Proceedings of the
44th IEEE Conference on Decision and Control, and
the European Control Conference 2005
Seville, Spain, December 12-15, 2005

Lyapunov Adaptive Stabilization of Parabolic PDEs— Part I: A Benchmark for Boundary Control

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Abstract— **We develop an adaptive controller for a benchmark parabolic PDE controlled from a boundary and containing an unknown destabilizing parameter affecting the interior of the domain. This design departs from prior approaches that impose relative degree or open-loop stability assumptions, or require domain-wide actuation. An adaptive design for our benchmark plant is a necessary step towards developing controllers for physical systems such as fluid, thermal, and chemical dynamics, where actuation can be only applied non-intrusively, the dynamics are unstable, and the parameters, such as the Reynolds, Rayleigh, Prandtl, or Peclet numbers are unknown because they vary with operating conditions. Our method builds upon our explicitly parametrized control formula in [26] to avoid solving a Riccati or Bezout equation at each time step.**

I. INTRODUCTION

While for linear finite dimensional systems many adaptive schemes have been proposed [8], adaptive control techniques have been developed for only a few classes of PDEs restricted by relative degree, stability, or domain-wide actuation assumptions. In this paper and its companion [16] we develop the first adaptive controllers for parabolic PDEs controlled from a boundary and containing unknown destabilizing parameters affecting the interior of the domain. Our control laws are given by explicit formulae and open the door for the use of a wealth of certainty equivalence and Lyapunov techniques developed for finite dimensional systems. They initiate an effort towards developing adaptive controllers for physical systems such as fluid, thermal, and chemical dynamics, where actuation can be only applied non-intrusively, the dynamics are unstable, and the parameters, such as the Reynolds, Rayleigh, Prandtl, or Peclet numbers are unknown because they vary with operating conditions. Our method builds upon our explicitly parametrized control formulae in [26] to avoid solving Riccati or Bezout equations at each time step.

a) Literature Overview: Early works on adaptive control of infinite-dimensional systems, surveyed by Logemann and Townley [22], were for plants stabilizable by non-identifier based high gain feedback, under a relative degree one assumption. Model reference (MRAC) type schemes were designed by Hong and Bentsman [7], Bohm, Demetriou, Reich, and Rosen [2], Solo and Bamieh [30], Orlov [23], and Bentsman and Orlov [1]. While the strength of these results are the proofs of identifiability of infinite dimensional parameter vectors, their limitation is that they require control action throughout the PDE domain. Other efforts such as Demetriou and Ito [5] and Wen and Balas [32] have relied on positive realness assumptions. Adaptive linear quadratic control with least-squares estimation was pursued by Duncan, Maslowski, and Pasik-Duncan [6] for linear stochastic evolution equations with unbounded input operators and exponentially stable dynamics. Adaptive control of nonlinear PDEs has also received some attention. Liu and Krstic [19] and Kobayashi [11] considered a Burgers equation with various parametric uncertainties; Kobayashi [13] also considered the Kuramoto-Sivashinsky equation. Jovanovic and Bamieh [9] designed adaptive controllers for nonlinear systems on lattices, which include applications like infinite vehicular platoons or infinite arrays of microcantilevers. An experimentally validated adaptive boundary controller for a flexible beam was presented by de Queiroz, Dawson, Agarwal, and Zhang [4].

b) The Results of the Paper: For several unstable parabolic PDE systems controlled from the boundary we assume that physical parameters like reaction, diffusion, or advection coefficients are unknown. No adaptive controllers for such models have been proposed, even though they are frequent in applications that incorporate thermal-fluid or chemically reacting dynamics. An obstacle to the development of adaptive controllers has been the lack of parametrized families of nonadaptive controllers. This obstacle was removed by Smyshlyaev and Krstic [26] who developed explicit formulae for boundary control of a class of parabolic PDEs that includes the problems considered here. Those formulae are not only explicit functions of the spatial coordinates of the PDE, but also depend explicitly on the physical parameters of the plant. This feature is absent from standard methods like LQR extensions to PDEs because parametrized solutions to Riccati equations cannot be obtained. While an adaptive version of an LQR approach would require a solution to a high-dimensional Riccati equation at each time step, our approach only requires that new parameter updates be plugged into the control formula.

For clarity, we present results for scalar and vector parameter problems. They can be extended to functional parameters as in [1], [2], [7], [23], [30]. This is illustrated briefly in Section VII and is a topic of a future paper [28]. With the controllers parametrized in the physical parameters, our schemes are of *indirect* type.

Three basic approaches to the design of parameter identifiers for adaptive control exist [17]: the Lyapunov approach, the passivity-based approach (pursued in [1], [2], [7], [23]), and the swapping approach. The Lyapunov approach, which ensures the best transient performance properties is seldom possible without changing the control law to compensate the potentially destabilizing effect of adaptation, even in the linear case. We exploit the structural opportunities within the class of PDEs we are considering and develop Lyapunov adaptation schemes.

Our Lyapunov design is inspired by an idea Praly [24] developed for adaptive nonlinear control under growth conditions. Since our PDE problems are linear, we have found a way to significantly simplify this approach, however, we retain its main feature—a logarithm weight on the plant state in the Lyapunov function. This results in a normalization of the update law by a norm on the plant state, which is uncommon for Lyapunov designs.

To avoid tedium and keep the concepts clear we present designs for the simplest classes of systems for which the concepts are nontrivial. We start in this paper with a benchmark reaction-diffusion problem with only the destabilizing reaction coefficient unknown. Then, in [16] it is shown how to deal with parametric uncertainties in boundary conditions or reaction terms involving boundary values. Finally, in [16] a solutions is shown to a reaction-advection-diffusion problem with all three coefficients unknown. A skilled designer can combine these tools to craft solutions to more general problems.

c) Notation: The spatial $L_2(0,1)$ norm is denoted by $\|\cdot\|$. The symbols $I_1(\cdot), I_2(\cdot), J_1(\cdot)$, etc., denote the corresponding Bessel functions.

II. Control Design for a System with an Unknown Reaction Coefficient

We start the paper with a design for a benchmark system and present extensions in subsequent sections. The benchmark system has a destabilizing reaction term and employs control only at the boundary. The unknown reaction coefficient is scalar, however, an extension to spatially-varying functional coefficients is discussed in Section VII. A problem with multiple parameters is also discussed in [16].

While this paper assumes availability of full state feedback, [16] presents designs that employ only boundary sensing.

Consider the following plant

$$u_t(x,t) = u_{xx}(x,t) + \lambda u(x,t)\,, \tag{1}$$

$$u(0,t) = 0\,, \tag{2}$$

where λ is an unknown constant parameter that can have any real value. High values of λ lead to instability for either boundary conditions $u(1,t)=0$ or $u_x(1,t)=0$. We use a Neumann boundary controller designed in [26] in the form[1]

$$u_x(1) = -\frac{\hat\lambda}{2}u(1) - \hat\lambda\int_0^1 \xi\frac{I_2\left(\sqrt{\hat\lambda(1-\xi^2)}\right)}{1-\xi^2}u(\xi)d\xi\,, \tag{3}$$

which employs the measurements of $u(x)$ for $x\in[0,1]$ and an estimate $\hat\lambda$ of λ. Consider an invertible change of variable

$$w(x) = u(x) - \int_0^x k(x,\xi,\hat\lambda)u(\xi)\,d\xi\,, \tag{4}$$

$$k(x,\xi,\hat\lambda) = -\hat\lambda\xi\frac{I_1\left(\sqrt{\hat\lambda(x^2-\xi^2)}\right)}{\sqrt{\hat\lambda(x^2-\xi^2)}}\,. \tag{5}$$

[1]In the sequel, to reduce notational overload, the dependence on time will be suppressed whenever possible.

The transformation (4) maps (1)–(3) into [29]

$$w_t = w_{xx} + \dot{\hat\lambda}\int_0^x \frac{\xi}{2}w(\xi)\,d\xi + \tilde\lambda w\,, \tag{6}$$

$$w(0) = 0\,, \tag{7}$$

$$w_x(1) = 0\,, \tag{8}$$

where $\tilde\lambda = \lambda - \hat\lambda$ is the parameter estimation error.

We will show that the update law

$$\dot{\hat\lambda} = \gamma\frac{\|w\|^2}{1+\|w\|^2}\,, \qquad 0<\gamma<1 \tag{9}$$

achieves regulation of $u(x,t)$ to zero for all $x\in[0,1]$, for arbitrarily large initial data $u(x,0)$ and for an arbitrarily poor initial estimate $\hat\lambda(0)$.

Theorem 1: Suppose that the system (1)–(3), (9) has a well defined classical solution for all $t\ge 0$. Then, for any initial condition $u_0\in H_1$ and any $\hat\lambda(0)\in\mathbb{R}$, the solutions $u(x,t)$ and $\hat\lambda(t)$ are uniformly bounded and $\lim_{t\to\infty}u(x,t)=0$ for all $x\in[0,1]$. Moreover, the following performance bounds hold in the closed-loop nonlinear system:

$$\begin{aligned} u(x,t)^2 \le\ & 32\left(1+3\lambda^2+\tilde\lambda(0)^2+\gamma\log\left(1+\|w(0)\|^2\right)\right) \\ & \times\left[\|w_x(0)\|^2 + 3\sqrt{\gamma}\left(1+\|w(0)\|^2\right)\mathrm{e}^{\frac{1}{\gamma}\tilde\lambda(0)^2}\right. \\ & \left.\times\left(\log\left(1+\|w(0)\|^2\right)+\frac{1}{\gamma}\tilde\lambda(0)^2\right)^{3/2}\right] \end{aligned} \tag{10}$$

for all $x\in[0,1]$, $t\ge 0$, and

$$\begin{aligned} &\int_0^\infty u(x,t)^2dt \le \\ &48\left(1+3\lambda^2+\tilde\lambda(0)^2+\gamma\log\left(1+\|w(0)\|^2\right)\right) \\ &\times\left(1+\|w(0)\|^2\right)\mathrm{e}^{\frac{1}{\gamma}\tilde\lambda(0)^2} \\ &\times\left(\log\left(1+\|w(0)\|^2\right)+\frac{1}{\gamma}\tilde\lambda(0)^2\right) \end{aligned} \tag{11}$$

for all $x\in[0,1]$.

Remark 1: While the bound (10) obviously quantifies the "peak transient" performance, the bound (11) quantifies the rate of convergence to zero.

Remark 2: The non-negative form of the adaptive law (9) is coincidental for this particular benchmark plant and it is further discussed in Section V.

Remark 3: It is also important to note that the update law (9) contains normalization. Normalization is uncommon in Lyapunov designs and is the result of including the logarithm in the Lyapunov function [24]. Normalization is necessary because the control law (3) is of certainty equivalence type—unlike the Lyapunov adaptive controllers in [17] which employ non-normalized adaptation and strengthened nonlinear controllers that compensate for time-varying effects of adaptation. An additional measure of preventing overly fast adaptation in (9) is the restriction on the adaptation gain ($\gamma<1$).

III. Proof of Theorem 1

Consider a Lyapunov function candidate

$$V = \frac{1}{2}\log\left(1+\|w\|^2\right) + \frac{1}{2\gamma}\tilde{\lambda}^2 . \qquad (12)$$

The time derivative along the solutions of (6)–(9) can be shown to be

$$\dot{V} = -\frac{\|w_x\|^2}{1+\|w\|^2} + \frac{\dot{\hat{\lambda}}}{2}\frac{\int_0^1 w(x)\left(\int_0^x \xi w(\xi)d\xi\right)dx}{1+\|w\|^2} \qquad (13)$$

(the calculation involves one step of integration by parts). Using the Cauchy-Schwartz inequality twice we obtain the following sequence of inequalities:

$$\begin{aligned}
&\left|\int_0^1 w(x)\left(\int_0^x \xi w(\xi)d\xi\right)dx\right| \\
&\leq \int_0^1 |w(x)|\left(\int_0^x \xi|w(\xi)|d\xi\right)dx \\
&\leq \int_0^1 |w(x)|\left(\int_0^x \xi^2 d\xi\right)^{1/2}\left(\int_0^x w(\xi)^2 d\xi\right)^{1/2}dx \\
&\leq \|w\|\int_0^1 |w(x)|\frac{1}{\sqrt{3}}x^{3/2}dx \\
&\leq \frac{\|w\|}{\sqrt{3}}\|w\|\left(\int_0^1 x^3 dx\right)^{1/2} \\
&\leq \frac{1}{2\sqrt{3}}\|w\|^2 . \qquad (14)
\end{aligned}$$

Using Poincare's inequality, one gets

$$\left|\int_0^1 w(x)\left(\int_0^x \xi w(\xi)d\xi\right)dx\right| \leq \frac{2}{\sqrt{3}}\|w_x\|^2 . \qquad (15)$$

Substituting this inequality and (9) into (13), we get

$$\dot{V} \leq -\left(1-\frac{\gamma}{\sqrt{3}}\right)\frac{\|w_x\|^2}{1+\|w\|^2} . \qquad (16)$$

This implies that $V(t)$ remains bounded for all time whenever $0<\gamma\leq\sqrt{3}$. From the definition of V it follows that $\|w\|$ and $\hat{\lambda}$ remain bounded for all time. However, we need to show that $w(x,t)$ is bounded for all time and for all x. To do this, consider

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|w_x\|^2 &= \int_0^1 w_x w_{xt}\,dx = -\int_0^1 w_{xx}w_t\,dx \\
&= -\int_0^1 w_{xx}^2\,dx - \tilde{\lambda}\int_0^1 w_{xx}w dx \\
&\quad -\frac{\dot{\hat{\lambda}}}{2}\int_0^1 w_{xx}\int_0^x \xi w(\xi)\,d\xi \\
&= -\|w_{xx}\|^2 + \tilde{\lambda}\int_0^1 w_x^2 dx \\
&\quad +\frac{\dot{\hat{\lambda}}}{2}\int_0^1 xww_x dx \\
&= -\|w_{xx}\|^2 + \tilde{\lambda}\|w_x\|^2 \\
&\quad +\frac{\dot{\hat{\lambda}}}{4}\left(w(1)^2-\|w\|^2\right) . \qquad (17)
\end{aligned}$$

Integration by parts was used several times to obtain the above equalities. Using Agmon's inequality (noting that $w(0)=0$), then Young's inequality, and finally Poincare's inequality (noting that $w_x(1)=0$), one gets that

$$w(1)^2 - \|w\|^2 \leq \|w_x\|^2 \leq 4\|w_{xx}\|^2 . \qquad (18)$$

Substituting (18) into (17), it follows that

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|w_x\|^2 &\leq -(1-\gamma)\|w_{xx}\|^2 + \tilde{\lambda}\|w_x\|^2 \\
&\leq \tilde{\lambda}\|w_x\|^2 . \qquad (19)
\end{aligned}$$

Integrating the last inequality, we obtain

$$\begin{aligned}
\|w_x(t)\|^2 &\leq \|w_x(0)\|^2 \\
&\quad +2\sup_{0\leq\tau\leq t}|\tilde{\lambda}(\tau)|\int_0^t \|w_x(\tau)\|^2 d\tau . \qquad (20)
\end{aligned}$$

To obtain this bound, on one hand we have from (12) and (16) that

$$\tilde{\lambda}(t)^2 \leq \tilde{\lambda}(0)^2 + \gamma\log\left(1+\|w(0)\|^2\right) . \qquad (21)$$

On the other hand,

$$\begin{aligned}
&\int_0^t \|w_x(\tau)\|^2 d\tau \\
&\leq \sup_{0\leq\tau\leq t}\left(1+\|w(\tau)\|^2\right)\int_0^t \frac{\|w_x(\tau)\|^2}{1+\|w(\tau)\|^2}d\tau . \qquad (22)
\end{aligned}$$

From (12) and (16) it follows that

$$1+\|w(\tau)\|^2 \leq \left(1+\|w(0)\|^2\right)\mathrm{e}^{\frac{1}{\gamma}\tilde{\lambda}(0)^2} . \qquad (23)$$

Integrating (16) we get

$$\begin{aligned}
&\int_0^t \frac{\|w_x(\tau)\|^2}{1+\|w(\tau)\|^2}d\tau \\
&\leq \frac{1}{2\left(1-\frac{\gamma}{\sqrt{3}}\right)}\left(\log\left(1+\|w(0)\|^2\right)+\frac{1}{\gamma}\tilde{\lambda}(0)^2\right) . \qquad (24)
\end{aligned}$$

Substituting (23) and (24) into (22), and then, along with (21), into (20), we get

$$\begin{aligned}
\|w_x(t)\|^2 &\leq \|w_x(0)\|^2 \\
&\quad +\frac{\sqrt{\gamma}}{1-\frac{\gamma}{\sqrt{3}}}\left(1+\|w(0)\|^2\right)\mathrm{e}^{\frac{1}{\gamma}\tilde{\lambda}(0)^2} \\
&\quad \times\left(\log\left(1+\|w(0)\|^2\right)+\frac{1}{\gamma}\tilde{\lambda}(0)^2\right)^{3/2} . \qquad (25)
\end{aligned}$$

By combining Agmon's and Poincare's inequalities (and using the fact that $w(0)=0$), we get $\max_{x\in[0,1]}|w(x)|^2 \leq 4\|w_x\|^2$, thus $w(x,t)$ is uniformly bounded.

Next, we prove regulation of $w(x,t)$ to zero. Using (6)–(8) and (14) we obtain

$$\frac{1}{2}\left|\frac{d}{dt}\|w\|^2\right| \leq \|w_x\|^2 + \left(|\tilde{\lambda}|+\frac{\gamma}{4\sqrt{3}}\right)\|w\|^2 . \qquad (26)$$

Since $\|w\|$ and $\|w_x\|$ have been proven bounded, it follows that $\frac{d}{dt}\|w\|^2$ is bounded, and thus $\|w(t)\|$ is uniformly continuous. By combining (22)–(24) with Poincare's inequality we also get that $\|w(t)\|^2$ is integrable in time over the infinite time interval. By Barbalat's lemma it follows that $\|w(t)\| \to 0$ as $t \to \infty$.

To show regulation also in the maximum norm, we note that, from Agmon's inequality, $|w(x,t)|^2 \leq 2\|w(t)\|\|w_x(t)\|$. Since $\|w_x\|$ is bounded and $\|w(t)\|$ has been shown convergent to zero, the regulation in maximum norm follows.

Having proved the boundedness and regulation of w, we now set out to establish the same for u. We start by noting that [26]

$$u(x) = w(x) + \int_0^x l(x,\xi,\hat{\lambda})w(\xi)d\xi\,, \tag{27}$$

where

$$l(x,\xi,\hat{\lambda}) = -\hat{\lambda}\xi \frac{J_1\left(\sqrt{\hat{\lambda}(x^2-\xi^2)}\right)}{\sqrt{\hat{\lambda}(x^2-\xi^2)}}\,. \tag{28}$$

It is straightforward to show that

$$\|u_x\|^2 \leq 2\left(1+\hat{\lambda}^2+4M\right)\|w_x\|^2\,, \tag{29}$$

where

$$M = \int_0^1 \left(\int_0^1 \left|l_x(x,\xi,\hat{\lambda})\right| d\xi\right)^2 dx \tag{30}$$

and

$$l_x(x,\xi,\hat{\lambda}) = \hat{\lambda}x\xi \frac{J_2\left(\sqrt{\hat{\lambda}(x^2-\xi^2)}\right)}{x^2-\xi^2}\,. \tag{31}$$

By mimicking the calculation in [26, Equation (101)], we get $\int_0^1 \left|l_x(x,\xi,\hat{\lambda})\right| d\xi \leq |\hat{\lambda}|x+1$, which implies

$$M \leq \int_0^1 \left(|\hat{\lambda}|x+1\right)^2 dx = \frac{1}{3}\hat{\lambda}^2 + |\hat{\lambda}| + 1 \leq \frac{\hat{\lambda}^2+3}{2}\,. \tag{32}$$

Thus, it follows that

$$\begin{aligned} \|u_x\|^2 &\leq 2\left(4+3\hat{\lambda}^2\right)\|w_x\|^2 \\ &\leq 8\left(1+3\lambda^2+\tilde{\lambda}^2\right)\|w_x\|^2\,. \end{aligned} \tag{33}$$

Noting that $u(x,t)^2 \leq 4\|u_x\|^2$ for all $(x,t) \in [0,1]\times[0,\infty)$, by combining (33), (21), and (25), and using the fact that $\frac{1}{1-\frac{\gamma}{\sqrt{3}}} < 3$ for $\gamma < 1$, we get (10), which proves uniform boundedness of u.

To prove regulation of $u(x,t)$ to zero for all $x \in [0,1]$, we start by noting that

$$\|u\|^2 \leq 2(1+L)\|w\|^2 \tag{34}$$

where

$$L = \max_{0\leq\xi\leq x\leq 1} l(x,\xi,\hat{\lambda})^2 \tag{35}$$

is finite whenever $\hat{\lambda}$ is finite (which we have proved using Lyapunov analysis). Since $\|w\|$ is regulated to zero, so is $\|u\|$. By Agmon's inequality $u(x,t)^2 \leq 2\|u\|\|u_x\|$, where $\|u_x\|$ is bounded by (33), (21), and (25). This completes the proof of regulation of u.

The bound (11) is obtained in a similar manner to (10), by combining (33) with (21)–(24).

IV. Well Posedness

Since the purpose of our paper is stabilization, we focus our effort on proving boundedness and regulation. As evident from Section III, this is not a routine task due to the nonlinear character of the closed-loop system

$$w_t = w_{xx} + \frac{\gamma}{2}\frac{\|w\|^2}{1+\|w\|^2}\int_0^x \xi w(\xi)\,d\xi + \tilde{\lambda}w \tag{36}$$

$$w(0) = w_x(1) = 0\,, \tag{37}$$

$$\dot{\hat{\lambda}} = -\gamma\frac{\|w\|^2}{1+\|w\|^2}\,. \tag{38}$$

The analysis of existence and uniqueness of solutions is even more involved. One of the steps in proving *global* existence and uniqueness of *classical* solutions is to prove boundedness of $w_t(t,x)$ and $w_{xx}(t,x)$, which proceeds as follows. It is first observed from the first line of (19) that $\|w_{xx}\|$ is square integrable over infinite time. The same property holds for $\|w_t\|$. It is then shown that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|w_t\|^2 + \|w_{tx}\|^2 = \\ &\tilde{\lambda}\|w_t\|^2 + \frac{\ddot{\hat{\lambda}}}{2}\int_0^1 w_t(x)\int_0^x \xi w(\xi)d\xi dx \\ &+\dot{\hat{\lambda}}\int_0^1 w_t(x)\left(\int_0^x \frac{\xi}{2}w_t(\xi)d\xi - w(x)\right)dx \end{aligned} \tag{39}$$

and

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|w_{tx}\|^2 + \|w_{txx}\|^2 = \\ &\tilde{\lambda}\|w_{tx}\|^2 + \frac{\ddot{\hat{\lambda}}}{2}\int_0^1 x w_{tx}(x)w(x)dx \\ &+\dot{\hat{\lambda}}\int_0^1 w_{tx}(x)\left(\frac{x}{2}w_t(x) - w_x(x)\right)dx\,, \end{aligned} \tag{40}$$

where

$$\ddot{\hat{\lambda}} = \frac{\gamma}{(1+\|w\|^2)^2}\frac{d}{dt}\|w\|^2 \tag{41}$$

is bounded because of (26). From the boundedness of $\|w\|, \|w_x\|, \dot{\hat{\lambda}}, \ddot{\hat{\lambda}}$ and the square integrability in time of $\|w\|, \|w_t\|$, by integrating (39) it follows that $\|w_t\|$ is bounded and $\|w_{tx}\|$ is square integrable. Then, by integrating (40) and using the square integrability of $\|w_x\|$ and the other functions mentioned above, it follows that $\|w_{tx}\|$ is bounded and $\|w_{txx}\|$ is square integrable. By Agmon's inequality, we get that $w_t(t,x)$ is uniformly bounded for all values of its arguments, and the same holds for $w_{xx}(t,x)$. Those properties are also valid in the original variable $u(t,x)$ using the smoothly invertible variable change (4)–(5).

Existence and uniqueness of appropriately defined *weak* solutions can be studied in the same way as in [19, Section 4]. One writes the system in the form of two integral equations, using the "heat equation" Green function for the PDE for w, and then applies the Banach fixed point theorem. The main difference in using that idea here would be that the Green function used in [19] was for Neumann boundary conditions at both ends, whereas in our case one boundary condition is Dirichlet and the other is Neumann, which would necessitate a slightly different Green function.

V. Parametric Robustness

Let us suppose that the adaptation is turned off, i.e., $\gamma = 0$, i.e., $\dot{\hat{\lambda}} \equiv 0$. Then the closed loop system is

$$w_t = w_{xx} + \left(\lambda - \hat{\lambda}\right) w \quad (42)$$

with boundary conditions $w(0) = 0, w_x(1) = 0$, where $\hat{\lambda}$ is constant. It can be shown that $\hat{\lambda} > \lambda - \frac{\pi^2}{4}$ is exponentially stabilizing, whereas $\hat{\lambda} < \lambda - \frac{\pi^2}{4}$ is destabilizing. Thus, if an upper bound on λ is known—denote it by $\bar{\lambda}$—then (3) is a stabilizing linear controller whenever $\hat{\lambda} \geq \bar{\lambda}$.

This robustness property explains why $\dot{\hat{\lambda}}$ (9) is nonnegative: overestimating $\hat{\lambda}$ cannot be harmful within the controller structure (3).[2] A caveat is that, in the presence of noise, $\hat{\lambda}$ will drift. In the update law (9) the estimate has nowhere to drift but up[3]. In implementation one would add leakage, deadzone, or projection [8] to prevent drift.

The frozen-parameter robustness is an unusual feature of (3). It is different than the "infinite gain margin" of inverse optimal controllers, which allow an arbitrary increase of a gain *in front of* the control law. Optimal controllers are not robust to changes in the plant parameter λ, whereas (3) is.

Viewing (3) as "high-gain" would be incorrect because it resorts to high gain only when λ generates a high number of unstable eigenvalues in the plant.

The form of gain that controller (3) is capable of employing should not be confused with adaptive high gain [22] where a multiplicative gain is tuned for a controller

$$u_x(1) = G\{Cu\} \quad (43)$$

where G is the gain and C is an operator such that $u_x(1) \mapsto Cu$ is relative degree one. For the present system, C independent of the unknown λ cannot be found, therefore, tuning of a multiplicative gain G could not be successful.

VI. An Alternative Approach

The use of a log in the Lyapunov function (12) was inspired by Praly's Lyapunov design in [24]. We do not follow it exactly because our PDEs are linear. It is however of interest to see what it results in, as it may have potential beyond our class of problems.

Let us start by denoting

$$A = \frac{\int_0^1 w(x)\left(\int_0^x \xi w(\xi) d\xi\right) dx}{1 + \|w\|^2} \quad (44)$$

$$B = 2\frac{A}{1 + \|w\|^2} \quad (45)$$

$$H = -A^2 + \frac{1}{1 + \|w\|^2}\int_0^1 \left(\left(\int_0^x \xi w(\xi) d\xi\right)^2 + w(x)\left(\int_0^x \xi\left(\int_0^\xi w(\eta) d\eta\right) d\xi\right)\right) dx\,. \quad (46)$$

This method employs two estimates working in tandem, $\hat{\lambda}$ and $\hat{\theta}$. A long Lyapunov based derivation, briefly justified after the statement of the theorem below, yields

$$\begin{aligned}\dot{\hat{\lambda}} = \gamma \frac{\beta\gamma}{\beta\gamma(1-\gamma H) - 1} \times \Bigg(& \frac{\frac{3}{2}\|w\|^2 + 2A\|w_x\|^2}{1 + \|w\|^2} \\ & - \left(\left(1 + \frac{1}{\gamma^2}\right) - \frac{1}{\beta\gamma^2}\right)\beta\gamma B\left(\hat{\lambda} - \hat{\theta} - \gamma A\right) \\ & - \sigma\left(\hat{\lambda} - \hat{\theta} - \gamma A\right)\Bigg)\end{aligned} \quad (47)$$

$$\dot{\hat{\theta}} = \gamma\left(\frac{2\|w\|^2}{1 + \|w\|^2} - \beta\gamma B\left(\hat{\lambda} - \hat{\theta} - \gamma A\right)\right). \quad (48)$$

We have written the two update laws in a way to highlight the similarities. The gains need to satisfy

$$\gamma < 3 \quad (49)$$

$$\beta > \frac{1}{\gamma(1 - \frac{\gamma}{3})} \quad (50)$$

$$\sigma > 0\,. \quad (51)$$

The conditions (49) and (50) are related to the fact that $|H| \leq \frac{1}{3}$.[4] These conditions ensure that the denominator in the first line of (47) remains positive.

Besides its complexity, a disadvantage of the update law (47) is that it employs $\|w_x\|$, i.e., it requires the measurement of the spatial derivative $u_x(x,t)$.

Theorem 2: Suppose that the system (1)–(3), (47), (48) has a well defined classical solution for all $t \geq 0$. Then, for any initial condition $u_0 \in L_2$ and any $\hat{\lambda}(0), \hat{\theta}(0) \in \mathbb{R}$, the spatial L_2 norm $\|u(t)\|$ remains bounded and the spatial H_1 norm $\|u_x(t)\|$ is square integrable over an infinite time interval. Moreover, the estimates $\hat{\lambda}(t), \hat{\theta}(t)$ are bounded.

The proof of this result employs a Lyapunov function

$$\begin{aligned}V = & \frac{\beta\gamma^2}{2}\frac{\beta\gamma + 1}{\beta\gamma - 1} + \log\left(1 + \|w\|^2\right) \\ & - \frac{1}{2\gamma}\left(\hat{\lambda} - \hat{\theta}\right)^2 + \frac{1}{2\gamma}\left(\lambda - \hat{\theta}\right)^2 \\ & + \frac{\beta}{2}\left(\hat{\lambda} - \hat{\theta} - \gamma A\right)^2.\end{aligned} \quad (52)$$

It is possible to prove that

$$V \geq \log\left(1 + \|w\|^2\right) + \frac{1}{2\gamma}\left(\left(\hat{\lambda} - \hat{\theta}\right)^2 + \frac{\beta\gamma - 1}{2}\left(\lambda - \hat{\theta}\right)^2\right), \quad (53)$$

i.e., V is positive definite around the equilibrium $w(x) \equiv 0, \hat{\lambda} = \hat{\theta} = \lambda$. Then, a *very* long calculation yields

$$\dot{V} = -2\frac{\|w_x\|^2}{1 + \|w\|^2}\,. \quad (54)$$

The properties stated in Theorem 2 readily follow.

[2]While the update law (9) can take the estimate $\hat{\lambda}$ only "up," the growth of the estimate stops as $\|w(t)\|$ goes to zero. Since $V(t)$ is nonincreasing and bounded from below (by zero), it has a limit. Hence $\tilde{\lambda}(t)^2$ has a limit. So does $\hat{\lambda}(t)$ and it is higher than $\lambda - \frac{\pi^2}{4}$. The size of $\hat{\lambda}(\infty)$ depends on the size of the initial condition u_0.

[3]This issue is no less critical with update laws that are sign-indefinite, however, with (9) it is obvious.

[4]A fairly obvious bound is $|H| \leq 3$ but a careful calculation in the vein of (14) can establish a tighter bound $|H| \leq \frac{1}{3}$.

VII. Extension to Spatially Dependent Coefficients

In the present paper we have considered only parameters without spatial variation. In a future paper [28] we will present an extension to spatially-varying problems [1], [2], [7], [23], [30]. For example, the design for the benchmark

$$u_t = u_{xx} + \lambda u \tag{55}$$

can be generalized to the plant

$$u_t = u_{xx} + \lambda(x) u \tag{56}$$

where $\lambda(x)$ is continuous. We design the adaptive controller

$$\begin{aligned} u(1) &= \int_0^1 \hat{k}(1,\xi) u(\xi) d\xi & (57) \\ \hat{\lambda}_t(t,x) &= \gamma \frac{u(t,x)\left(w(t,x) - \int_x^1 \hat{k}(\xi,x) w(t,\xi) d\xi\right)}{1+\|w(t)\|^2} & (58) \end{aligned}$$

where $\hat{\lambda}(t,x)$ is the online functional estimate of $\lambda(x)$, the state transformation is given by

$$w(x) = u(x) - \int_0^1 \hat{k}(x,\xi) u(\xi) d\xi \,,$$

and the kernel

$$\hat{k}(x,\xi) = \hat{k}_n(x,\xi)$$

is obtained recursively from

$$\begin{aligned} \hat{k}_0(x,\xi) &= -\frac{1}{2}\int_{\frac{x-\xi}{2}}^{\frac{x+\xi}{2}} \hat{\lambda}(\zeta)\, d\zeta & (59) \\ \hat{k}_{i+1}(x,\xi) &= \hat{k}_i(x,\xi) \\ &\quad + \int_{\frac{x-\xi}{2}}^{\frac{x+\xi}{2}} \int_0^{\frac{x-\xi}{2}} \hat{\lambda}(\zeta-\sigma)\, \hat{k}_i(\zeta+\sigma, \zeta-\sigma) \\ &\quad \times d\sigma d\zeta & (60) \end{aligned}$$

for each new update of $\hat{\lambda}(t,x)$. Stability is guaranteed for sufficiently small γ and sufficiently high n. The recursion (60) is convergent [18]. Several methods for its symbolic or numerical computation were proposed and illustrated in [26], noting that the computational effort is at least an order of magnitude lower than solving a Riccati equation.

Acknowledgements

I would like to thank Laurent Praly, Andrey Smyshlyaev, Petar Kokotovic, and Yuri Orlov for helpful suggestions, and Brian Anderson and Steve Morse for stimulating discussion. The work was supported by NSF grant CMS-0329662.

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