(t)$. Then, if A_∞ is Hurwitz, (7) is exponentially stable.

Proof: See, for example, [5]. ■

Consider the following system

$$z\frac{dx}{dz} = A(z)x, \tag{8}$$

where $z \in \mathbb{R}$ is the independent variable, and $A(z)$ is an $n\times n$ matrix valued function of z.

Definition 1: Suppose that by the variable transformation

$$z = Ce^{\nu t},\ C,\nu \in \mathbb{R},\ \nu < 0, \tag{9}$$

the system (8) is transformed into (7), where $A'(t) = \nu A(Ce^{\nu t})$. Then, (8) is said to be polynomially stable if (7) is exponentially stable.

Remark 1: It follows from the Lemma 1 if all eigenvalues of $A(0)$ are in the right half plane of the complex plane, (8) is polynomially stable.

Consider the system

$$z\frac{dx}{dz} = A(z)x + b(z)u, \tag{10}$$

where u is the control input and $b(z)$ is an $n\times 1$ vector valued function of z. For an $n\times n$ real matrix valued function $P(z)$ such that $\mathrm{Det}P(z) \neq 0, z \neq 0$, (10) is transformed by the variable transformation $x = P(z)y$ into

$$z\frac{dy}{dz} = \hat{A}(z)y + \hat{b}(z)u, \tag{11}$$

where

$$\hat{A}(z) = P^{-1}(z)A(z)P(z) - zP^{-1}(z)\frac{d}{dz}P(z), \tag{12}$$

$$\hat{b}(z) = P^{-1}(z)b(z). \tag{13}$$

The equation (12) is equivalent to

$$z\frac{d}{dz} - A(z) = P(z)\left(z\frac{d}{dz} - \hat{A}(z)\right)P(z)^{-1} \tag{14}$$

The mapping from $(A(z), b(z))$ to $(\hat{A}(z), \hat{b}(z))$ defined in (12)(13) is denoted by $(\hat{A}(z), \hat{b}(z)) = \mathcal{A}(A(z), b(z))$.

B. infinite dimensional matrix representation of the system

The notations in this section are based on [11].

We denote the set of all Laurent series about $z = 0$ and the set of all Taylor series about $z = 0$ be $\mathbb{C}((z)), \mathbb{C}((z))^+$ respectively.

Let $\mathcal{S}(n,m) = \mathbb{C}((z))^{n\times m}$ and $\mathcal{S}^+(n,m) = (\mathbb{C}((z))^+)^{n\times m}$.

Any element $A(z) \in \mathcal{S}(n,m)$ can be written as

$$A(z) = \sum_{k\in\mathbb{Z}} A_k z^k, \tag{15}$$

where

$$A_k = \begin{pmatrix} a_{1\ km+1} & a_{1\ km+2} & \cdots & a_{1\ km+m} \\ a_{2\ km+1} & a_{2\ km+2} & \cdots & a_{2\ km+m} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n\ km+1} & a_{n\ km+2} & \cdots & a_{n\ km+m} \end{pmatrix}, \tag{16}$$

where $a_{ij} \in \mathbb{C},\ i = 1,\cdots,n,\ j \in \mathbb{Z}$.

Any element $A(z) \in \mathcal{S}^+(n,m)$ can be written as

$$A(z) = \sum_{k=0}^{\infty} A_k z^k, \tag{17}$$

We consider the infinite dimensional vector space $V = \mathbb{C}((z))^n$. We take

$$\cdots, z^2\boldsymbol{e}_n, z\boldsymbol{e}_1, \cdots, z\boldsymbol{e}_n, \boldsymbol{e}_1, \boldsymbol{e}_2, \cdots, \boldsymbol{e}_n, z^{-1}\boldsymbol{e}_1, \cdots \tag{18}$$

as the bases of V ($\boldsymbol{e}_{-kn+i} = z^k\boldsymbol{e}_i,\ i = 1,\cdots,n,\ k \in \mathbb{Z}$), where $\boldsymbol{e}_i$ $(i = 1,\cdots,n)$ are the natural bases of n dimensional vector space.

Since $A(z) \in \mathcal{S}(n,m)$ is considered as a linear mapping in V, using the bases (18) it has a $\mathbb{Z}\times\mathbb{Z}$ matrix representation $\tilde{A}$

$$\tilde{A} = (a_{ij})_{i,j\in\mathbb{Z}};\ a_{i+n,j+m} = a_{ij}. \tag{19}$$

More explicitly, $\tilde{A}$ can be written as

$$\tilde{A} = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \\ \cdots & A_0 & A_1 & A_2 & \cdots \\ \cdots & A_{-1} & A_0 & A_1 & \cdots \\ \cdots & A_{-2} & A_{-1} & A_0 & \cdots \\ & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{20}$$

For $A(z) \in \mathcal{S}^+(n,n)$, a derivative operator $A(z) - z\frac{d}{dz}$ is considered as a linear mapping in V. Therefore, using the bases (18) it also has a $\mathbb{Z}\times\mathbb{Z}$ matrix representation $\mathcal{L}$

$$\mathcal{L} = (a_{ij})_{i,j\in\mathbb{Z}};\ a_{i+n,i+n} = a_{ii}+1,\ a_{i+n,j+n} = a_{ij}\ (i \neq j).$$

As in (20), $\mathcal{L}$can be written as

$$\mathcal{L} = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \\ \cdots & A_0 - I_n & A_1 & A_2 & \cdots \\ \cdots & 0 & A_0 & A_1 & \cdots \\ \cdots & 0 & 0 & A_0 + I_n & \cdots \\ & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{21}$$

where I_n is the $n\times n$ identity matrix.

Suppose $P(z) \in \mathcal{S}^+(n,n)$ satisfies $\mathrm{Det}P(z) \neq 0, z \neq 0$. Let $A(z) \in \mathcal{S}^+(n,n), b(z) \in \mathcal{S}^+(n,1)$ and $(\hat{A}(z), \hat{b}(z)) = \mathcal{A}(A(z), b(z))$. We denote the infinite dimensional matrix representation in V using the bases (18) of $A(z) - z\frac{d}{dz}, \hat{A}(z) - z\frac{d}{dz}, b(z), \hat{b}(z)$ and P by $\mathcal{L}, \hat{\mathcal{L}}, \tilde{b}, \tilde{\hat{b}}$ and $\tilde{P}$.

Then, the equations (14) and (13) are equivalent to

$$\mathcal{L} = \tilde{P}\hat{\mathcal{L}}\tilde{P}^{-1}, \tag{22}$$

$$\tilde{b} = \tilde{P}\tilde{\hat{b}}. \tag{23}$$

C. stabilization of the cascade system

Theorem 1: For $A(z) \in \mathcal{S}^+(n,n), b(z) \in \mathcal{S}^+(n,1)$, there exists $P(z) \in \mathcal{S}^+(n,n)$ such that the couple $(\hat{A}(z), \hat{b}(z))$ satisfies

$$\operatorname{rank}\begin{pmatrix}\hat{b}(0) & \hat{A}(0)\hat{b}(0) & \cdots & \hat{A}(0)^{n-1}\hat{b}(0)\end{pmatrix} = n, \quad (24)$$

where $(\hat{A}(z), \hat{b}(z)) = \mathcal{A}(A(z), b(z))$, if the vectors in V, $\{\tilde{b}, \mathcal{L}\tilde{b}, \cdots, \mathcal{L}^{n-1}\tilde{b}\}$ are independent, where $\tilde{b}, \mathcal{L}$ are the $\mathbb{Z}\times 1, \mathbb{Z}\times\mathbb{Z}$ matrix representations of $b(z), A(z) - z\frac{d}{dz}$ using the bases (18) respectively.

Proof: We define

$$\hat{\boldsymbol{e}}_1 = \tilde{b},\ \hat{\boldsymbol{e}}_2 = \mathcal{L}\tilde{b}, \cdots, \hat{\boldsymbol{e}}_n = \mathcal{L}^{n-1}\tilde{b}. \quad (25)$$

Using $\hat{e}_1, \cdots, \hat{e}_n$, we choose new bases of V as

$$\cdots, z^2\hat{\boldsymbol{e}}_n, z\hat{\boldsymbol{e}}_1, \cdots, z\hat{\boldsymbol{e}}_n, \hat{\boldsymbol{e}}_1, \hat{\boldsymbol{e}}_2, \cdots, \hat{\boldsymbol{e}}_n, z^{-1}\hat{\boldsymbol{e}}_1, \cdots \quad (26)$$

where $\hat{\boldsymbol{e}}_{-kn+i} = z^k\hat{\boldsymbol{e}}_i,\ i = 1, \cdots, n,\ k \in \mathbb{Z}$. Then, we have

$$\mathcal{L}\hat{\boldsymbol{e}}_1 = \hat{\boldsymbol{e}}_2,\ \cdots,\ \mathcal{L}\hat{\boldsymbol{e}}_{n-1} = \hat{\boldsymbol{e}}_n,\ \mathcal{L}\hat{\boldsymbol{e}}_n = \sum_{k=n}^{-\infty} c_k\hat{\boldsymbol{e}}_k, \quad (27)$$

where $c_k \in \mathbb{C}, k = n, n-1, \cdots$. The last equation follows from the fact the derivative operator $A(z) - z\frac{d}{dz}$ does't decrease the degree of the polynomial of the operand.

Let $\hat{\mathcal{L}}, \tilde{\tilde{b}}$ be the representations of $\mathcal{L}, \tilde{b}$ using the new bases (26) respectively. Then,$\hat{\mathcal{L}}$ can be written as

$$\hat{\mathcal{L}} = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \\ \cdots & \hat{A}_0 - I_n & \hat{A}_1 & \hat{A}_2 & \cdots \\ \cdots & 0 & \hat{A}_0 & \hat{A}_1 & \cdots \\ \cdots & 0 & 0 & \hat{A}_0 + I_n & \cdots \\ & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad (28)$$

where $\hat{A}_k \in \mathbb{C}^{n\times n}, k = 0, 1, 2, \cdots$ and

$$\hat{A}_0 = \begin{pmatrix} 0 & 0 & \cdots & 0 & c_1 \\ 1 & 0 & \cdots & 0 & c_2 \\ 0 & 1 & \cdots & 0 & c_3 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & c_n \end{pmatrix}. \quad (29)$$

On the other hand,$\tilde{\tilde{b}}(z)$ can be written as

$$\tilde{\tilde{b}} = \begin{pmatrix}\cdots & \hat{b}_1^T & \hat{b}_0^T & 0 & \cdots\end{pmatrix}^T, \quad (30)$$

where $\hat{b}_k \in \mathbb{C}^{n\times 1}, k = 0, 1, 2, \cdots$ and

$$\hat{b}_0 = \begin{pmatrix}1 & 0 & \cdots & 0\end{pmatrix}^T. \quad (31)$$

We define $\tilde{P}$ as

$$\tilde{P} = [\cdots, \hat{\boldsymbol{e}}_0, \hat{\boldsymbol{e}}_1, \cdots, \hat{\boldsymbol{e}}_n, \hat{\boldsymbol{e}}_{n+1}, \cdots]. \quad (32)$$

Let $\hat{A}, \hat{b}$ and P be the elements of $\mathcal{S}^+(n,n), \mathcal{S}^+(n,1)$ and $\mathcal{S}^+(n,n)$ such that the infinite dimensional matrix representations in V using the bases (26) of $\hat{A}(z) - z\frac{d}{dz}, \hat{b}$ and P are $\hat{\mathcal{L}}, \tilde{\tilde{b}}$ and $\tilde{P}$ respectively. Then, since $\mathcal{L}, \hat{\mathcal{L}}, \tilde{b}, \tilde{\tilde{b}}$ and $\tilde{P}$ satisfies (22) and (23), $A(z), \hat{A}(z), b(z), \hat{b}(z)$ and P satisfies (14) and (13), Moreover, it is easily verified that $\hat{A}(0) = \hat{A}_0$ and $\hat{b}(0) = \hat{b}_0$ satisfy (24). ■

We consider here the system

$$\frac{dx}{dt} = A''(z)x + b''(z)u, \quad (33)$$

$$\frac{dz}{dt} = \nu z, \quad (34)$$

where $u \in \mathbb{R}$ is the control input, $A''(z)$ is an $n \times n$ real matrix valued function of z, $b''(z)$ is an $n \times 1$ real vector valued function of z, and $\nu \in \mathbb{R}, \nu < 0$. From (33),(34), we have (10), where

$$A(z) = \nu^{-1}A''(z),\ b(z) = \nu^{-1}b''(z). \quad (35)$$

Theorem 2: Suppose $A(z), b(z)$ defined by (35) can be decomposed as $A(z) = A_A(z) + A_N(z), b(z) = b_A(z) + b_N(z)$ where $A_A(z) \in \mathcal{S}^+(n,n), b_A(z) \in \mathcal{S}^+(n,1)$ and $A_N(0) = 0, b_N(0) = 0$. Let $\mathcal{L}_A, \tilde{b}_A$ denote $\mathbb{Z}\times\mathbb{Z}$ and $\mathbb{Z}\times 1$ matrix representations in V using the bases (18) of $A_A(z) - z\frac{d}{dz}, b_A(z)$ respectively. Then there exists $F''(z) \in \mathcal{S}(1,n)$, such that (33) with $u = F''(z)x$ becomes exponentially stable, if the vectors in V, $\{\tilde{b}_A, \mathcal{L}_A\tilde{b}_A, \cdots, \mathcal{L}_A^{n-1}\tilde{b}_A\}$ are independent and

$$\lim_{z\to 0} P^{-1}(z)A_N(z)P(z) = 0,\ \lim_{z\to 0} P^{-1}(z)b_N(z) = 0 \quad (36)$$

where $P \in \mathcal{S}^+(n,n)$ whose $\mathbb{Z}\times\mathbb{Z}$ matrix representation using the bases (18) is

$$\tilde{P} = [\cdots, \hat{\boldsymbol{e}}_0, \hat{\boldsymbol{e}}_1, \cdots, \hat{\boldsymbol{e}}_n, \hat{\boldsymbol{e}}_{n+1}, \cdots]. \quad (37)$$

where

$$\hat{\mathbf{e}}_1 = \tilde{b}_A,\ \hat{\mathbf{e}}_2 = \mathcal{L}_A\tilde{b}_A, \cdots, \hat{\mathbf{e}}_n = \mathcal{L}_A^{n-1}\tilde{b}_A. \quad (38)$$

Proof: By the variable transformation $x = P(z)y$, (10) is transformed into (11). In this case $\hat{A} = \hat{A}_A + \hat{A}_N, \hat{b} = \hat{b}_A + \hat{b}_N$, where $(\hat{A}_A, \hat{b}_A) = \mathcal{A}(A_A, b_A)$ and

$$\hat{A}_N(z) = P^{-1}(z)A_N(z)P(z), \quad (39)$$

$$\hat{b}_N(z) = P^{-1}(z)b_N(z). \quad (40)$$

Since $\{\tilde{b}_A, \mathcal{L}_A\tilde{b}_A, \cdots, \mathcal{L}_A^{n-1}\tilde{b}_A\}$ are independent in V by the assumption, it follows from the theorem 1 that $(\hat{A}_A(z), \hat{b}_A(z))$ satisfies (24). Therefore, it follows from the well known fact in the linear control theory there exists F_0 such that all eigenvalues of $\hat{A}_A(0) + \hat{b}_A(0)F_0$ are in the right half plane of the complex plane. Consider the closed loop system of (11) with $u = F(z)y$, where $F(z) \in \mathcal{S}^+(1,n)$ such that $F(0) = F_0$, namely

$$z\frac{dy}{dz} = A_{cl}(z)y,\ A_{cl}(z) = \hat{A}(z) + \hat{b}(z)F(z). \quad (41)$$

From the assumption (36),

$$\lim_{z\to 0} A_{cl}(z) = \hat{A}_A(0) + \hat{b}_A(0)F(0). \quad (42)$$

It follows from Remark 1 that the closed loop system of (11) with $u = F(z)y$ is polynomially stable. This means that the closed loop system of (10) with $u = F(z)P^{-1}(z)x$ is also polynomially stable. Then, it follows from the definition 1 that the closed loop system of (33) with $u = F''(z)x$, where $F''(z) = F(z)P^{-1}(z)$, is exponentially stable. ■

Remark 2: The input u tends to 0 exponentially, since y tends to 0 exponentially.

Remark 3: The method to design the state feedback law in Theorem 2 is equivalent to one in [19] except that the system is reduced to a linear system with constant coefficient matrices by variable transformation in [19] while the coefficient matrices of the system still depends on variable z after variable transformation in Theorem 2.

D. example

Consider the system

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \\ z \end{pmatrix} = \begin{pmatrix} 1 & x_1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}, \tag{43}$$

where u, v are control inputs. It can be easily shown that (43) is maximally nonholonomic (controllable).

By choosing $v = -z$, (43) is transformed into the form of (33)(34)

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = -\begin{pmatrix} z & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u, \tag{44}$$

$$\frac{dz}{dt} = -z. \tag{45}$$

Dividing (44) by (45), we have

$$z\frac{dx}{dz} = A(z)x + b(z)u, \tag{46}$$

where

$$A(z) = \begin{pmatrix} z & 0 \\ 0 & 0 \end{pmatrix}, \; b(z) = -\begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{47}$$

Since $\tilde{b} = -\boldsymbol{e}_1 - \boldsymbol{e}_2$ and $\mathcal{L}\tilde{b} = -\boldsymbol{e}_{-1}$ are independent, we can construct new bases $\hat{\boldsymbol{e}}_1 = \tilde{b}$ and $\hat{\boldsymbol{e}}_2 = \mathcal{L}\tilde{b}$. Then, variable transformation matrix $P(z)$ is obtained as

$$P(z) = -\begin{pmatrix} 1 & z \\ 1 & 0 \end{pmatrix}. \tag{48}$$

Since

$$\mathcal{L}^2\tilde{b} = \boldsymbol{e}_{-1} - \boldsymbol{e}_{-3} = c_1\hat{\boldsymbol{e}}_1 + c_2\hat{\boldsymbol{e}}_2 - \boldsymbol{e}_{-3}, \tag{49}$$

where $c_1 = 0, c_2 = -1$, we set

$$W = \begin{pmatrix} -c_2 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}. \tag{50}$$

By the variable transformation $x = P(z)Wy$ we have

$$z\frac{dy}{dz} = \hat{A}(z)y + \hat{b}(z)u, \tag{51}$$

where $\hat{A}(z) = \hat{A}_0 + \hat{A}_1 z$, where

$$\hat{A}_0 = \begin{pmatrix} 0 & 1 \\ c_1 & c_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}, \; \hat{A}_1 = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}, \tag{52}$$

and $\hat{b} = \begin{pmatrix} 0 & 1 \end{pmatrix}^T$.

We set $F_0 = [-c_1 - a_1, \; -c_2 - a_2] = [-a_1, \; 1 - a_2]$ where a_1, a_2 is chosen so that the roots of $s^2 + a_2 s + a_1 = 0$ are in the right half plane of the complex plane.

Finally, the control input that stabilizes (44) is given as

$$\begin{aligned} u &= F(z)x = F_0 W^{-1} P(z)^{-1} x, && (53)\\ &= \frac{1}{z}\{(a_1 - a_2 + 1)x_1 \\ &\quad +((a_2 - a_1 - 1) + (a_2 - 1)z)x_2\}. && (54) \end{aligned}$$

Remark 4: It is easily verified that the variable transformation $x = P(z)y$ where $P(z)$ is in the form

$$P(z) = \begin{pmatrix} z^k & 0 \\ 0 & z^l \end{pmatrix}, \; k, l \in \mathbb{Z} \tag{55}$$

fails to satisfy (24).

III. Cascade system of a nonlinear system and a multivariable linear system

A. stabilization of the cascade system

We consider here the system

$$\frac{dx}{dt} = A'''(\xi)x + b'''(\xi)u, \tag{56}$$

$$\frac{d\xi}{dt} = C\xi, \tag{57}$$

where $\xi = [\xi^1, \cdots, \xi^p]^T \in \mathbb{R}^{p\times 1}$, $C \in \mathbb{R}^{p\times p}$, and $A'''(\xi), b'''(\xi)$ are $n \times n$ and $n \times 1$ matrix valued affine functions of ξ respectively.

Assumption 1: We assume that all eigenvalues of C are in the left half plane in the complex plane. We also assume that $\nu \in \mathbb{R}$ is the eigenvalue of C which has the smallest absolute value of the real part among the all eigenvalues of C.

Let $v_1 \in \mathbb{C}^{p\times 1}$ be the eigenvector corresponding to ν. We choose $v_2, \cdots, v_p$ so that $S = [v_1, \cdots, v_n]$ is nonsingular. We set $\xi = S\eta$,where $\eta = [\eta^1, \cdots, \eta^p]^T \in \mathbb{R}^{p\times 1}$. We define z by $z = \eta^1$.

We set $A_0 = A'''(0)$ and $b_0 = b'''(0)$. Then, we define $A_L(\xi) = A'''(\xi) - A_0$ and $b_L(\xi) = b'''(\xi) - b_0$.

Lemma 2: Under Assumption 1, along the trajectory of ξ determined by (57),

$$\lim_{t\to\infty}\frac{A_L(\xi)}{z} = A_L(v_1), \; \lim_{t\to\infty}\frac{b_L(\xi)}{z} = b_L(v_1) \tag{58}$$

holds.

Proof: From the definition of η, we have

$$\xi^j = v_1^j\eta^1 + v_2^j\eta^2 + \cdots + v_p^j\eta^p, \; j = 1, \cdots, p, \tag{59}$$

where $v_k = [v_k^1, \cdots, v_k^p]^T$ $(k = 1, \cdots, p)$. Since ν has the smallest absolute value of the real part among the all eigenvalues of C,

$$\lim_{t\to\infty} \frac{\xi^j}{\eta^1} = v_1^j. \tag{60}$$

∎

By the definition of z, (34) holds. Therefore, we have $z(t) = Ce^{\nu t}$, where $C \in \mathbb{R}$ depends on the initial value of z. From (56) and (34), we have (10), where

$$\begin{aligned} A(z) &= \nu^{-1} A'''(\nu^{-1} \ln(z/C)), && (61)\\ b(z) &= \nu^{-1} b'''(\nu^{-1} \ln(z/C)). && (62) \end{aligned}$$

Lemma 2 implies that along the trajectory of ξ determined by (57) we can assume that $A(z) = A_A(z) + A_N(z), b(z) = b_A(z) + b_N(z)$ where $A_A(z) \in \mathcal{S}^+(n,n), b_A(z) \in \mathcal{S}^+(n,1)$ are defined by

$$\begin{aligned} A_A(z) &= A_0 + A_L(v_1)z, && (63)\\ b_A(z) &= b_0 + b_L(v_1)z, && (64) \end{aligned}$$

and the matrices $A_N(z)$ and $b_N(z)$ are defined by

$$\begin{aligned} A_N(z) &= A(z) - A_A(z), && (65)\\ b_N(z) &= A(z) - b_A(z). && (66) \end{aligned}$$

The matrices $A_N(z), b_N(z)$ satisfies $A_N(0) = b_N(0) = 0$.

Theorem 3: Suppose $A_A(z), b_A(z), A_N(z)$ and $b_N(z)$ are defined by (63)(64)(65)(66). Let $\mathcal{L}_A, \tilde{b}_A$ denote $\mathbb{Z} \times \mathbb{Z}$ and $\mathbb{Z} \times 1$ matrix representations in V using the bases (18) of $A_A(z) - z\frac{d}{dz}, b_A(z)$ respectively. Then there exists $F'''(z) \in \mathcal{S}(1,n)$, such that (56) with $u = F'''(z)x$ becomes exponentially stable, if the vectors in V, $\{\tilde{b}_A, \mathcal{L}_A\tilde{b}_A, \cdots, \mathcal{L}_A^{n-1}\tilde{b}_A\}$ are independent and

$$\lim_{z\to 0} P^{-1}(z) A_N(z) P(z) = 0,\ \lim_{z\to 0} P^{-1}(z) b_N(z) = 0 \tag{67}$$

where $P \in \mathcal{S}^+(n,n)$ whose $\mathbb{Z} \times \mathbb{Z}$ matrix representation using the bases (18) is

$$\tilde{P} = [\cdots, \hat{\boldsymbol{e}}_0, \hat{\boldsymbol{e}}_1, \cdots, \hat{\boldsymbol{e}}_n, \hat{\boldsymbol{e}}_{n+1}, \cdots]. \tag{68}$$

where

$$\hat{\boldsymbol{e}}_1 = \tilde{b}_A,\ \hat{\boldsymbol{e}}_2 = \mathcal{L}_A\tilde{b}_A, \cdots, \hat{\boldsymbol{e}}_n = \mathcal{L}_A^{n-1}\tilde{b}_A. \tag{69}$$

Proof: This theorem can be proven similarly to Theorem 2 . ∎

B. example

We deal with the 3-dimensional second-order chained system with two control inputs described by the following equations:

$$\ddot{\zeta}_1 = w_1,\ \ddot{\zeta}_2 = w_2,\ \ddot{\zeta}_3 = \zeta_2 w_1. \tag{70}$$

To transform (70) into a cascade system of the form of (1)(2), we define new variables x, ξ and inputs u, v as

$$\begin{aligned} x = \quad & (x^1, x^2, x^3, x^4) = && (\zeta_2, \dot{\zeta}_2, \zeta_3, \dot{\zeta}_3), && (71)\\ \xi = \quad & (\xi^1, \xi^2) = && (\zeta_1, \dot{\zeta}_1), && (72) \end{aligned}$$

$$u = w_2,\ v = w_1. \tag{73}$$

Then, we have

$$\begin{aligned} \frac{dx}{dt} &= A'''(\xi)x + B'''(\xi)u, && (74)\\ &= \begin{pmatrix} 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ v(\xi) & 0 & 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0\\1\\0\\0 \end{pmatrix} u. && (75) \end{aligned}$$

and

$$\frac{d\xi}{dt} = C\xi + Dv = \begin{pmatrix} 0 & 1\\ 0 & 0 \end{pmatrix}\xi + \begin{pmatrix} 0\\1 \end{pmatrix} v \tag{76}$$

Since (C, D) is controllable, we can choose the eigenvalues of the closed loop system arbitrarily by state feedback $v(\xi) = G\xi$. Suppose G is chosen so that all eigenvalues of $C + DG$ are in the left half plane in the complex plane and ν is the real eigenvalue which has the smallest absolute value of the real part among all eigenvalues. We denote the eigenvector corresponding ν by v_1.

Then, we have

$$\frac{dz}{dt} = \nu z. \tag{77}$$

Dividing (74) by (77), we have

$$z\frac{dx}{dz} = A(z)x + b(z)u, \tag{78}$$

where

$$A(z) = \nu^{-1} A'''(\xi(z)),\ b(z) = \nu^{-1} b'''(\xi(z)). \tag{79}$$

We have

$$\begin{aligned} A_A(z) &= A_0 + A_L(v_1)z = \nu^{-1} \begin{pmatrix} 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ v(v_1)z & 0 & 0 & 0 \end{pmatrix},\\ b_A(z) &= b_0 + b_L(v_1)z = \nu^{-1} \begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix}^T,\\ A_N(z) &= A(z) - A_A(z) = \nu^{-1} \begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ v(\xi) - v(v_1)z & 0 & 0 & 0 \end{pmatrix},\\ b_N(z) &= 0. \end{aligned}$$

Since

$$\begin{aligned} \tilde{b}_A &= \nu^{-1}\boldsymbol{e}_2, && (80)\\ \mathcal{L}_A\tilde{b}_A &= \nu^{-2}\boldsymbol{e}_1, && (81)\\ \mathcal{L}_A^2\tilde{b}_A &= \nu^{-3} v(v_1)\boldsymbol{e}_0, && (82)\\ \mathcal{L}_A^3\tilde{b}_A &= \nu^{-4} v(v_1)\boldsymbol{e}_{-1} - \nu^{-3} v(v_1)\boldsymbol{e}_0, && (83) \end{aligned}$$

are independent, we can construct new bases

$$\hat{\boldsymbol{e}}_1 = \tilde{b}_A,\ \hat{\boldsymbol{e}}_2 = \mathcal{L}_A\tilde{b}_A,\ \hat{\boldsymbol{e}}_3 = \mathcal{L}_A^2\tilde{b}_A,\ \hat{\boldsymbol{e}}_2 = \mathcal{L}_A^3\tilde{b}_A. \tag{84}$$

Then, variable transformation matrix $P(z)$ is obtained as

$$P(z) = \begin{pmatrix} 0 & \nu^{-2} & 0 & 0 \\ \nu^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & \nu^{-4}v(v_1)z \\ 0 & 0 & \nu^{-3}v(v_1)z & -\nu^{-3}v(v_1)z \end{pmatrix}. \quad (85)$$

It follow from (60) that

$$P(z)^{-1}A_N(z)P(z) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{v(\xi)}{v(v_1)z} - 1 & 0 & 0 & 0 \end{pmatrix} \to 0,\ z \to 0. \quad (86)$$

Since

$$\mathcal{L}_A^4 \tilde{b}_A = c_1\hat{\boldsymbol{e}}_1 + c_2\hat{\boldsymbol{e}}_2 + c_3\hat{\boldsymbol{e}}_3 + c_4\hat{\boldsymbol{e}}_4, \quad (87)$$

where $c_1 = 0, c_2 = 0, c_3 = -1, c_4 = -2$, we set

$$W = \begin{pmatrix} -c_2 & -c_3 & -c_4 & 1 \\ -c_3 & -c_4 & 1 & 0 \\ -c_4 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 2 & 1 \\ 1 & 2 & 1 & 0 \\ 2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \quad (88)$$

By the variable transformation $x = P(z)Wy$ we have

$$\begin{aligned} \hat{A}_A &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ c_1 & c_2 & c_3 & c_4 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & -2 \end{pmatrix}, \\ \hat{b}_A &= \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix}^T. \end{aligned}$$

We set

$$\begin{aligned} F_0 &= [-c_1 - a_1,\ -c_2 - a_2,\ -c_3 - a_3,\ -c_4 - a_4], \\ &= [-a_1,\ -a_2,\ 1 - a_3,\ 2 - a_4], \end{aligned}$$

where a_1, a_2, a_3, a_4 is chosen so that the roots of

$$s^4 + a_4 s^3 + a_3 s^2 + a_2 s + a_1 = 0 \quad (89)$$

are in the right half plane of the complex plane.

Finally, the control input that stabilize (74) is given as

$$u = F(z)x = F_0 W^{-1} P(z)^{-1} x \quad (90)$$

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