Proceedings of the
44th IEEE Conference on Decision and Control, and
the European Control Conference 2005
Seville, Spain, December 12-15, 2005

Output Feedback Reference Governor for Nonlinear Systems

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Abstract— **This paper develops a new reference management algorithm for constrained nonlinear systems with unmeasurable states. Instead of the state itself, the present method utilizes an ellipsoidal region in which the state is guaranteed to lie. Such a region can be obtained by using the set-valued observer due to Scholte and Campbell [9]. The present method requires a solution of the optimization problem which may not be solved effectively. For ease of implementation, we introduce a relaxed problem which is always efficiently solvable. When neither noise nor disturbance are present and the reference is constant, we show sufficient conditions for the modified reference to be settled to the reference in a finite time, and consequently for the convergence of the state to the desired equilibrium. In the presence of noise and/or disturbance, we derive somewhat conservative conditions for the finite-time settling of the modified reference to the original one and for the convergence of the state to the neighborhood of the equilibrium. The effectiveness of the present method is demonstrated by a numerical example.**

I. Introduction

Most of practical control systems inherently have constraints due to the nonlinear characteristics of actuators or for the safety of hardwares. This can lead to the performance deterioration or even the instability of the closed-loop system, if not properly accounted for. Thus, when we deal with the tracking control problem, it is required not only to achieve a good tracking performance but also to avoid the violation of these constraints. One of the ways to simultaneously meet these requirements is to add an auxiliary mechanism called a *reference governor* or a *reference management algorithm* [1]–[3],[5]–[7] to the closed-loop system. The reference governor modifies the reference signal, and input the resulting signals to the closed-loop system.

Heretofore, various reference management algorithms have been proposed and most of them are aimed at linear systems [1],[3],[7]. Linear systems possess an important advantage that there is an algorithm to calculate a so-called *maximal output admissible set*, which is the maximal set consisting of the initial states satisfying the infinite-time constraints [4],[8]. In contrast, for nonlinear systems, there is no way to compute such a set.

Bemporad *et al.* [2] proposed the first reference management algorithm for nonlinear systems based on the fact that the initial states satisfying the constraints in enough long finite-time interval guarantee the infinite-time constraint fulfillment. However, this algorithm requires a great deal of on-line computational effort. Gilbert *et al.* [5],[6] provided algorithms with less computation using the inside approximation of the maximal output admissible set.

For linear systems, some output-feedback reference governors have been reported in the literature [1]. However, all the reference governors for nonlinear systems are state-feedback ones which assume the exact measurements of all the states. Since, in the real world, it is often the case that some states are not available for control or are contaminated by the noise, the application of the conventional methods is limited. To overcome this difficulty, this paper proposes an output-feedback reference governor for nonlinear systems. Though Gilbert and Kolmanovsky [6] considered the case where the closed-loop system suffers from the disturbance, the noisy measurement case was not considered. In contrast, the present method allows us to design the modified reference in consideration of the sensor noise.

The following notations are used throughout this paper. For vectors $g, h \in \mathbf{R}^m$, $g \geq h$ means $g_i \geq h_i$ for all i. $\mathbf{0}$ and $\mathbf{1}$ denote the column vectors with appropriate dimension whose elements are 0 and 1, respectively. For a positive definite symmetric matrix $K \in \mathbf{R}^{n \times n}$ and a vector $x_o \in \mathbf{R}^n$, $\Omega(x_o, K)$ is an ellipsoid

$$\Omega(x_o, K) = \{x \in \mathbf{R}^n | (x - x_o)^\top K^{-1}(x - x_o) \leq 1\}.$$

II. Problem Statement

Consider the following nonlinear closed-loop system Σ consisting of a plant Σ_p and a feedback controller Σ_c, which is given *a priori*. Suppose that Σ_c is not necessarily designed in consideration of constraints on state and control though, in general, the systems are required to fulfill these constraints as well as achieving a good control performance.

$$\begin{aligned} x(t+1) &= f(x(t), r(t)) + w(t), && (1)\\ y(t) &= h_1(x(t), r(t)), && (2)\\ z(t) &= h_2(x(t), r(t)) + v(t), && (3)\\ c(t) &= h_0(x(t), r(t)), && (4) \end{aligned}$$

where the function $f : \mathbf{R}^n \times \mathbf{R}^{p_1} \to \mathbf{R}^n$ and $h_2 : \mathbf{R}^n \times \mathbf{R}^{p_1} \to \mathbf{R}^{p_2}$ are assumed to be C^2 function.

The vector $x(t) \in \mathbf{R}^n$ is the state of Σ and is unknown vector which cannot be measured directly. Though its initial value $x(0)$ is also unknown, it is known to be confined in the ellipsoid $\Omega(x_0, K_0)$, namely

$$x(0) \in \Omega(x_0, K_0), \tag{5}$$

where the vector x_0 and symmetric positive definite matrix K_0 are known.

The vector $r(t) \in \mathbf{R}^{p_1}$ is the reference that belongs to a compact convex set R. Let the output of the reference

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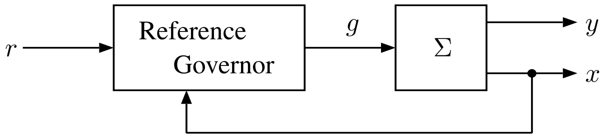

Fig. 1. Conventional method

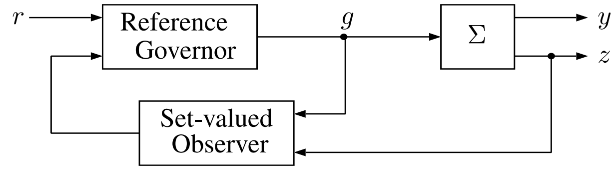

Fig. 2. Present method

governor be $g(t) \in \mathbf{R}^{p_1}$, and we refer to it as the modified reference. Moreover, $y(t) \in \mathbf{R}^{p_1}$ is the output to be controlled, $z(t) \in \mathbf{R}^{p_2}$ is the measurement, $w(t) \in \mathbf{R}^{n}$ is the disturbance, and $v(t) \in \mathbf{R}^{p_2}$ is the sensor noise. The disturbance and sensor noise are unknown deterministic signals and assumed to be bounded by ellipsoids given by

$$w(t) \in \Omega(\mathbf{0}, Q_w),\ v(t) \in \Omega(\mathbf{0}, Q_v)\ \forall t \geq 0, \qquad (6)$$

respectively. The vector $c(t) \in \mathbf{R}^{p_0}$ is the auxiliary output that describes state and control constraints. Namely, $c(t)$ must be constrained within a prescribed set $\mathcal{C}$ as

$$c(t) \in \mathcal{C} := \{c \in \mathbf{R}^{p_0} | c \leq \mathbf{0}\}\ \forall t \geq 0. \qquad (7)$$

A desired equilibrium state for the step reference $r(t) \equiv r \in R$ and the zero disturbance $w(t) \equiv 0$ is denoted by $x_e(r)$. Namely, $x_e(r)$ is a vector which satisfies $x_e(r) = f(x_e(r), r)$. Assume that $h_0(x_e(r), r) \in \mathcal{C}\ \forall r \in R$ and that, for simplicity, $x_e(r)$ is unique for any $r \in R$ and the controller is designed in such a way to stabilize $x_e(r), r \in R$.

The present algorithm provides the modified reference based on a function V and its sub-level set

$$S(r) = \{x \in \mathbf{R}^n | V(x, r) \leq 0\}. \qquad (8)$$

For any $\epsilon > 0$, define the set $S_\epsilon(r)$ as

$$S_\epsilon(r) = \{x \in \mathbf{R}^n | V(x, r) \leq -\epsilon\}.$$

Assumption 1:

(a) The function $V : \mathbf{R}^n \times R \to \mathbf{R}$ is continuous.

(b) There exists $\epsilon_0 > 0$ such that $x_e(r) \in S_{\epsilon_0}(r)\ \forall r \in R$.

(c) There exists a compact set $X \subset \mathbf{R}^n$ such that $S(r) \subset X\ \forall r \in R$.

(d) Let $\phi(t, x, r, w(\cdot))$ be the solution of (1) for the initial state $x(0) = x$, step reference $r(t) \equiv r \in R$ and disturbance $w(\cdot) \in \Omega(\mathbf{0}, Q_w)$. Then, $h_0(\phi(t, x, r, w(\cdot)), r) \in \mathcal{C}$ holds for all $w(\cdot) \in \Omega(\mathbf{0}, Q_w)$, $t \geq 0$, $r \in R$ and $x \in \mathbf{R}^n$ satisfying $V(x, r) \leq 0$.

(e) For any $r \in R$, the set $S(r)$ is a polyhedron or a convex set.

The function $V(x, r)$ can be constructed by the methods in [5] or [6]. State-feedback reference governors do not require the Assumption 1(e) and allow $S(r)$ of any form. Namely, the present algorithm restricts the construction of $S(r)$. However, the typical selection such as a polyhedron or an ellipsoid belongs to the class of this assumption.

In the real world, it is often the case that some states are not available for control. Nevertheless, most of the conventional reference governors [2], [3], [5]–[7] are state-feedback ones (Fig. 1), which require the exact measurements of all the states. This paper develops a new reference management algorithm which guarantees the constraints fulfillment even when some states are unmeasurable.

III. Reference Management Algorithm

The design of the modified reference is based on the references [5],[6]. Namely, for $\lambda \in [0, 1]$, we define $g(t; \lambda) = g(t-1) + \lambda(r(t) - g(t-1))$, and then the modified reference is given by $g(t) = g(t; \lambda(t))$, where $\lambda(t)$ is computed at each time step. In the following, we will see how to compute the desired $\lambda(t) \in [0, 1]$.

To begin with, at each time step, we calculate the maximum $\lambda^*(t)$ among $\lambda \in [0, 1]$ satisfying

$$x(t) \in S(g(t; \lambda)). \qquad (9)$$

That is to say, the reference governor aims at only making the modified reference close to the original one while satisfying the constraint (7). This implies that it is assumed that the closed-loop system achieves a good response characteristic in the absence of the constraint.

Since, in this paper, the state vector is not available, we cannot check the condition (9). On the other hand, for nonlinear systems with deterministic disturbances and/or noises, some recursive algorithms to compute the set within which the true state is guaranteed to fall have been proposed ([9] and references therein). Hereafter, we refer to the set as a *state existence region*. The present algorithm computes an ellipsoidal state existence region $\Omega(\hat{x}(t), K(t))$ by using the set-valued observer due to Scholte and Campbell [9] and utilizes the region instead of the true state vector $x(t)$ (Fig. 2). The initial condition for the set-valued observer is given by (5).

Recall that $x(t)$ lies within $\Omega(\hat{x}(t), K(t))$. Hence, we shall find the largest $\lambda^*(t)$ among $\lambda \in [0, 1]$ satisfying the following condition instead of (9).

$$\Omega(\hat{x}(t), K(t)) \subset S(g(t; \lambda)). \qquad (10)$$

This leads to the following optimization problem.

Optimization Problem OP

$$\begin{array}{rcll} \text{data} & : & r(t), \hat{x}(t), K(t), g(t-1) & (11) \\ \max\limits_{\lambda} & & \lambda & (12) \\ \text{subject to} & & \lambda \in [0, 1] \text{ and } (10) & (13) \end{array}$$

If there does not exist any $\lambda \in [0, 1]$ satisfying the constraint (13), let $\lambda(t) = 0$, namely $g(t) = g(t; 0) =$

$g(t-1)$. Furthermore, similarly to [6], we set $\epsilon, \delta > 0$, and admit small change of g satisfying

$$\|g(t, \lambda^*(t)) - g(t-1)\|_\infty < \delta \tag{14}$$

only when

$$\max_{x \in \Omega(\hat{x}(t), K(t))} V(x, g(t-1)) \leq -\epsilon. \tag{15}$$

Note that we set ϵ in such a way that $x_e(r) \in S_\epsilon(r)\ \forall r \in R$.

To sum up the above discussion, the reference management algorithm is described as follows.

[Reference Management Algorithm RMA]

Step 0: Let $t = 0, K(0) = K_0$ and $\hat{x}(0) = x_0$ and set $g(-1)$, $\epsilon > 0$ and $\delta > 0$.

Step 1: Compute $K(t)$ and $\hat{x}(t)$ by the set-valued observer[9].

Step 2:

- If there is a feasible solution of OP, compute the optimal solution $\lambda^*(t)$.
  - If $\lambda^*(t)$ satisfies (14) and (15) does not hold, let $\lambda(t) = 0$
  - otherwise let $\lambda(t) = \lambda^*(t)$.
- If there exists no feasible solution of OP let $\lambda(t) = 0$.

Step 3: Input $g(t) = g(t; \lambda(t))$ to Σ.

Step 4: $t = t + 1$ and go to Step 1.

If the set $S(r)$ is given by a polyhedron as well as the linear system case [1], the satisfaction of (10) for fixed λ can be easily checked. Hence OP is solved by using the grid search for λ over the interval $[0, 1]$. Additionally, the assessment of (15) is also easy. However, if $S(r)$ is a general convex set, checking (10) requires to solve

$$\max_{x \in \Omega(\hat{x}(t), K(t))} V(x, g(t; \lambda)) \leq 0, \tag{16}$$

even for fixed λ. In this case, it is difficult to compute even a sub-optimal solution of OP. Thus, we relax (10) by using a convex polyhedral approximation of $\Omega(\hat{x}(t), K(t))$. We compute a convex polyhedral approximation Π_0 of the unit sphere $\Omega(\mathbf{0}, I)$ and its vertices $\tilde{x}_0^i,\ i \in [1, N]$ *a priori*. At each time step t, let $\tilde{x}^i(t)$ be defined by

$$\tilde{x}^i(t) = \hat{x}(t) + K^{1/2}(t)\tilde{x}_0^i,\ i \in [1, N]. \tag{17}$$

Then, the polyhedron $\Pi(\hat{x}(t), K(t))$ formed by the vertices $\tilde{x}^i(t)$ becomes an outer approximation of $\Omega(\hat{x}(t), K(t))$. By the convexity of $S(r)$, $\tilde{x}^i(t) \in S(g(t; \lambda))\ \forall i \in [1, N]$ guarantees (10), and hence (9). As a result, we have to solve the following problem instead of OP. If the problem is feasible, then a sub-optimal solution can be computed by the grid search for λ over the interval $[0, 1]$.

Relaxed Optimization problem ROP

$$\begin{aligned}
\text{data} \quad &: \quad r(t), \tilde{x}^i(t), g(t-1) && (18)\\
\max_{\lambda} \quad & \quad \lambda && (19)\\
\text{subject to} \quad & \quad \lambda \in [0, 1] && (20)\\
& \quad \tilde{x}^i(t) \in S(g(t; \lambda))\ \forall i \in [1, N] && (21)
\end{aligned}$$

In accordance with this relaxation of the optimization problem OP, (15) is also replaced by

$$V(\tilde{x}^i(t), g(t-1)) \leq -\epsilon \quad \forall i \in [1, N]. \tag{22}$$

The resulting reference management algorithm is given as follows.

[Relaxed Reference Management Algorithm RRMA]

Replace Step 2 of RMA as follows.

- If there is a feasible solution of ROP, compute the optimal solution $\lambda^*(t)$.
  - If $\lambda^*(t)$ satisfies (14) and (22) does not hold, set $\lambda(t) = 0$.
  - Otherwise, set $\lambda(t) = \lambda^*(t)$.
- If there does not exist a feasible solution of ROP, set $\lambda(t) = 0$.

Theorem 1: Suppose that V satisfies Assumption 1 and $x(0) \in S(g(-1))$ holds. Then RRMA guarantees the following.

(i) At any time step $t \geq 0$, the state $x(t) \in \mathbf{R}^n$ and the modified reference $g(t) \in R$ is well-defined.

(ii) The constraint (7) is satisfied.

Proof: The statement (i) immediately follows from (1), the convexity of R and the above algorithm RRMA. Thus, we prove only (ii).

There are two possibilities : (a) $T = \emptyset$ (b) $T \neq \emptyset, T = \{t \geq 0 | \lambda(t) \neq 0\}$.

In the case of (a), $g(t) \equiv g(-1)$ holds. Hence $x(0) \in S(g(-1))$ and Assumption 1(d) guarantee the infinite-time constraint fulfillment.

We consider the case (b). If $\tilde{t} \in T$, then $\Omega(\hat{x}(\tilde{t}), K(\tilde{t})) \subset S(g(\tilde{t}))$, that is

$$x(\tilde{t}) \in S(g(\tilde{t})) \tag{23}$$

holds. Firstly, it follows from $x(0) \in S(g(-1))$ and Assumption 1(d) that $g(t) = g(-1)\ \forall t \in [0, t_{min})$, where $t_{min} = \min_{t \in T} t$. Any $\tilde{t} \in T$ is contained in either of the following two sets.

$$\begin{aligned}
T_1 &= \{\tilde{t} \in T \mid \exists \sigma \in (0, \infty) \text{ s.t. } \tilde{t} + \sigma \in T\}\\
T_2 &= \{\tilde{t} \in T \mid \lambda(t) = 0\ \forall t \geq \tilde{t}\}
\end{aligned}$$

Thus, it is sufficient to show

(b-1) For any $t_1 \in T_1, t_2 \in T(t_1 < t_2)$, the constraint over $[t_1, t_2]$ is satisfied.

(b-2) For $\tilde{t} \in T_2$, the constraint since $\tilde{t} + 1$ is satisfied.

The item (b-2) is obvious from (23), Assumption 1(d) and $g(\tilde{t}) = g(\tilde{t} + 1) = g(\tilde{t} + 2) = \cdots$.

We show (b-1). For any $\tilde{t} \in T_1$, there exists the minimal $\sigma > 0$ satisfying $\tilde{t} + \sigma \in T$. Then $g(\tilde{t} + \sigma - 1) = \cdots = g(\tilde{t} + 1) = g(\tilde{t})$ and $g(\tilde{t} + \sigma) \neq g(\tilde{t} + \sigma - 1)$ hold. The equation (23) and Assumption 1(d) guarantee the constraint fulfillment over $[\tilde{t}, \tilde{t}+\sigma-1]$. Moreover, $x(\tilde{t}+\sigma) \in S(g(\tilde{t}+\sigma))$ holds from $\tilde{t} + \sigma \in T$, implying the constraint fulfillment at time $\tilde{t}+\sigma$. Therefore, the constraint over $[\tilde{t}, \tilde{t}+\sigma]$ is satisfied. By setting $\tilde{t} = \tilde{t} + \sigma$ and repeating the above discussion, it is shown that the constraint over $[\tilde{t}, t]$ is satisfied for any

$t(\geq \tilde{t}) \in T$. Since this result is obtained for any $\tilde{t} \in T_1$, (b-1) holds.

Assembling these results completes the proof of (ii). ■

For the subsequent discussion, we give the following lemma.

Lemma 1: In ROP, if there exists a λ satisfying (20) and (21) and $\lambda^*(t) < 1$, then

$$\max_i V(\tilde{x}^i(t), g(t; \lambda^*(t))) = 0 \tag{24}$$

holds.

Remark 1: In the construction of Π_0, the larger N is set, the smaller conservativeness of the approximation is achieved. However, a large N increases the on-line computation of the reference management. Thus, we need to choose an appropriate N taking account of this trade-off.

Remark 2: The present method postulates that the functions f and h_2 are C^2 functions, which imposes a strong limitation on the controller design. However, suppose that the functions describing the dynamics of the plant are C^2 functions and the controller states are available without the sensor noise, (e.g. consider the situation of sampled-data control). Then the present algorithm can be applied to such a case, regardless of the structure of the controller, by adding the set-valued observer only to the plant.

Remark 3: With some appropriate modification similar to [6], the present algorithm can be applied to continuous-time systems. We regard the reference governor as a sampled-data system. However, it has a problem that there is no algorithm to compute the state existence region for nonlinear continuous-time systems, though the region is required at each sampling instant. Hence, we need an exact discrete-time model without discretization error.

IV. Convergence Analysis

This section discusses the convergence of the state $x(t)$ to the desired equilibrium $x_e(r)$ under the assumption that the reference is a step function $r(t) \equiv r \in R$. As a preparation, we make the following assumption. A sufficient condition for this assumption is given by so-called uniform observability. For more detail on this, the readers are recommended to refer to the reference [9].

Assumption 2: There exists a $\tilde{s} > 0$ such that $K(t) \leq \tilde{s}I \;\; \forall t \geq 0$.

Proposition 1: Suppose that Assumption 2 holds. If $w(t) \equiv v(t) \equiv \mathbf{0}$, then the center of the ellipsoid $\Omega(\hat{x}(t), K(t))$, $\hat{x}(t)$, converges to the true state vector $x(t)$ asymptotically. Furthermore, if $w(t)$ and/or $v(t)$ are not zero, then the upper bound of the error between $x(t)$ and $\hat{x}(t)$, namely $\|x(t) - \hat{x}(t)\|$, can be computed. In this paper, let the square of the upper bound be $\gamma > 0$.

A. Noise Free Case

Suppose that $w(t) \equiv \mathbf{0}$ and $v(t) \equiv \mathbf{0}$. We assume the following.

Assumption 3: For any $r(t) \equiv r \in R$ and $x(0) \in S(r)$ ($V(x(0), r) \leq 0$), $\phi(t, x(0), r, 0) \to x_e(r)$ $(t \to \infty)$ holds. That is, the set $S(r)$ is a region of attraction for $x_e(r)$.

Proposition 1 and Assumption 3 yield the next lemma.

Lemma 2: For any $g(t) \equiv g \in R$, the center of the ellipsoid $\Omega(\hat{x}(t), K(t))$ converges to $x_e(g)$ asymptotically. Namely, for any $\rho > 0$, there exists a $\bar{\tau} \in [0, \infty)$ such that $\hat{x}(t) \in \Omega(x_e(g), \rho I) \;\; \forall t \geq \bar{\tau}$.

We first define $V_m(s, \rho)$ by

$$V_m(s, \rho) = \max_{K \leq sI, i \in [1,N]} \tilde{x}_0^{i\top} K \tilde{x}_0^i + \rho. \tag{25}$$

The optimization problem in (25) is a linear matrix inequality problem, and can be solved by an existing convex programming algorithm.

Proposition 2: Suppose that $\hat{x}(t) \in \Omega(x_e(g), \rho I)$ and $K(t) < sI$ hold. Then we have

$$\Pi(\hat{x}(t), K(t)) \subset \Omega(x_e(g), V_m(s, \rho) I)$$

for any $g \in R$ and $\rho > 0$.

Next, let

$$\begin{aligned} \bar{s}(r, \rho) = & \max_{s>0} s \\ & \text{subject to } \Omega(x_e(r), V_m(s, \rho) I) \subset S_\epsilon(r) \end{aligned} \tag{26}$$

The optimization problem (26) can be solved by the bisection method over $(0, s_m]$ where s_m is a sufficiently large constant. The satisfaction of (26) is checked by using the polyhedral approximation of $\Omega(x_e(r), V_m(s, \rho) I)$ with sufficient large N.

Assumption 4: There exists $\tau^* \in [0, \infty)$ and $\rho^* > 0$ such that

$$K(t) \leq \bar{s}(r, \rho^*) I \;\; \forall t \geq \tau^* \tag{27}$$

For simplicity of notation, we hereafter denote $s^* = \min_{r \in R} \bar{s}(r, \rho^*)$, and $V_m^* = V_m(s^*, \rho^*)$. It is obvious from (26) that

$$\Omega(x_e(r), V_m^* I) \subset S_\epsilon(r) \;\; \forall r \in R. \tag{28}$$

From the above discussion, we obtain the following theorem.

Theorem 2: Let the reference be a step signal $r(t) \equiv r \in R$. Suppose that RRMA is applied to the closed-loop system Σ. If Assumption 1–4 and $x(0) \in S(g(-1))$ hold, then it follows for any $r \in R$ that

(i) there exists a $\tilde{t} \in [0, \infty)$ such that $g(t) = \bar{r} \;\forall t \geq \tilde{t}$,

(ii) the state $x(t)$ converges $x_e(r)$ asymptotically.

Proof: For a certain $\tilde{t} \geq 0$, if $g(\tilde{t}) = r$, then $g(t) = r \;\; \forall t \geq \tilde{t}$ clearly holds. Hence the theorem is false only if $\lambda(t) < 1 \;\; \forall t \geq 0$. We show that this leads to a contradiction.

Define

$$\begin{aligned} I_0 &= \{t \geq 0 \mid \lambda(t) = 0\}, \\ I_1 &= \{t \geq 0 \mid 0 < \lambda(t) < 1\} = I_2 \cup I_3, \\ I_2 &= \{t \geq 0 \mid \|g(t) - g(t-1)\|_\infty \geq \delta\}, \\ I_3 &= \{t \geq 0 \mid \|g(t) - g(t-1)\|_\infty < \delta \\ & \qquad \text{and } \max_i V(\tilde{x}^i(t), g(t-1)) < -\epsilon\}. \end{aligned}$$

Similarly to [6], I_2 is shown to be finite. We prove that the set I_0 is finite. For some time t_0, assume $t_0 \in I_1$ and $t_0 + t \in I_0 \;\forall t \in [0, t_f]$, that is, $g(t_0 + t) = g(t_0) \;\forall t \in [0, t_f]$, where $t_f = \max(\bar{\tau}, \tau^* - t_0) \in (0, \infty)$. Since $t_0 \in I_1$, we get $x(t_0) \in S(g(t_0))$. From this, Lemma 2 and $t_f \geq \bar{\tau}$, we get

$$\hat{x}(t_0 + t_f) \in \Omega(x_e(g(t_0)), \rho^* I). \quad (29)$$

Assumption 4 and $t_0 + t_f \geq \tau^*$ yield

$$K(t_0 + t_f) < s^* I. \quad (30)$$

Hence we see from (29), (30) and Proposition 2 that $\Pi(\hat{x}(t_0 + t_f), K(t_0 + t_f)) \subset \Omega(x_e(g(t_0)), V_m^* I)$. Moreover, (28) and $g(t_0) = g(t_0 + t_f)$ imply $\Pi(\hat{x}(t_0 + t_f), K(t_0 + t_f)) \subset S_\epsilon(g(t_0 + t_f))$, namely

$$V(\tilde{x}^i(t_0 + t_f), g(t_0 + t_f)) < -\epsilon \ \forall i \in [1, N]. \quad (31)$$

From the continuity of V and (31), there is a nonzero feasible solution of ROP at time $t_0 + t_f$. Furthermore, because $g(t_0 + t_f - 1) = g(t_0 + t_f)$,

$$V(\tilde{x}^i(t_0 + t_f), g(t_0 + t_f - 1)) < -\epsilon \quad (32)$$

holds and (22) is satisfied. Thus, $\lambda(t_0 + t_f)$ must not be zero and $t_0 + t_f \notin I_0$. This contradicts $t_0 + t \in I_0 \ \forall t \in [0, t_f]$, and there is a $t \in [0, t_f]$ such that $t_0 + t \notin I_0$. By the similar discussion, it is shown that $\{t \geq 0\} = I_0$ cannot occur. Namely, $\lambda(t) = 0$ does not continue infinitely. Therefore, I_0 is a finite set. This implies that I_3 must be an infinite set. From Lemma 1, it follows for any $t \in I_3$ that

$$\max_i V(\tilde{x}^i(t), g(t)) = 0 \quad (33)$$

$$V(\tilde{x}^i(t), g(t-1)) < -\epsilon \ \forall i \in [1, N]. \quad (34)$$

By the continuity of V, there exists $\tilde{\delta} > 0$ such that

$$|V(x, g_1) - V(x, g_2)| < \epsilon \quad (35)$$

for all g_1, g_2 and $x \in X$ satisfying $\|g_1 - g_2\|_\infty < \tilde{\delta}$. Furthermore, since the set R is compact and $0 < \lambda(t) < 1 \ \forall t \in I_3$, if I_3 is infinite then there exist an infinite subset $\tilde{I}_3 \subset I_3$ and $\bar{g}$ such that $t \in \tilde{I}_3$ and $t \to \infty$ imply $g(t) \to \bar{g} \in R$. Since $g(t)$ approaches r monotonically,

$$\|g(t) - g(t-1)\|_\infty < \tilde{\delta} \quad (36)$$

holds regardless of whether $t-1$ is contained in $\tilde{I}_3$ or not, for sufficient large $t \in \tilde{I}_3$. For such a large $t \in \tilde{I}_3$, define $i(t) = \arg\max_i V(\tilde{x}^i(t), g(t))$. Then we get $V(\tilde{x}^{i(t)}(t), g(t)) = 0$ and $\tilde{x}^{i(t)}(t) \in S(g(t)) \subset X$. From this, (35) and (36), it follows that $V(\tilde{x}^{i(t)}(t),\ g(t-1)) > -\epsilon$. This contradicts (34) and (i) is proved.

Statement (ii) is obvious because of (i), Assumption 3 and $x(\tilde{t}) \in \Omega(\hat{x}(\tilde{t}), K(\tilde{t})) \subset S(r)$. ∎

Assumptions 1–4 and $x(0) \in S(g(-1))$ are a sufficient condition for the convergence of the state to the desired equilibrium. Roughly speaking, these imply that the set-valued observer is stable and the size of $\Omega(\hat{x}(t), K(t))$ is smaller than that of $S(r)$. Therefore, $s^* = \min_{r \in R} \bar{s}(r, \rho^*)$ determines the size of the state existence region $\Omega(\hat{x}(t), K(t))$ for the state to converge.

Remark 4: It follows that $\max_{K \leq sI, i \in [1,N]} \tilde{x}_0^{i\top} K \tilde{x}_0^i \to s \ (N \to \infty)$. Thus, by replacing $V_m(s, \rho)$ with $s + \rho$, we obtain a sufficient condition for the convergence of the state when RMA is applied.

B. Noisy Case

When the disturbance and/or noise is present, Assumption 3 is unrealistic. In order to cope with the noisy case, we make the following assumption instead.

Assumption 5: For any $w \in \Omega(\mathbf{0}, Q_w)$, $x(0) \in S(r)$ and $r(t) \equiv r \in R$, there exist $\bar{\tau} \in [0, \infty)$ and $\mu > 0$ such that $\phi(t, x(0), r, w) \in \Omega(x_e(r), \mu I) \ \forall t \geq \bar{\tau}$.

Similarly to Lemma 2, from Proposition 1 and Assumption 5, if $x(0) \in S(g)$ for any $g(t) \equiv g \in R$, there is a $\bar{\tau} \in [0, \infty)$ such that $\hat{x}(t) \in \Omega(x_e(g), (\mu + \gamma) I) \ \forall t \geq \bar{\tau}$. Hence, replacing ρ by $\mu + \gamma$ in Theorem 2 yields the following theorem immediately.

Assumption 6: For any $r \in R$, there exists $\tau^* \in [0, \infty)$ such that

$$K(t) \leq \bar{s}(r, \mu + \gamma) I \ \forall t \geq \tau^* \quad (37)$$

Theorem 3: Let the reference be a step signal $r(t) \equiv r \in R$. Suppose that RRMA is applied to the closed-loop system Σ. If Assumption 1 2, 5 and 6 and $x(0) \in S(g(-1))$ hold, then it follows for any $r \in R$ that

- (i) there exists $\tilde{t} \in [0, \infty)$ such that $g(t) = r \ \forall t \geq \tilde{t}$,
- (ii) the state $x(t)$ converges into a sphere $\Omega(x_e(r), \mu I)$ in a finite time.

V. Numerical Example

Consider the bilinear plant Σ_p given by

$$\begin{aligned} x_p(t+1) &= x_p(t) + (1 + 0.4 x_p(t)) u(t), \\ y(t) &= x_p(t), \\ z(t) &= x_p(t) + v(t). \end{aligned}$$

Suppose that the output y has a constraint $y(t) \leq 27 \ \ \forall t \geq 0$ and the control input u has a constraint $|u(t)| \leq 0.3 \ \forall t \geq 0$ due to saturation. Let the sensor noise v be a bounded noise satisfying $|v(t)| \leq 0.1 \ \forall t \geq 0$. Apply the following linear controller Σ_c without taking account of the constraints.

$$\begin{aligned} x_c(t+1) &= 0.5 x_p(t) + 0.2 (r(t) - z(t)) \\ u(t) &= 0.2 x_c(t) \end{aligned}$$

Then the resulting closed-loop system Σ is described as

$$\begin{aligned} x(t+1) &= A x(t) + F x^\top(t) T x(t) + B r(t) - B v(t), \\ y(t) &= C_1 x(t), \\ z(t) &= C_1 x(t) + v(t), \\ c(t) &= C_0 x(t) + D_0, \end{aligned}$$

where

$$\begin{aligned} x(t) &= \begin{bmatrix} x_p(t) \\ x_c(t) \end{bmatrix}, A = \begin{bmatrix} 1.0 & 0.2 \\ -0.2 & 0.5 \end{bmatrix}, \\ F &= \begin{bmatrix} 0.08 \\ 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0.2 \end{bmatrix}, T = \begin{bmatrix} 0 & 0.5 \\ 0.5 & 0 \end{bmatrix}, \\ C_1 &= \begin{bmatrix} 1 & 0 \end{bmatrix}, c(t) = \begin{bmatrix} y(t) - 27 \\ u(t) - 0.3 \\ -u(t) - 0.3 \end{bmatrix}, \\ C_0 &= \begin{bmatrix} 1 & 0 \\ 0 & 0.2 \\ 0 & -0.2 \end{bmatrix}, D_0 = \begin{bmatrix} -27 \\ -0.3 \\ -0.3 \end{bmatrix}. \end{aligned}$$

The equilibrium $x_e(r)$ is uniquely given by $x_e(r) = \begin{bmatrix} r & 0 \end{bmatrix}^\top$. Let the set R be $R = \{r \in \mathbf{R} | 0 \leq r \leq 25\}$. Then $x_e(r)$ is stable for all $r \in R$.

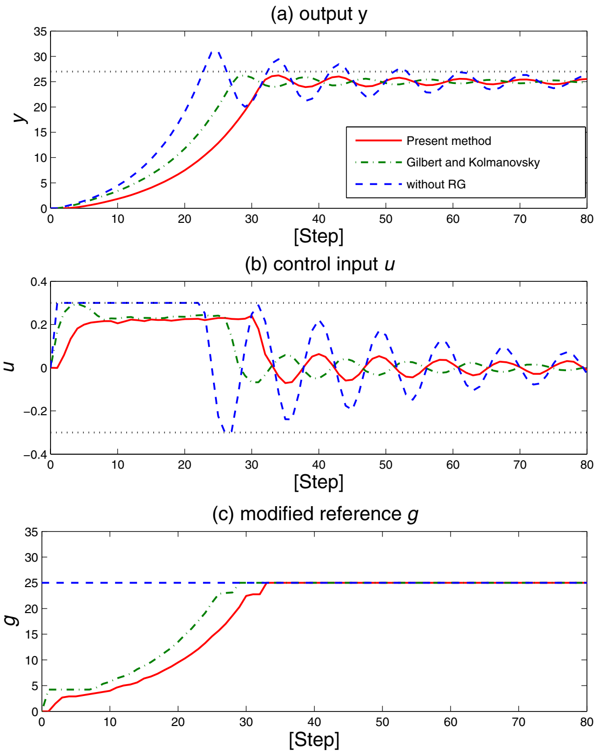

Fig. 3. Simulation result

We construct the function $V(x, r)$ as

$$V(x, r) = (x - x_e(r))^\top P(r)(x - x_e(r)) - 1, \qquad (38)$$

where $P(r)$ is determined by using the method of [6]. Firstly, we compute $S(r_i)$ for $r_i = 0.5i, i = 0, 1, 2, \cdots, 50$ in such a way that it is an inner approximation of the set of constraints admissible initial states for $r(t) \equiv r_i$. Next, $P(r),\ r \in R$ is constructed by the linear interpolation.

The simulation results are shown in Fig. 3, where we set $r = 25$, $x_0 = x_e(0) = \mathbf{0}$, $K_0 = P(0)$, $\epsilon = 10^{-3}$, $\delta = 0.1$, $N = 15$, and we input a random noise v satisfying $|v(t)| \le 0.1\ \forall t \ge 0$. Figs. 3 (a) – (c) illustrate the responses of y, u, g, respectively. The red solid curves show the responses by the present method, blue dashed curves show the responses without a reference governor, green dash-dotted curves show the responses by the method of Gilbert and Kolmanovsky [6] (where we assume that all the states are available), and black dotted curves show the bounds of the constraints. We solve the ROP by the grid search with grid width 10^{-2}.

We see from Fig. 3 that, without a reference governor, the control input is saturated and the constraint on y is violated. In contrast, the present method satisfies both constraints on u and y in spite of the unmeasurable states. Though, by necessity, the response of the modified reference is slow as compared to the direct measurement case, the deterioration is not so large and the modified reference tracks the original one rapidly.

The above numerical computations were preformed over MATLAB running on the PC with Pentium–IV 3.4[GHz] processor. The present algorithm took 0.1756[s] per step on average and there is room for improvement in terms of reduction of the computational time. For example, when we use the bisection method instead of the grid search, the computational time is drastically reduced to 3×10^{-3}[s]. Note that $\lambda^*(t)$ obtained by the bisection method may not be a sub-optimal solution of ROP since the connectedness of the feasible area of ROP is not always assured (in the case of this example, it is assured).

VI. Conclusion

This paper has proposed an output-feedback reference governor for nonlinear systems. The estimation of the state vector is realized by the set-valued observer [9] and the resulting ellipsoidal state existence region is used instead of the true states. Furthermore, the convergence of the states with the present algorithm has been analyzed. We have first considered the noise-free case, and derived a sufficient condition for the asymptotic convergence of the state to the desired equilibrium. Next, we have derived, for noisy case, a sufficient condition for the settling of the state within a neighborhood of the desired equilibrium. The effectiveness of the present method has been shown by a numerical example.

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