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Randomized control of open quantum systems

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Abstract— **The problem of open-loop dynamical control of generic open quantum systems is addressed. In particular, I focus on the task of effectively switching off environmental couplings responsible for unwanted decoherence and dissipation effects. After revisiting the standard framework for dynamical decoupling via deterministic controls, I describe a different approach whereby the controller intentionally acquires a random component. An explicit error bound on worst-case performance of stochastic decoupling is presented.**

I. INTRODUCTION

The need for accurately controlling the dynamics of a quantum-mechanical system is central to a variety of tasks ranging across contemporary physics, engineering, and information sciences [1], [2], [3]. In particular, motivated by both continuous experimental advances in nanoscale devices and the challenge to practically implement fault-tolerant quantum information processing, control strategies for *open* quantum systems undergoing realistic irreversible dynamics [4] play an increasingly prominent role.

Dynamical decoupling techniques offer a versatile control toolbox for open quantum-system engineering [5], [6], [7]. In its essence, a decoupling protocol consists in a sequence of *open-loop* transformations on the target system (control pulses in the simplest setting), designed in such a way that the effect of unwanted dynamics is coherently averaged out in the resulting controlled evolution. Applied to the removal of unwanted couplings between the target system and its surrounding environment, this paves the way to a general strategy for decoherence control and error-suppressed quantum computation purely based on unitary control means.

Both within formulations of the decoupling problem and more general coherent-control settings, the restriction to purely *deterministic* control fields has provided a most natural starting point. In a way, this finds ample justification in the fact that non-deterministic effects (such as stochastic noise and/or random control imperfections) typically deteriorate system performance, motivating the effort for designing intrinsically robust decoupling schemes [8] and for assessing open-loop fault-tolerance thresholds [9]. Yet, no fundamental reasons exist for not lifting such a restriction, by *purposefully allowing stochasticity* in the underlying control design. Beside being conceptually intriguing on its own, it is worth recalling that notable examples may be found of situations where noise and randomness might have a beneficial rather than detrimental effect. Of special relevance are phenomena like the self-averaging of intermolecular interactions in gases and liquids via random microscopic motions [10] and quantum stochastic resonance [11], or the idea of dissipation-assisted quantum computation [12].

A first step toward exploring randomized quantum control was recently taken by Viola and Knill [13], confirming in principle the possibility of enhanced system performance as compared to deterministic control in relevant scenarios. It is the purpose of this paper to further elucidate the random decoupling framework, by first presenting a general control-theoretic formulation and contrast it to the standard deterministic one (Section II), and then discuss in detail a quantitative error bound on stochastic control performance (Section III). Final remarks conclude in Section IV.

II. FORMULATION OF THE CONTROL PROBLEM

A. Quantum-control systems

The standard open-loop control problem for an isolated, *closed* quantum system S defined on a state space $\mathcal{H}_S$ of dimension $d_S < \infty$ is described (in units where $\hbar = 1$) by a bilinear control system of the form [14]

$$
\begin{aligned}
\frac{dU(t)}{dt} &= -i\Big(H_0 + H_c(t)\Big)U(t)\,, \\
H_c(t) &= \sum_{\ell=1}^{m} H_\ell u_\ell(t)\,.
\end{aligned}
\tag{1}
$$

Here, $U(t)$ is the evolution operator (or *propagator*) of the system, whereas $H_0 \equiv H_S$, H_ℓ represent the internal (or *drift*) Hamiltonian, and the applied control Hamiltonians, respectively. Both H_0 and the H_ℓ are Hermitian operators on $\mathcal{H}_S$ which, without loss of generality, may be assumed to be traceless. The time dependence of the overall control Hamiltonian $H_c(t)$ is modeled through the real functions $u_\ell(t)$, which typically represent electromagnetic fields and are the control inputs of the problem. A broad separation between *deterministic* and *stochastic* control systems may be drawn depending on whether each control input is a deterministic function of time or some randomness is allowed for at least one input. The state of S is described in general by a Hermitian, positive operator ρ_S on $\mathcal{H}_S$, normalized with respect to the trace norm in such a way

that $\mathrm{tr}_S(\rho_S) = 1$. In what follows, I will assume that S is initially in a *pure state*, described by a one-dimensional projector π_S of the form $\pi_S = |\psi\rangle\langle\psi|$, with $|\psi\rangle \in \mathcal{H}_S$.

It is convenient to focus directly on the *control propagator* $U_c(t)$ as the basic object for control design,

$$U_c(t) = \mathcal{T}\exp\left\{-i\int_0^t du H_c(u)\right\}, \qquad (2)$$

where the symbol $\mathcal{T}$ denotes as usual time ordering. By effecting a canonical transformation to a time-dependent frame that continuously follows the applied control,

$$\tilde{\rho}_S(t) = U_c^\dagger(t)\rho_S(t)U_c(t), \qquad (3)$$

the explicit action of the control field is removed from the dynamics. The control problem of Eq. (1) takes the form

$$\begin{aligned}
\frac{d\tilde{U}(t)}{dt} &= -i\tilde{H}(t)\tilde{U}(t), \\
\tilde{H}(t) &= U_c^\dagger(t)H_0 U_c(t),
\end{aligned} \qquad (4)$$

in terms of the propagator $\tilde{U}(t)$ for the transformed state,

$$\tilde{\rho}_S(t) = \tilde{U}(t)\tilde{\rho}_S(0)\tilde{U}^\dagger(t), \quad \tilde{U}(t) = U_c^\dagger(t)U(t). \qquad (5)$$

I will refer to the formulations of Eqs. (1), (4) as *physical* and *logical* frame formulations, respectively. While from the mathematical point of view the logical frame description has the disadvantage of being highly non-linear in the control inputs, Eq. (4) makes it very convenient to directly map properties of the desired effective evolution back into design constraints for $U_c(t)$, and viceversa. If the control strategy is *cyclic*, that is $U_c(t + T_c) = U_c(t)$ for $T_c > 0$, and H_0 is time-independent as assumed so far, the periodicity of the control field is transferred to the logical Hamiltonian $\tilde{H}(t)$, and an exact representation of the controlled evolution in terms of *average Hamiltonian theory* exists [6], [10],

$$\tilde{U}(t) = e^{-i\overline{H}t}, \quad \overline{H} = \sum_{\kappa=0}^{\infty} \overline{H}^{(\kappa)}, \qquad (6)$$

each term $\overline{H}^{(\kappa)}$ being computed from the Magnus series for $\tilde{H}(t)$. As it turns out, the logical formulation is also particularly useful in situations where the control strategy directly incorporates *symmetry* criteria.

For a realistic *open* quantum system, the influence of the surrounding environment may modify the dynamics in two important ways. (i) S may couple to a *classical* environment, effectively resulting into a (possibly random) time-dependent modification of the system parameters, in particular $H_S \mapsto H_S(t)$. Deterministic time-dependent quantum control systems have been recently investigated in [15]. (ii) S may couple to a *quantum* environment E, that is a second quantum subsystem defined on a state space $\mathcal{H}_E$ of dimension $d_E \gg d_S$ and characterized by an internal Hamiltonian H_E. Let $\mathbf{I}_{S,E}$ denote the identity operator on $\mathcal{H}_{S,E}$, respectively. The drift Hamiltonian $H_0(t) \equiv H_{SE}(t)$

of a general open quantum system may then be expressed as

$$H_0(t) = H_S(t) \otimes \mathbf{I}_E + \mathbf{I}_S \otimes H_E + \sum_a J_a(t) \otimes B_a, \qquad (7)$$

where the B_a's are linearly independent environment operators and, without loss of generality, we may assume the coupling operators (or *error generators*) to be traceless. In typical situations, both the exact time dependence of $H_S(t)$ and $J_a(t)$, as well as the exact form of H_E, B_a are unknown. If $\rho_{SE}(t)$ denotes the *joint* state of the composite S, E system, the evolution of S alone is now described by the *reduced* state obtained by a partial trace over E,

$$\rho_S(t) = \mathrm{tr}_E(\rho_{SE}(t)). \qquad (8)$$

In general, the evolution of an initially pure state π_S of S under the Hamiltonian (7), followed by (i) the ensemble average over the resulting time histories and/or (ii) the partial trace (8), results in a *mixed* state of S, $\mathrm{tr}(\rho_S^2(t)) < 1$. This implies genuinely *non-unitary*, irreversible dynamics for S, which physically accounts for quantum decoherence and dissipation effects [4].

For an open system, a control problem *formally* similar to (1) may still be formulated for the combined propagator $U(t)$ of S plus E, provided that the action of the controller is explicitly restricted to the *system variables only* that is,

$$H_c(t) \equiv H_c(t) \otimes \mathbf{I}_E, \quad U_c(t) \equiv U_c(t) \otimes \mathbf{I}_E. \qquad (9)$$

Two frame transformations may be relevant in the open system context. The transformation to a logical frame, which explicitly removes the applied control Hamiltonian, is effected as before,

$$\tilde{\rho}_{SE}(t) = U_c^\dagger(t)\rho_{SE}(t)U_c(t), \qquad (10)$$

leading to a control problem formally similar to (4), with

$$\begin{aligned}
\tilde{H}_{SE}(t) &= \left[U_c^\dagger(t)H_S(t)U_c(t)\right] \otimes \mathbf{I}_E + \mathbf{I}_S \otimes H_E + \\
&+ \sum_a \left[U_c^\dagger(t)J_a(t)U_c(t)\right] \otimes B_a.
\end{aligned} \qquad (11)$$

If a formulation which also removes the evolution due to H_E is needed, a simultaneous canonical transformation to a logical interaction frame is effected on the environment variables,

$$\tilde{\rho}_{SE}'(t) = U_E^\dagger(t)\tilde{\rho}_{SE}(t)U_E(t), \quad U_E(t) = e^{-iH_E t}. \qquad (12)$$

The corresponding propagator $\tilde{U}'(t)$ still satisfies an equation similar to (4), where now

$$\begin{aligned}
\tilde{H}_{SE}'(t) &= \left[U_c^\dagger(t)H_S(t)U_c(t)\right] \otimes \mathbf{I}_E + \\
&+ \sum_a \left[U_c^\dagger(t)J_a(t)U_c(t)\right] \otimes \left[U_E^\dagger(t)B_a U_E(t)\right].
\end{aligned} \qquad (13)$$

The various propagators are related to each other as follows:

$$U(t) = U_c(t)\tilde{U}(t) = U_c(t)U_E(t)\tilde{U}'(t). \qquad (14)$$

B. Control tasks and performance indicators

A *dynamical control* problem may be regarded as a steering problem for the evolution operator of the target system in the appropriate frame. For an open system, a task of critical importance is decoherence control, which effectively requires the suppression of the error generators $J_a(t)$. In particular, a *decoupling problem* consists in determining a control configuration $\{H_\ell, u_\ell(t)\}$ such that for a given evolution time $T > 0$ the joint propagator factorizes e.g.,

$$\tilde{U}(T) = \tilde{X}_S(T) \otimes U_E(T) , \qquad (15)$$

in the logical frame, $\tilde{X}_S(T)$ being a unitary operator on S. Notice that Eq. (15) implies decoupling in the physical frame as well. The simplest decoupling objective, on which I will focus henceforth, corresponds to identity design on S (the so-called *no-op* gate in quantum computation terminology [3], or complete decoupling or annihilation in decoupling terminology [6], [7]), whereby

$$\tilde{U}(T) = \mathbf{I}_S \otimes U_E(T) . \qquad (16)$$

If both H_S and the J_a are constant in time, and $U_c(t)$ is periodic, then the logical Hamiltonian (11) is also periodic and the above equation, once fulfilled at time $T = T_c$, remains valid for arbitrary times $T_N = NT_c$, $N \in \mathbf{N}$. Under these conditions, the logical and physical frames overlap for every N, and the controlled evolution reads as

$$\tilde{\rho}_S(T_N) = \rho_S(T_N) = \rho_S(0) = \pi_S = |\psi\rangle\langle\psi| . \qquad (17)$$

Thus, arbitrary initial states of S are *stroboscopically preserved* in both the logical and the physical frames. If either H_S or J_a are time-varying, and/or the control strategy is *acyclic*, it is still meaningful to require that

$$\tilde{\rho}_S(T) = \rho_S(0) = \pi_S , \quad T > 0, \ \forall \pi_S . \qquad (18)$$

For stochastic control, the above objective is further relaxed to *average state preservation in the logical frame* that is,

$$\mathbf{E}\{\tilde{\rho}_S(T)\} = \rho_S(0) = \pi_S , \quad T > 0, \ \forall \pi_S , \qquad (19)$$

with $\mathbf{E}\{\ \}$ denoting ensemble expectation. Clearly, control schemes involving random operations are intrinsically acyclic, the control path practically never returning the system to the physical frame. If, however, the past control trajectory is recorded, this may be exploited to bring the state of S back to the physical frame at any time if desired.

In order to quantify the accuracy of a given control procedure at achieving the intended objective, suitable performance indicators are needed. Let $\pi_S^\perp = \mathbf{I}_S - |\psi\rangle\langle\psi|$ denote the orthogonal complement of π_S in $\mathcal{H}_S$. Then the above task (19) is achieved if and only if, on average, the logical (reduced) state of the system has zero component along $\pi_S^\perp$ (irrespective of the state of the environment). This naturally suggests to consider, for each pure initial state π_S, the following *a priori error probability*,

$$\epsilon_T(\pi_S) = \mathbf{E}\left\{\mathrm{tr}_S\left(\pi_S^\perp \tilde{\rho}_S(T)\right)\right\} . \qquad (20)$$

Note that $\epsilon_T(\pi_S) \geq 0$ for all π_S follows from the fact that both $\pi_S^\perp$ and $\tilde{\rho}_S(T)$ are Hermitian semi-positive definite operators. A *worst-case pure state error probability* may then be defined by maximizing over pure states that is,

$$\epsilon_T = \mathrm{Max}_{\pi_S \in \mathcal{H}_S}\{\epsilon_T(\pi_S)\} . \qquad (21)$$

C. Control assumptions and group-theoretical design

Control design is strongly influenced by the class of available controls. A particularly simple scenario is provided by so-called *quantum bang-bang controls* [5], [6], whereby the control inputs $u_\ell(t)$ are able to be turned on and off impulsively with unbounded strength, so as to implement sequences of effectively instantaneous control pulses. While such idealized assumptions must (and can [8]) be significantly weakened for realistic applications, the bang-bang setting provides the most convenient starting point for discussing stochastic schemes.

Pictorially, it is helpful to visualize a control protocol in terms of the path that $U_c(t)$ follows in the space of unitary transformations on S. For bang-bang controls, such a path is described as a piecewise constant time dependence, with jumps between consecutive values corresponding to the application of an instantaneous control kick. In particular, a large class of decoupling schemes may be obtained by constraining such values to belong to a discrete subgroup $\mathcal{G}$ of unitary operators, the so-called *decoupling group* [6]. Let $\mathcal{G} = \{g_\ell\}$, where g_ℓ, $\ell = 0, \ldots, |\mathcal{G}| - 1$, $g_0 = \mathbf{I}_S$, denote group elements [1]. *Cyclic decoupling according to $\mathcal{G}$ over T_c* is implemented by sequentially steering $U_c(t)$ through each of the $|\mathcal{G}|$ group elements that is,

$$U_c[(j-1)\Delta t + s] = g_j , \quad s \in [0, \Delta t) , \qquad (22)$$

with $\Delta t = T_c/|\mathcal{G}|$ and $j = 1, \ldots, |\mathcal{G}|$. One can prove that, in a *fast control limit* where

$$T_c \to 0 , \ M \to \infty , \quad T = MT_c > 0 , \qquad (23)$$

the leading contribution to the average Hamiltonian resulting from $\tilde{H}_{SE}(t)$ in Eq. (11) is given by

$$\overline{H}_{SE}^{(0)} = \overline{H}_S \otimes \mathbf{I}_E + \mathbf{I}_S \otimes H_E + \sum_a \overline{J}_a \otimes B_a ,$$

$$\overline{X} = \frac{1}{T_c}\int_0^{T_c} dt\, U_c^\dagger(t) X U_c(t) . \qquad (24)$$

The advantage of group-based decoupling scheme is that the above time averages are directly mapped, via Eq. (22), to averages over the control group $\mathcal{G}$, effectively implying a *symmetrization of the controlled dynamics according to $\mathcal{G}$* [6], [17], [16]. If, in particular, the action of $\mathcal{G}$ is *irreducible*, then by Schur's lemma

$$\overline{X} = \frac{1}{|\mathcal{G}|}\sum_{g_\ell \in \mathcal{G}} g_\ell^\dagger X g_\ell = \frac{\mathrm{tr}(X)}{d_S}\mathbf{I}_S = 0 , \qquad (25)$$

---

[1] I am identifying an abstractly defined decoupling group with its image under a *projective* representation in $\mathcal{H}_S$. Loosely speaking, $\mathcal{G}$ is a "group up to phase factors", in general. This is irrelevant for the present discussion.

immediately implying complete decoupling as in Eq. (16).

While cyclic schemes may be very powerful and conceptually simple, they are only applicable (at least in the simple formulation presented here) to time-independent control systems. Also, because averaging requires traversing *all* of $\mathcal{G}$, they tend to become very inefficient as the size of $\mathcal{G}$ grows. The basic idea that underlies *random decoupling according to* $\mathcal{G}$ is to replace sequential cycling with *random sampling over* $\mathcal{G}$. In the simplest kind of protocols, the value of the propagator $U_c(t)$ is determined by a group element which is picked uniformly at random in $\mathcal{G}$ that is,

$$\text{Prob}\,(g_\ell) = \frac{1}{|\mathcal{G}|}\,, \quad \forall g_\ell \in \mathcal{G}\,. \qquad (26)$$

Thus, both the past control operations and the times at which they are effected are known, but the future control path is random. Under these conditions, no average Hamiltonian formulation is viable, and averaging effects emerge through *ensemble* rather than *time* averages,

$$\langle\langle X(t)\rangle\rangle = \mathbf{E}\left\{U_c^\dagger(t)X(t)U_c(t)\right\}\,. \qquad (27)$$

Under the uniformity assumption, such expectation values again reduce to averages over $\mathcal{G}$, leading to the possibility of *stochastic averaging*,

$$\langle\langle X(t)\rangle\rangle = \frac{1}{|\mathcal{G}|}\sum_{g_\ell \in \mathcal{G}} g_\ell^\dagger X(t)g_\ell = 0\,. \qquad (28)$$

The two key questions to address for random decoupling are to understand whether stochastic protocols are indeed capable of achieving decoupling and, if so, how they perform compared to deterministic counterparts. We focus here on the first question, by presenting an explicit derivation of an error bound for randomized control directly within the open-system context [2].

III. RANDOM DECOUPLING

A. General error bounds

We begin by recalling a few preliminary facts.

Remark 3.1: Let $||A||_2 = \text{Max}\,|\text{eig}\,(\sqrt{A^\dagger A})|$ denote the operator 2-norm of A. Then (see e.g. [18])
(i) $||A||_2 = \text{Max}\,|\text{eig}\,(A)|\,, \ \forall A = A^\dagger$;
(ii) $||AB||_2 \le ||A||_2 ||B||_2\,, \ \forall A, B$;
(iii) If U is unitary, $||U^\dagger A U||_2 = ||A||_2\,, \ \forall A$.

Lemma 3.2: Let A be any rank-1 operator on $\mathcal{H}_S$. Then

$$|\text{tr}(A)| \le ||A||_2$$

Proof: A may be represented as $A \simeq |v\rangle\langle v|$, for a d-dimensional complex vector $|v\rangle = [v_1, \ldots, v_d]$ with norm $||v|| = \sum_k |v_k|^2$. Then

$$|\text{tr}(A)| = |v_1| \le ||v|| = \text{Max}\,|\text{eig}\,(\sqrt{|v\rangle\langle v|})| = ||A||_2\,.$$

Q.E.D.

Theorem 3.3: Let S be an open quantum system described by a Hamiltonian of the form (7). Suppose that the control protocol satisfies the following assumptions:
(i) (*Irreducibility*) $\mathcal{G}$ acts irreducibly on $\mathcal{H}_S$.
(ii) (*Uniformity*) $U_c(t)$ is uniformly random for each t.
(iii) (*Independence*) For any $t, s > 0$, $U_c(t)$ and $U_c(t+s)$ are independent for $s > \Delta t$.

If, in addition, the total interaction Hamiltonian is uniformly bounded in time,

$$\left|\left|H_S(t) \otimes \mathbf{I}_E + \sum_a J_a(t) \otimes B_a(t)\right|\right|_2 < k\,, \quad \forall t\,, \qquad (29)$$

then

$$\epsilon_T = O\left(T\Delta t\,k^2\right) \quad \text{for} \quad T\Delta t\,k^2 \ll 1\,. \qquad (30)$$

Proof: Let π_S be an arbitrary pure state of S. The first step is to cast the pure-state error probability (20) in a more convenient form to bound. By purifying the initial state of E if necessary, we may assume that $\rho_{SE}(0) = \pi_S \otimes \pi_E$, both $\pi_{S,E}$ being one-dimensional projectors. By using the definition of partial trace and the cyclicity property of the full trace, we have

$$
\begin{aligned}
\epsilon_T(\pi_S) &= \mathbf{E}\left\{\text{tr}_S\left(\pi_S^\perp \tilde{\rho}_S(T)\right)\right\} & (31)\\
&= \mathbf{E}\left\{\text{tr}_{SE}\left(\pi_S^\perp \otimes \mathbf{I}_E \tilde{\rho}_{SE}(T)\right)\right\}\\
&= \mathbf{E}\left\{\text{tr}_{SE}\left(\pi_S^\perp \otimes \mathbf{I}_E \tilde{U}(T)\pi_S \otimes \pi_E \tilde{U}^\dagger(T)\right)\right\}\\
&= \mathbf{E}\left\{\text{tr}_{SE}\left(\pi_S^\perp \otimes \mathbf{I}_E \tilde{U}'(T)\pi_S \otimes \pi_E \tilde{U}'(T)^\dagger\right)\right\}\,,
\end{aligned}
$$

where the relation (14) has been used, and $U_E(t)$ drops. Let $H'_{SE}(t)$ denote the interaction Hamiltonian of Eq. (29). Then the task is to bound the error in implementing identity design on the logical interaction propagator at time T,

$$\tilde{U}'(T) = \mathcal{T}\exp\left\{-i\int_0^T du\tilde{H}'_{SE}(u)\right\}\,, \qquad (32)$$

with $\tilde{H}'_{SE}(t) = U_c^\dagger(t)H'_{SE}(t)U_c(t)$ given in Eq. (13).
The above propagator may be expressed as follows:

$$\tilde{U}'(T) = \sum_{n=0}^\infty I_n(T)\,, \qquad (33)$$

$$I_n(T) = (-i)^n \int_{0 \le u_1 \ldots \le u_n \le T} d\mathbf{u}\,\tilde{H}'_{SE}(u_n)\ldots\tilde{H}'_{SE}(u_1)\,, \qquad (34)$$

and similarly for $\tilde{U}(T)^\dagger$, with $d\mathbf{u} = du_1 \ldots du_n$. Thus, we need to calculate

$$
\begin{aligned}
\epsilon_T(\pi_S) &= \\
&= \mathbf{E}\left\{\text{tr}_{SE}\left(\sum_{n,m=0}^\infty \pi_S \otimes \pi_E I_m(T)^\dagger \pi_S^\perp \otimes \mathbf{I}_E I_n(T)\right)\right\}\,.
\end{aligned}
$$

The contributions with $n = 0$ or $m = 0$ vanish because of $\pi_S^\perp$ and π_S cancel each other upon exploiting the cyclicity of the trace. Because $\epsilon_T(\pi_S) \ge 0$,

$$
\begin{aligned}
|\epsilon_T(\pi_S)| &\le \\
&\le \sum_{n,m \ge 1}\left|\mathbf{E}\left\{\text{tr}_{SE}\left(\pi_S \otimes \pi_E I_m(T)^\dagger \pi_S^\perp \otimes \mathbf{I}_E I_n(T)\right)\right\}\right|.
\end{aligned}
$$

Under the assumption of sufficiently smooth behavior, the expectation may be moved under the integral. Fix a pair of integers $n, m \geq 1$, then the relevant contribution is

$$\int_{W^{(n,m)}} d\mathbf{u}\, d\mathbf{t}\, \mathbf{E} \left\{ \pi_S \otimes \pi_E \tilde{H}'_{SE}(t_1) \dots \tilde{H}'_{SE}(t_m) \pi_S^\perp \otimes \mathbf{I}_E \right.$$
$$\left. \tilde{H}'_{SE}(u_n) \dots \tilde{H}'_{SE}(u_1) \right\} , \quad (35)$$

where the integration region $W^{(n,m)} = \{(\mathbf{u}, \mathbf{t}) \,|\, 0 \leq u_1 \dots \leq u_n \leq T;\ 0 \leq \dots \leq t_m \leq T\}$. Let $W_1^{(n,m)}(\Delta t) \subset W^{(n,m)}$ denote the subset of points satisfying that u_ℓ, t_ℓ are each time-ordered and *no* u_ℓ or t_ℓ is further away than Δt from the rest, and let $W_2^{(n,m)}(\Delta t) \subset W^{(n,m)}$ denote the remaining region. Because, within $W_2^{(n,m)}(\Delta t)$, at least one of the integrating variables is more than Δt away from all the other variables, the independence assumption (iii) allows the expectation relative to such a variable to be taken separately. By the uniformity assumption (ii) on $U_c(t)$ for all t, and by the tracelessness assumption on $H'_{SE}(t)$ for all t, such an expectation vanishes. Therefore, $W_1^{(n,m)}(\Delta t)$ is the only subset of points contributing to the expectation in Eq. (35). Let $d\mathbf{w}^{(n,m)}$ denote the corresponding integration measure. Then

$$\epsilon_T(\pi_S) \leq \sum_{n,m \geq 1} \int_{W_1^{(n,m)}} d\mathbf{w}^{(n,m)}$$
$$\left| \mathbf{E} \left\{ \mathrm{tr}_{SE} \Big(\pi_S \otimes \pi_E \tilde{H}'_{SE}(t_1) \dots \pi_S^\perp \otimes \mathbf{I}_E \dots \tilde{H}'_{SE}(u_1) \Big) \right\} \right|$$
$$\leq \sum_{n,m \geq 1} \int_{W_1^{(n,m)}} d\mathbf{w}^{(n,m)}$$
$$\mathbf{E} \left\{ \left| \mathrm{tr}_{SE} \Big(\pi_S \otimes \pi_E \tilde{H}'_{SE}(t_1) \dots \pi_S^\perp \otimes \mathbf{I}_E \dots \tilde{H}'_{SE}(u_1) \Big) \right| \right\} ,$$

where in the second step Jensen's inequality has been used. By noticing that the argument of the trace is a rank-1 operator, Lemma 3.2 may be used to simplify

$$\epsilon_T(\pi_S) \leq \sum_{n,m \geq 1} \int_{W_1^{(n,m)}} d\mathbf{w}^{(n,m)}$$
$$\mathbf{E} \left\{ \left\| \pi_S \otimes \pi_E \tilde{H}'_{SE}(t_1) \dots \pi_S^\perp \otimes \mathbf{I}_E \dots \tilde{H}'_{SE}(u_1) \right\|_2 \right\} ,$$
$$\leq \sum_{n,m \geq 1} \mathrm{Vol}(W_1^{(n,m)}) k^{n+m} ,$$

where the inequality (ii) in the Remark 3.1 and the uniform bound k for $H'_{SE}(t)$ in (29) have been used, and $\mathrm{Vol}(W_1^{(n,m)})$ is the volume of $W_1^{(n,m)}$. Note that the dependence upon π_S has disappeared at this point.

The above volume may be estimated through a combinatorial argument. First, notice that given the two ordered lists $0 \leq u_1 \leq \dots \leq u_n \leq T$, $0 \leq t_1 \leq \dots \leq t_m \leq T$, there are $\binom{n+m}{m}$ different merged orderings. Fix a particular one. Then each element needs to be either within Δt of the next one or of the previous one. Make a choice for the odd-numbered elements, the first element being labeled 1. There are at most $2^{\lceil (n+m)/2 \rceil}$ such choices. For each of them the contribution to the volume may be bounded by

ordering the even-numbered elements, then by inserting the odd ones, ignoring the ordering constraint now. Finally,

$$\mathrm{Vol}(W_1^{(n,m)}) \leq \binom{n+m}{m} \frac{T^{\lfloor (n+m)/2 \rfloor} (2\Delta t)^{\lceil (n+m)/2 \rceil}}{(\lfloor (n+m)/2 \rfloor)!}$$
$$\leq 2^{\lceil (n+m)/2 \rceil} T^{\lfloor (n+m)/2 \rfloor} (2\Delta t)^{\lceil (n+m)/2 \rceil}$$
$$\equiv V_{nm} , \quad (36)$$

where the inequalities $\binom{n+m}{m} \leq 2^{n+m-1}$ (for $n + m \geq 2$), and $\lfloor (n+m)/2 \rfloor! \geq 2^{\lfloor (n+m)/2 \rfloor - 1}$ have been exploited.

The last step is to sum over n, m:

$$\mathrm{Max}_{\pi_S}\{\epsilon_T(\pi_S)\} \equiv \epsilon_T \leq \sum_{n,m=1}^\infty V_{nm} k^{n+m} . \quad (37)$$

This may be done by considering separately the four partial sums where both n and m have the same (even or odd) parity, or they have opposite (even-odd or odd-even) parity, respectively, and by evaluating the $\lfloor \rfloor$, $\lceil \rceil$ in Eq. (36) accordingly. Lengthy but straightforward calculations yield

$$\epsilon_T \leq (4T\Delta t k^2) \frac{1 + 8\Delta t k + 4T\Delta t k^2}{(1 - 4T\Delta t k^2)^2} = O(T\Delta t k^2) , \quad (38)$$

for values of $T\Delta t k^2 \ll 1$, as quoted in Theorem 3.3.

Q.E.D.

Remark 3.4: By setting all the coupling operators $J_a = 0$, the error bound for random decoupling of a closed or classically time-dependent control system is obtained.

According to the above Theorem, the performance of stochastic control can be made arbitrarily high by appropriate design, in particular by choosing a sufficiently small Δt in the present setting. Remarkably, this implies the possibility to *arbitrarily suppress on average decoherence in the logical frame*. Note that, unlike deterministic decoupling, stochastic schemes place *no* restriction on the time dependence of $H_0(t)$, only on the maximum eigenvalue of the interaction part, $H'_{SE}(t)$. The latter, however, may diverge in physical situations involving infinite-dimensional environments. Thus, appropriate care is needed to properly define the relevant strength k in such situations [19], [20]. Physically, the parameter k^{-1} is of the order of the *shortest correlation time* present in the interaction to be removed. While this provides the relevant time scale to the purposes of obtaining an *upper* error bound, *lower* or *typical* error bounds may be better in specific situations, depending on the details of both the system and the environment.

B. Example: Control of a single noisy qubit

A simple illustrative example is provided by a single two-state system (a qubit) dissipatively coupled to a quantum reservoir. In this case $\mathcal{H}_S = \mathbf{C}^2$ and a basis for the traceless operators on S is given by the Pauli operators, σ_α, $\alpha = x, y, z$. Consider for simplicity a time-independent open-system dynamics. Eq. (7) takes then the form

$$H_0 = \omega_0 \sigma_z \otimes \mathbf{I}_E + \mathbf{I}_S \otimes H_E + \sum_\alpha \sigma_\alpha \otimes B_\alpha , \quad (39)$$

where σ_z represents the energy eigenbasis of the isolated qubit, and ω_α, B_a are appropriate real parameters and Hermitian environment operators, respectively. Complete decoupling may be achieved in the deterministic setting by cycling the control propagator through a (projectively represented) error group for the qubit[3] that is, $\mathcal{G}_P \simeq \{\mathbf{I}_S, \sigma_x, \sigma_y, \sigma_z\}$. Thus, $T_c = 4\Delta t$, and Eq. (22) yields

$$U_c(t) = \begin{cases} \mathbf{I}_S & t \in \Delta t_1\,, \\ \sigma_x & t \in \Delta t_2\,, \\ \sigma_y & t \in \Delta t_3\,, \\ \sigma_z & t \in \Delta t_4\,. \end{cases}$$

In practice, this corresponds to a series of four equally spaced bang-bang so-called π- (or $180°$-) pulses, alternating between the $\hat{x}$ and $\hat{z}$ axes. In terms of the control inputs introduced in (1), a π-pulse along the α axis may be performed by applying a linearly polarized oscillating field

$$H_\alpha u_\alpha(t) = \sigma_\alpha\, V(t) \cos[\omega(t - t_P)]\,,$$
$$V(t) = V[\theta(t - t_P) - \theta(t - t_P - \tau)]\,, \quad V > 0\,,$$

where $\omega = \omega_0$ on resonance, t_P, τ are the time at which the pulse is applied and its duration, respectively, and $2V\tau = \pi$ with $\tau \to 0$, $V \to \infty$ to satisfy the bang-bang requirement.

For random decoupling over the Pauli group $\mathcal{G}_P$, the control prescription (26) corresponds to applying a sequence of π-pulses with are randomly drawn from $\mathcal{G}_P$ that is, each of the Pauli operators is applied with probability 0.25 at times $t_j = j\Delta t$, $j \in \mathbf{N}$. Physically, the relevant strength parameter k may be associated to the high-frequency cutoff ω_c that is contained in the reservoir power spectrum and determines its frequency response. In general, however, additional time scales related to both ω_0 and the temperature affect the overall control performance. Thus, according to the worst-case bound of Eq. (30), decoherence suppression at time T is achieved provided Δt is made sufficiently small with respect to ω_c^{-1}. Remarkably, an exact solution for the stochastically controlled dynamics may be obtained in the special case where $B_x = B_y = 0$, corresponding to pure decoherence. A detailed analysis of this limiting situation is reported in [21].

IV. CONCLUSION

I have discussed a control-theoretic formulation which explicitly invokes random control design, and which is applicable to arbitrary finite-dimensional, time-dependent open quantum control systems. I focused on random decoupling design for decoherence suppression as a relevant case study, and showed how arbitrarily low error rates may be achieved in principle. Further study is needed to both explore concrete applications of randomized schemes and assess their full potential, as well as to integrate random design within existing control settings. Beside pointing to a still largely unexplored territory in the theory and practice of quantum control, the ideas presented here might allow to

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[3]The abstract decoupling group is $\mathcal{Z}_2 \times \mathcal{Z}_2$ in this case.

take advantage of novel perspectives, as offered for instance by noisy quantum games [22] or randomized algorithms for classical uncertain systems [23]. It is my hope that the results presented here will prompt the control theory community to further investigate the interplay between randomness and coherence in quantum dynamical systems.

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