Proceedings of the
44th IEEE Conference on Decision and Control, and
the European Control Conference 2005
Seville, Spain, December 12-15, 2005

General Solution to Standard H^∞ Control Problems for a Class of Infinite-dimensional Systems

Kenji Kashima† and Yutaka Yamamoto‡

***Abstract*— This paper considers H^∞-control of infinite-dimensional systems whose transfer functions are expressible as the cascade connection of a rational transfer matrix and a scalar (possibly irrational) inner function. This class of systems is very suitable for describing many control problems in practice, when weighting functions are rational and plants have at most finitely many unstable modes. We show that this problem can be reduced to two (matrix-valued) Riccati equations and additional rank conditions. Furthermore, the obtained controller structure is completely characterized by the inner function and controllers for the finite-dimensional part. A numerical example is given to illustrate the result.**

I. INTRODUCTION

Since mid-1980's the H^∞-control of infinite-dimensional systems, such as delay-differential systems, have been studied extensively. For example, solutions via operator equations have been given in [1], [15]. From the computational point of view, the skew-Toeplitz approach and the so-called AAK (Adamjan-Arov-Krein) theory are more attractive in that they yield finitary rank conditions for optimality [5], [7], [10], [14], [16], [18]. The crux of the theory lies in the assumptions that

- the weighting functions are rational, and
- the plant have only finitely many unstable poles[1].

In particular, for the one-block problem of finding

$$\inf_{Q\in H^\infty} \|W + mQ\|_\infty \tag{1}$$

for an inner function m and a stable rational function W, a beautiful formula, the so-called Zhou-Khargonekar formula, has been established; see Section IV, [16], [18] and references therein. This Hamiltonian-based formula gives a finite rank condition for the one-block problem, in spite of its infinite-dimensionality.

It is well-known that a large class of H^∞ control problems for systems with finitely many unstable poles and rational weights reduce to the one-block problem above [5]. However, required performances of the obtained overall system are not originally given in the form of the one-block problem. Hence we are required to reduce the original problem to the standard one to apply the formula. This reduction step sometimes includes complicated manipulations of weighting functions. Moreover, this computation may make unclear the structure of resulting controllers.

In view of this, we attempt to capture the H^∞-control of infinite-dimensional systems in a more general framework. If we employ an infinite-dimensional generalized plant in which the finiteness assumptions noted above are not imposed, similar simple results will not be obtained, i.e., it is inevitable that two operator-valued Riccati equations appear [15]. In this paper, to make use of this finite-dimensionality explicitly, we consider a system consisting of a rational transfer matrix and a scalar, but not necessarily rational, inner function. A large class of control problems of infinite-dimensional systems can be described by such systems. In Section II, we formulate the H^∞ control problem for a class of infinite-dimensional systems in this framework. In Section III, we show that this problem can be separated into a finite-dimensional H^∞ control problem and a specific one-block problem. Finite rank conditions for this one-block problem are derived in Section IV.

NOTATION AND CONVENTION

As usual, H^p and H^p_- denote the Hardy spaces on the open right- and left-half complex plane, respectively. The orthogonal projections from $L^2(j\mathbb{R}) := H^2 \oplus H^2_-$ to H^2 (H^2_-) are denoted by $\pi^+[\cdot]$ ($\pi^-[\cdot]$). Let $q^\sim(s) := q(-s)$. For an inner function m, $H(m)$ is the orthogonal complement of mH^2 on H^2. It is known ([8]) that

$$H(m) = \{x \in H^2 : m^\sim x \in H^2_-\}. \tag{2}$$

The maximal singular value of a matrix is denoted by $\|\cdot\|$. For a linear mapping T, $\mathrm{Im}\, T$ and T^* represent the image subspace and the adjoint operator, respectively. For two subsets X, Y of a set, $X/Y := \{x \in X : x \notin Y\}$.

Definition 1: Let $m(s)$ be an inner function. Then the set of matrices $A \in \mathbb{R}^{n\times n}$ such that $m^\sim(s)$ is analytic in a neighborhood of every eigenvalue of A is denoted by $\mathcal{M}_m^{n\times n}$, or $\mathcal{M}_m$ when the size is clear from the context.

Preliminary results on matrix functions are in Appendix.

Definition 2: Let G_1 and G_2 be transfer matrices whose sizes are $(m_1 + m_2) \times (p_1 + p_2)$ and $p_2 \times m_2$, respectively. We say that G_2 internally stabilizes G_1, if the nine transfer matrices[2] from w, u_1 and u_2 to z, v_1 and v_2 in Figure 1 are all in H^∞. The transfer function from w to z is denoted by $\mathcal{F}_l(G_1, G_2)$.

II. PROBLEM FORMULATION

The problem investigated in this paper is as follows:

†Corresponding author. Graduate School of Information Science and Engineering, Tokyo Institute of Technology, Tokyo, 152-8552, JAPAN. E-mail address: kashima@mei.titech.ac.jp

‡Graduate School of Informatics, Kyoto University, Kyoto, 606-8501, JAPAN. E-mail address: yy@i.kyoto-u.ac.jp

[1] In [7], the plants with infinitely many unstable poles and finitely many unstable zeros are considered.

[2] In particular, when $m_1 = p_1 = 0$, we say that G_2 internally stabilizes G_1 if four transfer matrices from u_1 and u_2 to v_1 and v_2 belong to H^∞.

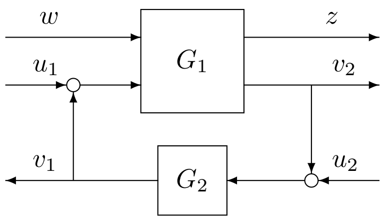

Fig. 1. Block diagram for the definition of the internal stability

Problem 1: Given a rational transfer matrix Σ

$$\Sigma(s) = \left[\begin{array}{c:c} \Sigma_{11} & \Sigma_{12} \\ \hdashline \Sigma_{21} & \Sigma_{22} \end{array}\right] = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right], \quad (3)$$

an inner function $m(s)$ and a prespecified performance level $\gamma > 0$, determine whether there exists a controller C which internally stabilizes $\Sigma_{\rm inf}$ given by

$$\Sigma_{\rm inf}(s) := \left[\begin{array}{c:c} \Sigma_{11} & \Sigma_{12} \\ \hdashline \Sigma_{21} & \Sigma_{22} \end{array}\right] \left[\begin{array}{c:c} I & \\ \hdashline & mI \end{array}\right] \quad (4)$$

and guarantees

$$\left\| \mathcal{F}_l\left(\Sigma_{\rm inf}, C\right) \right\|_\infty < \gamma. \quad (5)$$

If such a controller exists, find all admissible controllers.

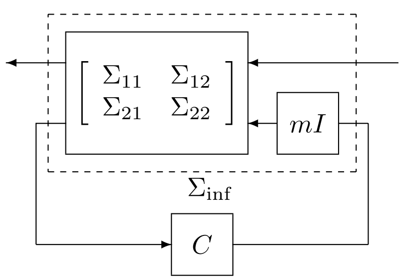

Fig. 2. Block diagram for Problem 1

This rational Σ encompasses into one weighting functions and unstable modes of the plant to be controlled, which are assumed to be finite-dimensional in the skew-Toeplitz approach. In this framework, we can formulate various H^∞ problems for infinite-dimensional systems with finitely many unstable poles and rational weights. To see this, let us consider the mixed sensitivity optimization problem for plants which can be factorized as

$$P(s) = P_r(s)m(s)P_o(s) \quad (6)$$

where m is inner, P_o is outer and P_r is rational. Note that any plant of this form has at most finitely many unstable poles. Then, for stable rational weights W_s and W_t, the corresponding weighted mixed sensitivity is

$$\begin{bmatrix} W_s(1-PC)^{-1} \\ W_t(1-PC)^{-1}PC \end{bmatrix} = \mathcal{F}_l\left(\Sigma_{\rm inf}, C\right), \quad (7)$$

where $\Sigma_{\rm inf}$ is given by (4) with

$$\Sigma(s) := \left[\begin{array}{c:c} W_s & W_sP_r \\ 0 & W_tP_r \\ \hdashline 1 & P_r \end{array}\right].$$

Here we ignored the outer part P_o which can be absorbed into controllers. It should be stressed that other reduction preprocess as in conventional results is not needed.

This formulation is the same as in [11], [12], [13], [17] when $m(s) = e^{-hs}$ for $h > 0$. In this case, Problem 1 represents the H^∞-control of finite-dimensional systems with delayed measurements and/or control inputs, and has been studied extensively. In [11], [13], [17], the delay is regarded as constraints on causality of the controller, and then a Hamiltonian-based solution is obtained for this problem. The result in this paper can be viewed as a generalization of this result. While the discussion here is parallel to that in [11], [13], [17], the generalization is not trivial, since e^{-hs} is a special inner function which is an entire function with no unstable zeros and hence free from some issues such as unstable pole-zero cancellations.

Hereafter we impose the following standard assumption on system matrices in Σ.

Assumption 1 ([3]):

1) (C_2, A, B_2) is stabilizable and detectable.
2) $\begin{bmatrix} A - j\omega I & B_2 \\ C_1 & D_{12} \end{bmatrix}$ and $\begin{bmatrix} A - j\omega I & B_1 \\ C_2 & D_{21} \end{bmatrix}$ are of row- and column-full rank for any $\omega \in \mathbb{R}$, respectively.
3) $D_{12}^T \begin{bmatrix} C_1 & D_{12} \end{bmatrix} = \begin{bmatrix} 0 & I \end{bmatrix}$ and $\begin{bmatrix} B_1 \\ D_{21} \end{bmatrix} D_{21}^T = \begin{bmatrix} 0 \\ I \end{bmatrix}$.
4) $D_{11} = 0$ and $D_{22} = 0$.

These assumptions except for 1) are easily removed by standard techniques. Under this assumption, Problem 1 with $m(s) = 1$ is solvable as follows [3]:

Theorem 1: Given Σ in (3) satisfying Assumption 1. Define two Hamiltonian matrices given by

$$\begin{aligned} H &:= \begin{bmatrix} A & \gamma^{-2}B_1B_1^T - B_2B_2^T \\ -C_1^TC_1 & -A^T \end{bmatrix}, \\ J &:= \begin{bmatrix} A^T & \gamma^{-2}C_1^TC_1 - C_2^TC_2 \\ -B_1B_1^T & -A \end{bmatrix}. \end{aligned}$$

Then a controller $K(s)$ which internally stabilizes Σ and satisfies $\left\| \mathcal{F}_l\left(\Sigma, K\right) \right\|_\infty < \gamma$ exists if and only if the following three conditions hold:

1) $H \in \mathrm{dom}(\mathrm{Ric})$ and $X := \mathrm{Ric}(H) \geq 0$.
2) $J \in \mathrm{dom}(\mathrm{Ric})$ and $Y := \mathrm{Ric}(J) \geq 0$.
3) $\rho(XY) < \gamma^2$.

Moreover, when these conditions hold, all such controllers are given by

$$K(s) = \mathcal{F}_l\left(M, Q\right), \quad (8)$$

where $Q \in H^\infty$ satisfies $\|Q\|_\infty < \gamma$ and

$$M = \left[\begin{array}{c|cc} \hat{A} & -ZL & ZB_2 \\ \hline F & 0 & I \\ -C_2 & I & 0 \end{array}\right], \quad (9)$$

with

$$\begin{aligned} \hat{A} &:= A + \gamma^{-2}B_1B_1^TX + B_2F + ZLC_2, \\ F &:= -B_2^TX, \\ L &:= -YC_2^T, \\ Z &:= (I - \gamma^{-2}YX)^{-1}. \end{aligned}$$

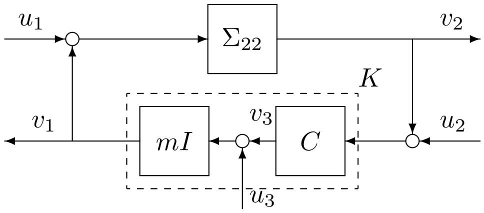

Fig. 3. Block diagram for the proof of Lemma 1

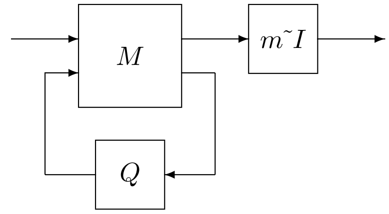

Fig. 4. Structure of controllers

In [3], the free parameter $Q \in H^\infty$ in (8) is rational to obtain finite-dimensional controllers. In this paper, we admit irrational Q, since we do not assume the finite-dimensionality of obtained controllers. Concerning stabilizability of Σ_{inf}, we impose another assumption.

Assumption 2: The matrix A belongs to $\mathcal{M}_m$.

This guarantees that there exist no unstable pole-zero cancellations between Σ and m. In fact, this assumption along with Assumption 1, 1) gives a sufficient condition for the existence of an internally stabilizing controller of Σ_{inf}.

III. Structure of Controllers

In this section, we show that Problem 1 can be separated into Problem 1 with $m(s) = 1$ and an additional one-block problem.

A. Internal stability

First, it is easily verified

$$\mathcal{F}_l(\Sigma_{\text{inf}}, C) = \mathcal{F}_l(\Sigma, mC). \tag{10}$$

Hence (5) is equivalent to $\|\mathcal{F}_l(\Sigma, K)\|_\infty < \gamma$ where $K = mC$. For the constraints on the internal stability, we can show the following:

Lemma 1: A controller C internally stabilizes Σ_{inf} if and only if $K := mC$ internally stabilizes Σ and

$$G_{32} := (I - mC\Sigma_{22})^{-1}C \in H^\infty. \tag{11}$$

Proof: Let $G_{ji} (i, j = 1, 2, 3)$ denote the transfer matrices from u_i to v_j in Figure 3. We first show the following three conditions are equivalent.

1) For any pair of $i, j (i, j = 1, 2, 3)$, $G_{ji} \in H^\infty$,
2) For any pair of $i, j (i, j = 2, 3)$, $G_{ji} \in H^\infty$,
3) For any pair of $i, j (i, j = 1, 2)$, $G_{ji}, G_{32} \in H^\infty$.

By straightforward computations, we can show that

- if $G_{3i} \in H^\infty$ then $G_{1i} \in H^\infty$,
- if $G_{i1} \in H^\infty$ then $G_{i3} \in H^\infty$,

for $i = 1, 2, 3$ and $G_{11} = G_{33}$. We show 2)$\Longrightarrow$1) only, since 3)$\Longrightarrow$ 1) follows similarly. It is sufficient to show that G_{21} and G_{31} belong to H^∞. Since G_{22} is stable, if $G_{21} = (I + G_{22})\Sigma_{22}$ has unstable poles, they are poles of Σ_{22}, i.e., eigenvalues of A. On the other hand, $G_{23} = mG_{21}$ is in H^∞. This means that all unstable poles of G_{21}, if one exists, must be canceled by the multiplication by mI. However, by the assumption $A \in \mathcal{M}_m$, no eigenvalue of A is a zero of m. Therefore G_{21} has no unstable poles and belongs to H^∞. Similarly $G_{31} \in H^\infty$ follows from $G_{31} = G_{32}\Sigma_{22} = \tilde{m}G_{33}$.

The equivalence of 2) and 3) claims that C internally stabilizes $m\Sigma_{22}$ if and only if $K = mC$ internally stabilizes Σ_{22} and $G_{32} \in H^\infty$. As is well-known, Σ is internally stabilized by K if and only if so is Σ_{22}. Similarly we can show that C internally stabilizes Σ_{inf} if and only if C internally stabilizes its (2,2)-block $m\Sigma_{22}$ [9]. ■

From this lemma and (10), we can conclude as follows:

- The three conditions in Theorem 1 are necessary for the existence of a controller required in Problem 1.
- The controller C is, if one exists, given by
$$C = \tilde{m}\mathcal{F}_l(M, Q) \tag{12}$$
for $Q \in H^\infty$ such that $\|Q\|_\infty < \gamma$; see Figure 4.
- In comparison to the case $m(s) = 1$, the optimal performance is deteriorated by m, because $G_{32} \in H^\infty$ is required.

Therefore we hereafter assume that the three conditions in Theorem 1 are all satisfied.

B. Constraints on admissible controllers

In this subsection, we show what constraints are imposed on C by (11). Intuitively, (11) is a condition which guarantees that C is causal and that there exist no unstable pole-zero cancellations between C and m. To discuss this in detail, we introduce the following definition.

Definition 3: Let m be an inner function and ϵ be a positive constant. Then

$$\mathcal{N}_m(\epsilon) := \{\lambda \in \mathbb{C}_+ : |m(\lambda)| < \epsilon\}. \tag{13}$$

Roughly speaking, $\mathcal{N}_m(\epsilon)$ is a neighborhood of zeros of m. Hence, the following lemma is a direct consequence of Definition 1 [9]:

Lemma 2: Let m be an inner function and $A \in \mathcal{M}_m$. For any given positive constant δ, there exists $\epsilon > 0$ such that in $\mathcal{N}_m(\epsilon)$, $(sI - A)^{-1}$ is analytic and

$$\epsilon\|(sI - A)^{-1}\| < \delta.$$

Under these notations, we obtain the following:

Lemma 3: Suppose that C is given by (12) for $Q \in H^\infty$ such that $\|Q\|_\infty < \gamma$. Then (11) holds if and only if there exists $\epsilon > 0$ such that C is analytic and bounded in $\mathcal{N}_m(\epsilon)$.

Proof: (Necessity) Solving (11) for C yields

$$C = G_{32}(I + m\Sigma_{22}G_{32})^{-1}. \tag{14}$$

From Lemma 2, we can take positive ϵ and δ such that Σ_{22} is analytic and

$$\epsilon\|G_{32}\|_\infty\|\Sigma_{22}\| < \delta < 1$$

in $\mathcal{N}_m(\epsilon)$. Therefore the desired result follows[3], according to equation (14) and $\|m\Sigma_{22}G_{32}\| < \delta < 1$ in $\mathcal{N}_m(\epsilon)$.

(Sufficiency) Since C is in the form of (12), $G_{11}, G_{12} \in H^\infty$ from Theorem 1. Hence $m\tilde{} G_{12}$ and $(I + G_{11})C$, both of which are equal to G_{32}, are analytic and bounded in $\mathbb{C}_+ \backslash \mathcal{N}_m(\epsilon)$ and in $\mathcal{N}_m(\epsilon)$, respectively. This means $G_{32} \in H^\infty$. ■

This lemma, as expected, claims that C must be analytic in a neighborhood of every zero of $m(s)$, i.e., C is causal and any unstable pole of C is not canceled by the plant.

C. Reduction to one-block problem

Lemma 3 will leads us further into a consideration of the class of admissible controllers, that is, constraints imposed on Q in order that the corresponding C satisfy the condition in Lemma 3. We now define two functions that play a crucial role in the following discussions.

Definition 4: Let m be an inner function and $(A, B, C, 0)$ with $A \in \mathcal{M}_m$ be a realization of W. Then we define

$$W^{(m)} := \left[\begin{array}{c|c} A & m\tilde{}(A)B \\ \hline C & 0 \end{array}\right] \tag{15}$$

and

$$\pi^m[W] := W - mW^{(m)}. \tag{16}$$

When $m = e^{-hs}$ for $h > 0$, $\pi^m[\cdot]$ is same as the h-*truncation* in [11], [13], [17]. The h-truncation is the operator which truncates the impulse response to its restriction on $[0, h]$. As another example, take

$$m(s) = \frac{s-2}{s+2}, \quad W(s) = \left[\begin{array}{c|c} 1 & 1 \\ \hline 1 & 0 \end{array}\right] = \frac{1}{s-1}.$$

Clearly the A-matrix of W is in $\mathcal{M}_m$. By Definition 4,

$$\pi^m[W] = \frac{1}{s-1} + \frac{s-2}{s+2} \cdot \frac{3}{s-1} = \frac{4}{s+2}.$$

In this case, $\pi^m[W]$ is stable and belongs to $H(m)$. More generally, the following lemma ([9]) is a consequence of Lemma 6 in Appendix.

Lemma 4: Under Definition 4, $\mathrm{Im}(\pi^m) \subset H(m) \cap H^\infty$.

Remark 1: For stable $W \in H^2$, we have $W^{(m)} \in H^2$ and then $W - \pi^m[W] = mW^{(m)}$ is in mH^2. Therefore $\pi^m[\cdot]$ gives the orthogonal projection from H^2 to $H(m)$.

By invoking this $\pi^m[\cdot]$, we derive a one-block problem equivalent to that of finding Q satisfying the conditions in Lemma 3. Equality (12) can be rewritten as

$$Q = \mathcal{F}_l(N, mC) \tag{17}$$

where

$$\begin{aligned} N &:= \left[\begin{array}{c:c} N_{11} & N_{12} \\ \hdashline N_{21} & N_{22} \end{array}\right] = \left[\begin{array}{c:c} 0 & I \\ \hdashline I & 0 \end{array}\right] M^{-1} \left[\begin{array}{c:c} 0 & I \\ \hdashline I & 0 \end{array}\right] \\ &= \left[\begin{array}{c|cc} \bar{A} & ZL & -ZB_2 \\ \hline F & 0 & I \\ -C_2 & I & 0 \end{array}\right] \end{aligned}$$

[3]If a square matrix Δ of complex functions is analytic and $\|\Delta\| < \delta < 1$ in a domain, then $(I - \Delta)^{-1}$ is analytic and $\|(I - \Delta)^{-1}\| < 1/(1 - \delta)$ in the same domain.

and

$$\bar{A} := \hat{A} - ZLC_2 - ZB_2F.$$

We now make a technical assumption.

Assumption 3: The three matrices $\bar{A}$, $\bar{A} + ZLC_2$ and $\bar{A} + ZB_2F$ belong to $\mathcal{M}_m$, and

$$N_{11} = \left[\begin{array}{c|c} \bar{A} & ZL \\ \hline F & 0 \end{array}\right]$$

gives a minimal realization of N_{11}.

Under the assumption that $\bar{A} \in \mathcal{M}_m$, we have

$$\begin{aligned} Q &= N_{11} + mN_{12}C(I - mN_{22}C)^{-1}N_{21} \\ &= \pi^m[N_{11}] + m\phi \end{aligned} \tag{18}$$

where

$$\phi := N_{11}^{(m)} + N_{12}C(I - mN_{22}C)^{-1}N_{21}. \tag{19}$$

Another assumption that $\bar{A} + ZLC_2, \bar{A} + ZB_2F \in \mathcal{M}_m$ is used in the following lemma. The minimality of the realization of N_{11} will be invoked in the next section.

Lemma 5: Under Assumption 3, suppose that $Q \in H^\infty$. Then there exists $\epsilon > 0$ such that C given by (12) is analytic and bounded in $\mathcal{N}_m(\epsilon)$ if and only if ϕ in (19) is in H^∞.

Proof: Note that $m\phi$ belongs to H^∞ by Lemma 4, (18) and $Q \in H^\infty$.

(Necessity) Four transfer matrices $N_{11}^{(m)}, N_{12}, N_{21}$ and N_{22} are rational and have the common A-matrix $\bar{A}$ which belongs to $\mathcal{M}_m$ by Assumption 3. By a similar argument to the proof of Lemma 3, we can show that there exists ϵ such that ϕ in (19) is analytic and bounded in $\mathcal{N}_m(\epsilon)$. Therefore, the desired claim follows from the fact $m\phi \in H^\infty$.

(Sufficiency) From (19), C can be rewritten as

$$\left\{mN_{12}^{-1}(\phi - N_{11}^{(m)})N_{21}^{-1}N_{22} + I\right\}^{-1} N_{12}^{-1}(\phi - N_{11}^{(m)})N_{21}^{-1}. \tag{20}$$

It suffices to show that we can take $\epsilon > 0$ such that C in this form is analytic and bounded in $\mathcal{N}_m(\epsilon)$. Similarly to the proof of Lemma 3, the desired result follows from the fact that N_{12} and N_{21} are invertible and their realizations are given by

$$N_{12}^{-1} = \left[\begin{array}{c|c} \bar{A} + ZB_2F & ZB_2 \\ \hline F & I \end{array}\right],$$

$$N_{21}^{-1} = \left[\begin{array}{c|c} \bar{A} + ZLC_2 & ZL \\ \hline C_2 & I \end{array}\right],$$

whose A-matrices are in $\mathcal{M}_m$ by Assumption 3. ■

This lemma means that C given by (12) internally stabilizes $\Sigma_{\inf}$ if and only if $\|Q\|_\infty < \gamma$ and $Q = \pi^m[N_{11}] + m\phi$ for some $\phi \in H^\infty$. Thus Problem 1 reduced to a specific one-block problem as follows:

Problem 2: Given $\gamma > 0$, Σ in (3) and an inner function $m(s)$ satisfying Assumptions 1 and 2. Assume that three conditions in Theorem 1 and Assumption 3 are satisfied. Then determine whether there exists $\phi \in H^\infty$ such that

$$\|\pi^m[N_{11}] + m\phi\|_\infty < \gamma. \tag{21}$$

IV. Solution to the specific one-block problem

We have shown that γ is achievable in Problem 1 if and only if three conditions in Theorem 1 are satisfied and

$$\inf_{\phi\in H^\infty} \|\pi^m [N_{11}] + m\phi\|_\infty < \gamma.$$

Hence let us consider the following:

Problem 3: Let m be an inner function and $(A, B, C, 0)$ with $A \in \mathcal{M}_m$ be a minimal realization of W. Then find

$$\rho_{\rm opt} := \inf_{\phi\in H^\infty} \|\pi^m [W] + m\phi\|_\infty. \tag{22}$$

When W is stable, $W^{(m)}$ is also stable by its definition. Then, by taking $\phi' = \phi + W^{(m)} \in H^\infty$, we can easily show that $\rho_{\rm opt}$ in (22) is equal to the infimum in (1). In this case, we can compute $\rho_{\rm opt}$ by the Zhou-Khargonekar formula: Define the ρ-dependent Hamiltonian matrix H_ρ by

$$H_\rho := \begin{bmatrix} A & BB^T/\rho \\ -C^TC/\rho & -A^T \end{bmatrix}. \tag{23}$$

and suppose that $H_\rho \in \mathcal{M}_m$. Then $\rho_{\rm opt}$ is the maximum ρ that makes $m\tilde{}(H_\rho)|_{22}$ singular, where $M|_{22}$ denotes the $(2, 2)$-block of matrix M. We attempt to extend this for unstable W.

When $m(s) = e^{-hs}$ with $h > 0$, $\rho_{\rm opt}$ in (22) is given by $L^2[0, h]$-induced norm of W. For the computational issue of this $L^2[0, h]$-induced norm, see [2], [11] for the Hamiltonian-based method or [4] for the bisection algorithm based on the Fourier series expansion of $L^2[0, h]$.

For simplicity, we denote $\Theta := \pi^m [W] \in H(m) \cap H^\infty$. Recall that $\rho_{\rm opt}$ is given by the operator norm of the Hankel operator $\Gamma_{m\tilde{}\Theta} : H^2 \to H^2_- : x \mapsto \pi^-[m\tilde{}\Theta x]$. It is shown that this operator norm is the maximal singular value of $\Gamma_{m\tilde{}\Theta}$ under a mild assumption [5]. Therefore we consider singular value equations of this operator. It follows from (2) that $\operatorname{Im}\Gamma_{m\tilde{}\Theta} \subset m\tilde{}H(m)$ and $\operatorname{Im}(\Gamma_{m\tilde{}\Theta})^* \subset H(m)$.

Theorem 2: Let $x \in H(m)$ and $y \in m\tilde{}H(m)$. Then $y = \Gamma_{m\tilde{}\Theta}x$ and $x = (\Gamma_{m\tilde{}\Theta})^*y$ if and only if there exist $\xi, \zeta \in \mathbb{R}^n$ satisfying

$$\begin{aligned} y &= m\tilde{}Wx - C(sI - A)^{-1}\xi && (24)\\ x &= mW\tilde{}y - B^T(sI + A^T)^{-1}\zeta. && (25)\end{aligned}$$

Proof: We show only the equivalence of $y = \Gamma_{m\tilde{}\Theta}x$ and (24). We can obtain the equivalence of $x = (\Gamma_{m\tilde{}\Theta})^*y$ and (25) similarly.

(Necessity) Denote $B = \begin{bmatrix} B_1 & \cdots & B_l \end{bmatrix}$ and $x(s) = \begin{bmatrix} x_1 & \cdots & x_l \end{bmatrix}^T$ where $x_i \in H(m)$. We can show[4] that

$$\xi = \sum_{i=1}^{l} (m\tilde{}x_i)(A) \cdot B_i \in \mathbb{R}^n \tag{26}$$

satisfies

$$m\tilde{}Wx - C(sI - A)^{-1}\xi \in m\tilde{}H(m) \subset H^2_- \tag{27}$$

and

$$W^{(m)}x - C(sI - A)^{-1}\xi \in H(m) \subset H^2. \tag{28}$$

[4]This follows from Lemma 6 in Appendix; see [9] for the detailed proof.

Therefore we have

$$\begin{aligned} y &= \pi^- \left[m\tilde{}Wx - W^{(m)}x\right] \\ &= \pi^- \left[m\tilde{}Wx - C(sI - A)^{-1}\xi\right] \\ &\quad -\pi^- \left[W^{(m)}x - C(sI - A)^{-1}\xi\right], \end{aligned}$$

and hence (24) follows from (27) and (28).

(Sufficiency) Suppose that there exists $\xi \in \mathbb{R}^n$ satisfying (24). Note that (27) holds, since $my \in H(m)$. Furthermore it can be proved ([9]) that (28) holds for any $\xi \in \mathbb{R}^n$ satisfying (27). By the converse argument of the proof of the necessity, $y = \Gamma_{m\tilde{}\Theta}x$ is obtained. ■

Theorem 2 characterizes the Schmidt pair x and y by two finite-dimensional vectors ξ and ζ. Furthermore this result is exactly the same as that in the case of the one-block problem of finding infimum in (1), i.e., Proposition 2.8 in [16]. Hence, by the same discussion in [16], we obtain the following theorem. The proof is omitted for the brevity; see [9] for its proof.

Theorem 3: Let $\rho > 0$. Assume that H_ρ defined by (23) is in $\mathcal{M}_m$. Then ρ is a singular value of the Hankel operator $\Gamma_{m\tilde{}\Theta}$ if and only if $m\tilde{}(H_\rho)|_{22}$ is not of full rank.

We can now propose a solution to the the standard H^∞ control problem posed in Problem 1. The following is the main result of this paper.

Theorem 4: Given a prespecified performance level $\gamma > 0$, a rational transfer matrix Σ in (3) and an inner function $m(s)$ satisfying Assumptions 1 and 2. The three conditions in Theorem 1 are necessary for the existence of a controller C which internally stabilizes $\Sigma_{\rm inf}$ and satisfies (5). Suppose that these conditions and Assumption 3 are satisfied. Suppose also that

$$H_\rho := \begin{bmatrix} \bar{A} & \rho^{-1}ZLL^TZ^T \\ -\rho^{-1}F^TF & -\bar{A}^T \end{bmatrix} \in \mathcal{M}_m$$

for any $\rho \geq \gamma$. Then there exists such C if and only if the essential norm ([5]) of $\Gamma_{m\tilde{}\pi^m[N_{11}]}$ is less than γ and $m\tilde{}(H_\rho)|_{22}$ is of full rank for any $\rho \geq \gamma$. Moreover, when these conditions are satisfied, all such controllers are given by (12) and (18) for arbitrary $\phi \in H^\infty$ satisfying (21).

Remark 2: We do not need to verify the rank condition for all $\rho \geq \gamma$. This is because $m\tilde{}(H_\rho)|_{22}$ is automatically of full rank for $\rho > \|\pi^m [N_{11}]\|_\infty$.

V. Example

Consider the mixed sensitivity optimization problem in (7). Weighting functions and plant are same as in [7], i.e.,

$$W_s = \frac{0.1s + 1}{s + 0.4}, \quad W_t = 0.5$$

and

$$P = P_r \cdot m = \frac{s + 3}{s - 3} \cdot \frac{(s + 1) + 2(s - 3)e^{-0.5s}}{(s - 1)e^{-0.5s} + 2(s + 3)}.$$

Here m is an inner function with infinitely many unstable zeros [7].

We now solve this problem according to Theorem 4. Figures 5 and 6 show the minimal singular values of $m\tilde{}(H_\rho)|_{22}$

for $\rho \geq \gamma$, when $\gamma = 2.80, 2.90$. According to Theorem 4, $\gamma = 2.80$ is not achievable, since there exists a $\rho > \gamma$ for which $m^{\sim}(H_\rho)|_{22}$ is not of full rank. On the other hand, $\gamma = 2.90$ is achievable, since $m^{\sim}(H_\rho)|_{22}$ is nonsingular for any $\rho \geq \gamma$.

Remark 3: The optimal mixed sensitivity given in [7] is 0.5584, which does not satisfy the estimate above. This is because a wrong upper bound for the achievable performance was employed in [7]. However, the formulae in [7] are all correct. In fact, searching for the optimal value with an appropriate performance bound yields an optimal mixed sensitivity around 2.85.

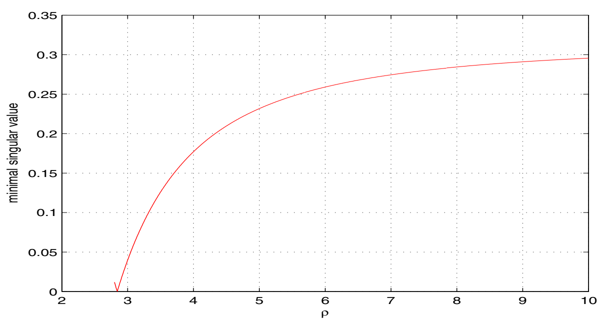

Fig. 5. Minimal singular values of $m^{\sim}(H_\rho)|_{22}$ versus ρ for $\gamma = 2.80$

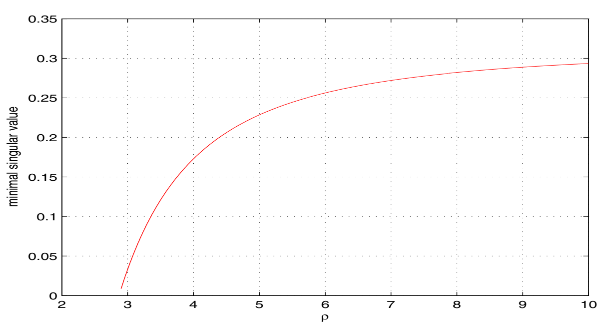

Fig. 6. Minimal singular values of $m^{\sim}(H_\rho)|_{22}$ versus ρ for $\gamma = 2.90$

VI. Conclusion

In this paper, we formulated the H^∞ control problems for a class of infinite-dimensional systems in terms of rational transfer matrix and a scalar (possibly infinite-dimensional) inner function. This representation maintains the advantage of the finiteness of both the weighting functions and also the number of unstable modes of the given plant. We have shown that this problem can be reduced to two (matrix-valued) Riccati equations and additional rank conditions. The obtained controller structure is completely characterized by the inner function and the finite-dimensional controllers given in [3].

References

[1] R. F. Curtain and H. J. Zwart, An Introduction to Infinite-Dimensional Linear Systems Theory, Springer-Verlag, 1995.

[2] T. Chen and B. Francis, Optimal Sampled-Data Control Systems, Springer-Verlag, 1995.

[3] J. C. Doyle, K. Glover, P. P. Khargonekar and B. A. Francis, "State-space solutions to standard H_2 and H_∞ control problems," *IEEE Trans. Autom. Control*, vol. 47, pp. 831–847, 1989.

[4] G. E. Dullerud, "Computing the L_2-induced norm of a compression operator," *Syst. Contr. Lett.*, vol. 37, pp. 87–91, 1999.

[5] C. Foias, H. Özbay and A. Tannenbaum, Robust Control of Infinite Dimensional Systems: Frequency Domain Methods, LNCIS, vol. 209, Springer-Verlag, 1996.

[6] G. H. Golub and C. F. van Loan, Matrix Computations, The Johns Hopkins University Press, 1989.

[7] S. Gümüşsoy and H. Özbay, "On the mixed sensitivity minimization for systems with infinitely many unstable modes," *Syst. Contr. Lett.*, vol. 53, pp. 211–216, 2004.

[8] K. Hoffman, Banach Spaces of Analytic Functions, Prentice Hall, 1962.

[9] K. Kashima, "General solution to standard H^∞ control problems for infinite-dimensional systems," Ph. D. Dissertation, Kyoto University, 2005

[10] K. Kashima and Y. Yamamoto, "Equivalence characterization of invariant subspaces of H^2 and applications to optimal sensitivity problem," *Proc. 42nd IEEE Conference on Decision and Control*, 2003

[11] G. Meinsma, L. Mirkin and Q. -C. Zhong, "Control of systems with I/O delay via reduction to a one-block problem," *IEEE Trans. Autom. Control*, vol. 47, pp. 1890–1895, 2002.

[12] G. Meinsma and H. Zwart, "On H_∞ control of dead-time systems," *IEEE Trans. Autom. Control*, vol. 45, pp. 272–285, 2000.

[13] L. Mirkin, "On the extraction of dead-time controllers and estimators from delay-free parameterizations," *IEEE Trans. Autom. Control*, vol. 48, pp. 543–753, 2003.

[14] O. Toker and H. Özbay, "H^∞ optimal and suboptimal controllers for infinite dimensional SISO plants," *IEEE Trans. Autom. Control*, vol. 40, pp. 751–755, 1995.

[15] B. van Keulen, H_∞-control for Infinite-Dimensional Systems: a state-space approach, Birkhäuser, 1993.

[16] Y. Yamamoto, K. Hirata and A. Tannenbaum, "Some remarks on Hamiltonians and the infinite-dimensional one block H^∞ problem," *Syst. Contr. Lett.*, vol. 29, pp. 111–117, 1996.

[17] Q. -C. Zhong, "On standard H_∞ control of processes with a single delay," *IEEE Trans. Autom. Control*, vol. 48, pp. 1097–1103, 2003.

[18] K. Zhou and P. P. Khargonekar, "On the weighted sensitivity minimization problem for delay systems," *Syst. Contr. Lett.*, vol. 8, pp. 307–312, 1987.

Appendix

A. Matrix functions

When a scalar function $f(s)$ is analytic in a neighborhood of any eigenvalue of a matrix $A \in \mathbb{R}^{n\times n}$, we can define[5] a matrix $f(A) \in \mathbb{R}^{n\times n}$. Therefore, when $x \in H(m)$ and $A \in \mathcal{M}_m$, matrix functions $m^{\sim}(A)$ and $(m^{\sim}x)(A)$ are both well-defined.

Lemma 6: Let m be an inner function, $X \in \mathcal{M}_m^{n\times n}$ and $M_1, M_2 \in \mathbb{R}^{n\times p}$. Then

$$\Phi(s) := (sI - X)^{-1}(M_1 - m(s)M_2)$$

is analytic in a neighborhood of every eigenvalue of X if and only if

$$m^{\sim}(X)M_1 = M_2.$$

Proof: This result can be shown by the same argument as in the proof of Theorem 2.3 in [16]; see [9] for the detailed proof. ■

[5]There are several equivalent ways of defining $f(A)$; see e.g. [6].