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Time Minimal Trajectories for two-level Quantum Systems with Drift

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Abstract— **On a two-level quantum system driven by an external field, we consider the population transfer problem from the first to the second level, minimizing the time of transfer, with bounded field amplitude. On the Bloch sphere (i.e. after a suitable Hopf projection), this problem can be attacked with techniques of optimal syntheses on 2-D manifolds.**

Let $(-E, E)$ be the two energy levels, and $|\Omega(t)| \leq M$ the bound on the field amplitude. For each values of E and M, we provide the explicit expression of the time optimal trajectory steering the state one to the state two in terms of a parameter that should be computed numerically.

For $M << E$, every time optimal trajectory is bang-bang and in particular the corresponding control is periodic with frequency of the order of the resonance frequency $\omega_R = 2E$.

On the other side, for $M > E$ the time optimal trajectory steering the state one to the state two is bang-bang with exactly one switching. Fixed E we also prove that for $M \to \infty$ the time needed to reach the state two tends to zero.

Finally we compare these results with some known results of Khaneja, Brockett and Glaser and with those obtained in the Rotating Wave Approximation.

Keywords: Control of Quantum Systems, Optimal Synthesis on the Bloch Sphere, Minimum Time

I. Introduction

In this paper we apply techniques of optimal synthesis on 2-D manifolds to the population transfer problem in a two-level quantum system (e.g. a spin 1/2 particle) driven by an external field (e.g. a magnetic field along a fixed axis). Two-level systems are the simplest quantum mechanical models interesting for applications (see for instance [3], [7]). The dynamics is governed by the time dependent Schrödinger equation (in a system of units such that $\hbar = 1$):

$$i\frac{d\psi(t)}{dt} = H(t)\psi(t), \tag{1}$$

where $\psi(.) = (\psi_1(.), \psi_2(.))^T : [0, T] \to \mathbb{C}^2$ is such that $\sum_{j=1}^{2} |\psi_j(t)|^2 = 1$ (i.e. $\psi(t)$ belongs to the sphere $S^3 \subset \mathbb{C}^2$), and:

$$H(t) = \begin{pmatrix} -E & \Omega(t) \\ \Omega(t) & E \end{pmatrix}, \tag{2}$$

where E, is a real number ($\pm E$ represent the energy levels of the system). The <u>control</u> $\Omega(.)$, that we assume to be a real function, different from zero only in a fixed interval, represents the external pulsed field. In the following we call <u>drift term</u>, the Hamiltonian with no external fields (i.e., the term $diag(-E, E)$).

The aim is to induce a transition from the first level (i.e., $|\psi_1|^2 = 1$) to the second level (i.e., $|\psi_2|^2 = 1$), minimizing the time of transfer, with bounded field amplitude:

$$|\Omega(t)| \leq M, \quad \text{for every } t \in [0, T],$$

where T is the time of the transition and M is a positive real constant representing the maximum amplitude available.

<u>Remark</u> **1:** *This problem was studied also in [9], but with quadratic cost $\int_0^T \Omega(t)^2\, dt$ and with no bound on the control. In this case, optimal solutions can be expressed in terms of Elliptic functions.*

It is a standard fact to eliminate an irrelevant global factor of phase by projecting the system on a two dimensional real sphere S^2 (called the Bloch Sphere) by means of an Hopf map. In this way the Schrödinger equation (1), (2) becomes the single input affine system (after setting $u(t) = \Omega(t)/M$):

$$\dot{y} = F_S(y) + uG_S(y), \quad \text{where:} \tag{3}$$

$$y = (y_1, y_2, y_3) \in \mathbb{R}^3, \quad \textstyle\sum_{j=1}^{3} y_j^2 = 1 \tag{4}$$

$$|u| \leq 1, \tag{5}$$

$$F_S(y) := k\cos(\alpha)(-y_2, y_1, 0)^T\ , \tag{6}$$

$$G_S(y) := k\sin(\alpha)(0, -y_3, y_2)^T\ , \tag{7}$$

with $\alpha = \arctan(M/E) \in]0, \pi/2[$, while the constant k is given by $k = 2E/\cos(\alpha) = 2\sqrt{M^2 + E^2}$.

Normalizations. *In the following, to simplify the notations, we normalize $k = 1$. This normalization corresponds to a reparametrization of the time. More precisely, if T is the minimum time to steer the state one to the state two for the system with $k = 1$, the corresponding minimum time for the original system is simply $\frac{T}{2\sqrt{M^2+E^2}}$. We come back to the original value of k only in Section III-C.*

The vector fields $F_S(y)$ and $G_S(y)$ describe rotations respectively around the axes y_3 and y_1. Now the state one is represented by the point $y_3 = 1$ (called the <u>north pole</u>) and the state two by the point $y_3 = -1$ (called <u>the south pole</u>). The optimal control problem is then to connect the north pole to the south pole in minimum time. As usual we assume the control $u(.)$ to be a measurable function satisfying (5) almost everywhere. The corresponding trajectory is a Lipschitz continuous function $y(.)$ satisfying (3) almost everywhere.

The most important and powerful tool to study optimal trajectories is the well known Pontryagin Maximum Principle (in the following PMP, see for instance [2]). It is a first order necessary condition for optimality and generalizes the Weierstraß conditions of Calculus of Variations to problems with non-holonomic constraints. For each optimal trajectory, the PMP provides a lift to the cotangent bundle that is a solution to a suitable pseudo–Hamiltonian system.

Even if the PMP is powerful, giving a complete solution to an optimization problem remains extremely difficult. First, one is faced with the problem of integrating a Hamiltonian system. Second, one should manage with "non Hamiltonian solutions" of the PMP, the so called abnormal extremals. Finally, even if one is able to find all the solutions of the PMP it remains the problem of selecting, among them, the optimal trajectories. For these reasons, usually, one can hope to find a complete solution of an optimal control problem for very special costs, dynamics and in low dimension only.

Two dimensional minimum time problems in control affine form, (like the problem (3)–(7)) are nice cases for which the analysis can be pushed much further, thanks to the theory developed in [5] (see also the references therein). In this paper we take advantage of that theory to restrict the set of candidate optimal trajectories. The optimal trajectories are then identified, by requiring that they respect certain crucial symmetries of the system. More precisely, for $M > E$ the time optimal trajectories steering the state one to the state two are bang-bang with exactly one switching, and we give the exact expressions of the corresponding optimal controls. In particular, fixed E we see that for $M \to \infty$ the time needed to reach the state two tends to zero.
On the other side, for $M << E$, every time optimal trajectory is periodic (and in particular bang-bang) with frequency of the order of the resonance frequency $\omega_R = 2E$, and can be selected among a finite set of trajectories which corresponds to solutions of suitable equations.

Remark **2:** *If we were describing either a spin 1/2 particle driven by two magnetic fields (one along the x axis and one along the y axis) or a two-level molecula driven by an external field in the Rotating Wave Approximation (RWA for short, see for instance [3]), then our Hamiltonian would contain complex controls:*

$$H(t) = \begin{pmatrix} -E & \Omega(t) \\ \Omega^*(t) & E \end{pmatrix}, \tag{8}$$

where $(^*)$ indicates the complex conjugation involution. In this case the minimum time problem with bounded controls (i.e., $|\Omega(t)| \leq M$) is easier, since it is possible to eliminate the drift term by a unitary change of coordinates and a change of controls (interaction picture). This problem has been studied in [6], [9]. The simplest time optimal trajectory, steering the system from the state one to the state two, corresponds to controls in resonance with the energy gap $2E$, and with maximal amplitude i.e.

$$\Omega(t) = Me^{i(2E)t}$$

The quantity $\omega_R = 2E$ is called the resonance frequency. In this case, the time T_{RWA} of transfer is proportional to the inverse of the laser amplitude. More precisely $T_{RWA} = \pi/(2M)$, see for instance [6]. In Section III-C the minimum time of transfer for the Hamiltonians (2) and (8) are compared.
In Section II, we recall some basic properties of optimal trajectories for the system (3)–(7), that were already obtained in [4]. In Section III we state our main results and in III-C we compare these results with some known results of Khaneja, Brockett and Glaser and with those obtained in the Rotating Wave Approximation. In Section IV we give an idea of the techniques we used.

II. KNOWN RESULTS

Recall that we have normalized $k = 1$. Note that, since the system (3)–(7) is Lie bracket generated on a compact manifold and the set of velocities is compact and convex then, for each pair of points p and q belonging to S^2, there exists a time optimal trajectory joining p to q.

The minimum time problem for the control system (3), (7), although with different purposes, has been partially studied in [4]. In particular it was proved that every time optimal trajectory is a finite concatenation of bang arcs (i.e., corresponding to control a.e. constantly equal to $+1$ or -1) and singular arcs (i.e., corresponding to singularities of the End point mapping, see for instance [5], that in our case correspond to controls a.e. vanishing) with some special structure. More precisely:

Definition 1: *A control $u : [a, b] \to [-1, 1]$ is said to be bang-bang if $u(t) \in \{-1, 1\}$ a.e. in $[a, b]$. Moreover, if $u(t) \in \{-1, 1\}$ and $u(t)$ is constant for almost every $t \in [a, b]$, then u is called a bang control. If $u(t) = 0$ for almost every $t \in [a, b]$, then u is called a singular control.*

A switching time of u is a time $t \in [a, b]$ such that, for every $\varepsilon > 0$, u is not bang on $(t - \varepsilon, t + \varepsilon) \cap [a, b]$. A control with a finite number of switchings is called regular bang-bang. A trajectory of the control system (3)–(7) is a bang trajectory, singular trajectory, bang-bang trajectory, regular bang-bang trajectory respectively, if it corresponds to a bang control, singular control, bang-bang control, regular bang-bang control respectively.

In the sequel, we use the following convention. The letter B refers to a bang arc and the letter S refers to a singular arc. A concatenation of bang and singular arcs is labeled by the corresponding letter sequence, written in order from left to right. Sometimes, we will use a subscript to indicate the time duration of an arc, so that we use B_t to refer to a bang arc defined on an interval of length t and, similarly, S_t for a singular arc defined on an interval of length t.

Using the PMP and the Theory developed in [5], in [4] (see also [1]) it was proved the following:

Proposition 1: *Consider the control system (3)–(7). Then:*

A. *every time optimal trajectory is a finite concatenation of bang and singular arc;*
B. *if a time optimal trajectory contains a bang arc B_t, then $t < 2\pi$;*
C. *if a time optimal trajectory contains a singular arc, then it is of the type $B_t S_s B_{t'}$, with $s \leq \frac{\pi}{\cos(\alpha)}$, $t, t' \geq 0$. Moreover the support of singular arcs lies on the set (called equator) $y_3 = 0$.*
D. *if a time optimal trajectory is bang-bang, then the time duration $\bar{T}$ along an interior bang arc is the same for all interior bang arcs and verifies $\pi \leq \bar{T} < 2\pi$.*

From **C.** it follows that the first and the last arc on optimal trajectory connecting the north with the south pole are not

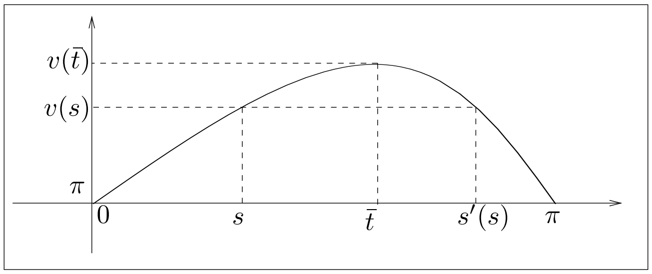

Fig. 1. Graph of $v(.)$ when $\alpha = \pi/6$

singular. The following proposition (see [4] for the proof) gives more details on the optimal trajectories starting at the north pole in the case $\alpha < \pi/4$.

Proposition 2: *Consider the control system (3)–(7), and assume $\alpha < \pi/4$. Then the optimal trajectories starting from the north pole are of the form $B_{s_i} B_{v(s_i)} \cdots B_{v(s_i)} B_{s_f}$, where $s_i \in [0, \pi]$, $s_f \in [0, v(s_i)]$ and*

$$v(s_i) = \pi + 2\arctan\left(\frac{\sin(s_i)}{\cos(s_i) + \cot^2(\alpha)}\right). \qquad (9)$$

Note that the function $v(.)$ is such that $v(0) = v(\pi) = \pi$ and moreover it is increasing on the interval $[0, \bar{t}]$ and decreasing on $[\bar{t}, \pi]$, where $\bar{t} = \arccos(-\tan^2(\alpha))$, moreover if α is small the maximum of $v(.)$ is $v(\bar{t}) = 2\arccos(-\tan^2(\alpha)) \sim \pi + 2\alpha^2$ (see Figure 1).

III. Main Results

A. The $\alpha \geq \pi/4$ Case

In the case $\alpha \geq \pi/4$, there are exactly four optimal trajectories steering the state one to the state two. They are easily described by the following:

Proposition 3: *Consider the control system (3)–(7), and assume $\alpha \geq \pi/4$. Then the optimal trajectories steering the north pole to the south pole are bang-bang with only one switching. More precisely they are the four trajectories corresponding to the four controls*

$$u^{(1)} = \begin{cases} u = 1, \ t \in [0, s_A] \\ u = -1, \ t \in]s_A, T] \end{cases}, \quad u^{(2)} = \begin{cases} 1, \ t \in [0, s_B] \\ -1, \ t \in]s_B, T] \end{cases}$$

$$u^{(3)} = \begin{cases} -1, \ t \in [0, s_A] \\ 1, \ t \in]s_A, T] \end{cases}, \quad u^{(4)} = \begin{cases} -1, \ t \in [0, s_B] \\ 1, \ t \in]s_B, T] \end{cases}$$

where:

$$s_A = \pi - \arccos(\cot^2(\alpha)), \ s_B = \pi + \arccos(\cot^2(\alpha)),$$

$$\textit{and} \quad T = 2\pi.$$

One can easily check that the switchings described in Proposition 3 occur on the equator ($y_3 = 0$).

B. The $\alpha < \pi/4$ Case

If $\alpha < \pi/4$, the situation is more complicated. From Proposition 2, we know that every optimal trajectory starting at the north pole has the form $B_{s_i} \underbrace{B_{v(s_i)} \cdots B_{v(s_i)}}_{n-1 \ times} B_{s_f}$ where the function $v(s_i)$ is given by formula (9). (In the following we do not specify if the first bang corresponds to control $+1$ or -1, since, as a consequence of the symmetries of the problem, if $u(t)$ is an optimal control steering the north pole to the south pole, $-u(t)$ steers the north pole to the south pole as well.)

It remains to identify one or more values of s_i, s_f and the corresponding number of switchings n for this trajectory to reach the south pole.

Next, given $s \in [0, \pi]$ such that $s \neq \bar{t} = \arccos(-\tan^2(\alpha))$ we call $s'(s)$ the unique solution to the equation $v(s) = v(s'(s))$ with $s'(s) \neq s$ and we define $s'(\bar{t}) = \bar{t}$ (see also Figure 1). Considering the symmetries of the problem, one can prove that if $\alpha < \pi/4$, s_f is equal either to s_i or to $s'(s_i)$. This fact is described by Lemma 1 below.

In the following we describe how to identify candidate triples (s_i, s_f, n) for which the corresponding trajectory steers the north pole to the south pole in minimum time. There are two kind of candidate optimal trajectories.

- $s_f = s'(s_i)$, called TYPE-1-candidate optimal trajectories
- $s_f = s_i$ called TYPE-2-candidate optimal trajectories

Define the following functions:

$$\theta(s) = 2\arccos\left(\sin^2\left(\frac{v(s)}{2}\right)\cos(2\alpha) - \cos^2\left(\frac{v(s)}{2}\right)\right) \qquad (10)$$

$$\beta(s) = 2\arccos(\sin(\alpha)\cos(\alpha)(1 - \cos(s))) \qquad (11)$$

Proposition 4: (TYPE-1-trajectories) *Let $\alpha < \pi/4$ and $s \in [0, \pi]$. Fixed α, the following equation for the couple (s, n):*

$$\mathcal{F}(s) := \frac{2\pi}{\theta(s)} = n, \qquad (12)$$

has either two or zero solutions. More precisely if (s, n) is a solution to equation (12), then $(s'(s), n)$ is the second one, and the trajectories $B_s \underbrace{B_{v(s)} \cdots B_{v(s)}}_{n-1} B_{s'(s)}$ and $B_{s'(s)} \underbrace{B_{v(s)} \cdots B_{v(s)}}_{n-1} B_s$ are TYPE-1-candidate optimal trajectories.

Proposition 5: (TYPE-2-trajectories) *Let $\alpha < \pi/4$ and $s \in [0, \pi]$. Fixed α, the following equation for the couple (s, n):*

$$\mathcal{G}(s) := \frac{2\beta(s)}{\theta(s)} + 1 = n, \qquad (13)$$

has exactly two solutions. More precisely these solutions have the form (s_1, n), $(s_2, n+1)$ and the trajectories $B_{s_1} \underbrace{B_{v(s_1)} \cdots B_{v(s_1)}}_{n-1} B_{s_1}$ and $B_{s_2} \underbrace{B_{v(s_2)} \cdots B_{v(s_2)}}_{n} B_{s_2}$ are TYPE-2-candidate optimal trajectories.

In Figure 2 the graphs of the functions (13) and (12) are drawn for a particular value of α, namely $\alpha = 0.13$.

Propositions 4 and 5 select a set of (possibly coinciding) 4 or 8 candidate optimal trajectories (half of them starting with

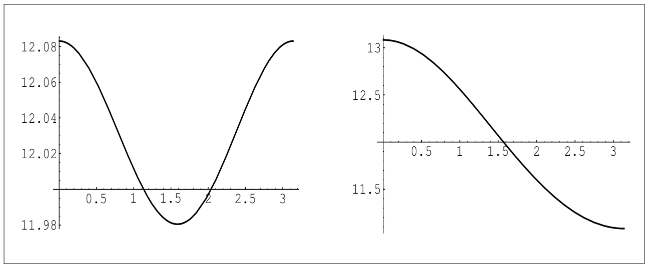

Fig. 2. Graph of the functions $\mathcal{G}$ and $\mathcal{F}$ with $\alpha = 0.13$

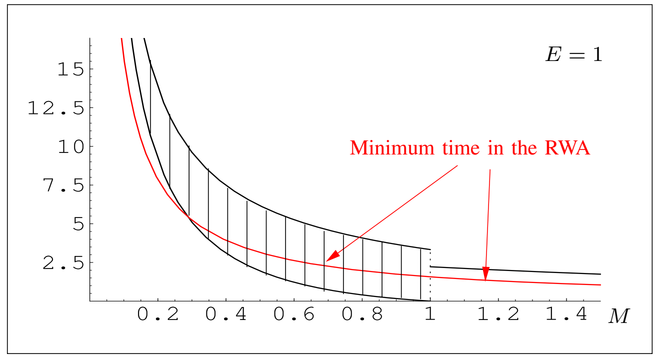

Fig. 3. Estimate on the minimum time to reach the state two and comparison with the time needed in the RWA

control $+1$ and the other half with control -1) corresponding to triples (s_i, s_f, n) that can be easily computed numerically.

Then the optimal trajectories can be easily selected. Notice that there are at least two optimal trajectories steering the north to the south pole (one starting with control $+1$ and the other with control -1).

In the particular case in which $\pi/(2\alpha)$ is an integer number $\bar{n}$ one can see that TYPE-1 candidate optimal trajectories coincide with some of TYPE-2 candidate optimal trajectories. They are of the type $B_\pi \underbrace{B_\pi ... B_\pi}_{\bar{n}-2} B_\pi$ or of the type $B_s \underbrace{B_{v(s)} ... B_{v(s)}}_{\bar{n}-1} B_s$ for some $s \in]0, \pi[$.

Otherwise if $\pi/(2\alpha)$ is not an integer number, define:

$$m := [\frac{\pi}{2\alpha}], \qquad r := \frac{\pi}{2\alpha} - m \in [0, 1[$$

where $[.]$ denotes the integer part. One can prove the following:

Proposition 6: *There exists $\bar{r}(m) \in]0, 1[$ such that:*

- *if $r \in [0, \bar{r}(m)]$ then equation (12) admits exactly two solutions that are both optimal, while TYPE-2 candidate optimal trajectories are not.*
- *if $r \in]\bar{r}(m), 1[$ then equation (12) does not admit any solution.*

The claims on existence of solutions of the previous propositions come from the fact that $\mathcal{F}(0) = \mathcal{F}(\pi) = \frac{\pi}{2\alpha}$ and the only minimum point of $\mathcal{F}$ occurs at $\bar{s} = \pi - \arccos(\tan^2(\alpha))$.

It turns out that the image of $\mathcal{F}$ is a small interval whose length is of order α^3 and therefore equation (12) has a solution only if α is close enough to $\frac{\pi}{2n}$ for some integer number n.

On the other hand it is possible to estimate the derivative of $\mathcal{G}$ with respect to s showing that it is negative in the open interval $]0, \pi[$. Therefore, since $\mathcal{G}(0) = \frac{\pi}{2\alpha} + 1$ and $\mathcal{G}(\pi) = \frac{\pi}{2\alpha} - 1$, equation (13) has always two solutions (if $\frac{\pi}{2\alpha}$ is an integer number then the trajectories corresponding to the solutions $s_i = 0$ and $s_i = \pi$ coincide).

Using the previous analysis one can easily prove the following:

Proposition 7: *If N is the number of switchings of an optimal trajectory joining the north to the south pole, then*

$$\frac{\pi}{2\alpha} - 1 \leq N < \frac{\pi}{2\alpha} + 1.$$

Using these inequalities and the fact that the function $2s + \left(\frac{\pi}{2\alpha} - 1\right) v(s)$ is increasing on $[0, \pi]$, one can give a rough estimate of the time needed to reach the south pole:

Proposition 8: *The total time T of an optimal trajectory joining the north to the south pole satisfies the inequalities:*

$$\frac{\pi^2}{2\alpha} - 2\pi < T < \frac{\pi^2}{2\alpha} + \pi.$$

C. Comparison with results in the RWA and with [8]

In this section we come back to the original value of k i.e. $k = 2E/\cos(\alpha) = 2\sqrt{M^2 + E^2}$, and we compare the time necessary to steer the state one to the state two for our model and the model (in the RWA) described in Remark 2.

For our model we have the following:

- for $\alpha \geq \pi/4$ then $T = 2\pi/k = \pi/\sqrt{M^2 + E^2}$;
- for $\alpha < \pi/4$ then T is estimated by

$$\frac{1}{k}\left(\frac{\pi^2}{2\alpha} - 2\pi\right) < T < \frac{1}{k}\left(\frac{\pi^2}{2\alpha} + \pi\right).$$

On the other hand, for the model in the RWA, we have $T_{RWA} = \pi/(2M)$ (cfr. Remark 2). Fixed $E = 1$, in Figure 3 the times T and T_{RWA} as function of M are compared. Notice that although T_{RWA} is bigger than the lower estimate of T in some interval, we always have $T_{RWA} \leq T$. This is due to the fact that the admissible velocities of our model are a subset of the admissible velocities of the model in the RWA.

Notice that, fixed $E = 1$, for $M \to 0$ we have $T \sim \pi^2/(4M)$, while for $M \to \infty$, we have $T \sim \pi/M$. In other words:

- for $M \to 0$ we have $T \sim (\pi/2)T_{RWA}$,
- for $M \to \infty$ we have $T \sim 2T_{RWA}$.

Remark **3:** For $M << E$ (i.e. for α small) $v(s) \sim \pi/(2E)$. It follows that a time optimal trajectory connecting the north to the south pole (in the interval between the first and the last bang) is periodic with period $P \sim \pi/E$ i.e. with a frequency of the order of the resonance frequency $\omega_R = 2E$ (see Figure 4). On the other side if $M > E$ then

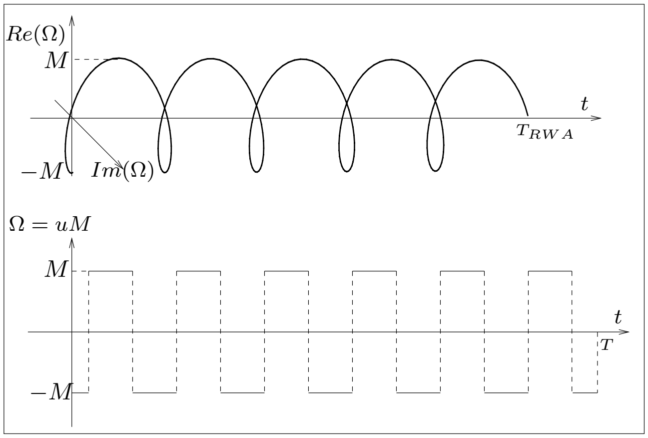

Fig. 4. Comparison between the optimal strategy for our system and in the RWA

the time optimal trajectory connecting the north with the south pole is the concatenation of two pulses. Notice that if $M >> E$, the time of transfer is of the order of π/M and therefore tends to zero as $M \to \infty$. It is interesting to compare this result with a result of Khaneja, Brockett and Glaser, for a two level system, but with no bound on controls (see [8]). They estimate the infimum time to reach every point of whole group $SU(2)$ in π/E.

Indeed for our model it is possible to prove that, for $M \to \infty$, not every point of the Bloch sphere can be reached from the state one in an arbitrarily small time, but this is the case for the state two, as we discussed above.

IV. Sketch of the Proofs

In this section we give an idea of the techniques we used to prove our results. The key point is described by the following Lemma which states a property of optimal trajectories as a consequence of the symmetries of the problem. Recall that we have normalized $k = 1$. Before stating the Lemma we note that it is possible to extend Proposition 2 to the case $\alpha \geq \pi/4$, assuming that $s_i \in [0, \arccos(-\cot^2(\alpha))[$, and in this case $v(.)$ is an increasing function on its interval of definition.

Lemma 1: *Every optimal bang-bang trajectory, connecting the north to the south pole, with more than one switching is such that $v(s_i) = v(s_f)$ where s_i is the first switching time and s_f is the time needed to steer the last switching point to the south pole.*

Proof of the lemma. Consider the problem of connecting the south pole to the north pole in minimum time through the system

$$\dot{z} = F'_S(z) + uG'_S(z) \tag{14}$$

where $z \in S^2$ and $F'_S(z) = -F_S(z)$, $G'_S(z) = -G_S(z)$.
The trajectories of system (14) coincide with those of the system (3)–(7), but the velocity is reversed. Therefore the optimal trajectories for the new problem coincide with the optimal ones for the system (3)–(7) connecting the north pole to the south pole, and the time between two switchings is the same. Moreover, if we perform the change of coordinates $(z_1, z_2, z_3) \to (y_1, y_2, y_3) = (-z_1, z_2, -z_3)$, then the new problem becomes exactly the starting problem, and so we deduce that, if we have more than one switching, it must be $v(s_i) = v(s_f)$.

Proof of Proposition 3. First one can easily see that the only possible trajectories steering the north to the south pole, with only one switching are those described by the proposition. So, since by Proposition 2 the total time for trajectories with more than two switchings is larger than 2π, it remains to compare our candidate with the trajectories containing a singular arc and with those with exactly two switchings. In the first case the trajectories must be of the type $B_t S_s B_{t'}$ and the only possibility is $t = t' = \pi - \arccos(\cot^2(\alpha))$, while the total time is $2\pi - 2\arccos(\cot^2(\alpha)) + 2\arccos(\cot(\alpha))/\cos(\alpha)$ which is larger than 2π.
One can observe that $v(.)$ is an increasing function if $\alpha > \pi/4$ and therefore, if we apply Lemma 1, we obtain that for an optimal trajectory with more than one switching it must be $s_i = s_f$. In particular the bang-bang trajectories with exactly two switchings joining the north pole to the south pole and with $s_i = s_f < \pi$ can be explicitly determined and their corresponding total time is $2\pi + 2\arcsin\left(\frac{1}{2\sin(\alpha)}\right)$.

Proof of Propositions 4 and 5. If $\alpha < \pi/4$ then $v(0) = v(\pi) = \pi$, moreover $v(.)$ is increasing between 0 and $\arccos(-\tan^2\alpha)$ and decreasing between $\arccos(-\tan^2\alpha)$ and π (see Figure 1).
Therefore, given $\alpha < \pi/4$ and $s \in [0, \pi]$ with $s \neq \arccos(-\tan^2\alpha)$, there exists one and only one time $s'(s) \in [0, \pi]$ different from s, such that $v(s) = v(s'(s))$.
Notice that s and $s'(s)$ satisfy the nice property

$$s + s'(s) = v(s). \tag{15}$$

Indeed both s and $s'(s)$ satisfy the following equation in $t \in [0, \pi]$:

$$\cot\left(\frac{1}{2}v(s)\right) = -\frac{\sin(t)}{\cos(t) + \cot^2(\alpha)} \Rightarrow$$

$$\Rightarrow \quad \cos\left(\frac{1}{2}v(s) - t\right) = -\cos\left(\frac{1}{2}v(s)\right)\cot^2(\alpha).$$

Therefore, since $\frac{1}{2}v(s) - t \in [-\pi, \pi] \quad \forall s, t \in [0, \pi]$ and $s'(s) \neq s$, it must be:

$$s'(s) - \frac{1}{2}v(s) = \frac{1}{2}v(s) - s \quad \Rightarrow \quad s + s'(s) = v(s).$$

So we deduce that there are two possible cases:

(♣) $s_f = s'(s_i)$
(♠) $s_f = s_i$

The description of candidate optimal trajectories is simplified by the following Lemma, of which we skip the proof.

Lemma 2: Set:

$$Z(s) = \frac{1}{\rho}\begin{pmatrix} 0 & \cot\left(\frac{1}{2}v(s)\right) & -\sin(\alpha) \\ -\cot\left(\frac{1}{2}v(s)\right) & 0 & 0 \\ \sin(\alpha) & 0 & 0 \end{pmatrix}$$

where $\rho = \sqrt{\cot^2\left(\frac{1}{2}v(s)\right) + \sin^2(\alpha)}$. Then, if $\theta(s)$ is defined as in (10), and $X^+ := F_S + G_S$, $X^- := F_S - G_S$, we have $e^{\theta(s)Z(s)} = e^{v(s)X^-}e^{v(s)X^+}$.

Notice that the matrix $Z(s) \in so(3)$ is normalized in such a way that the map $t \mapsto e^{tZ(s)} \in SO(3)$ represents a rotation around the axes $R(s) = \left(0, \sin(\alpha), \cot(\frac{1}{2}v(s))\right)^T$ with angular velocity equal to one.

Let us study the two possible cases described above:

(♣) Suppose that the optimal trajectory starts with $u = -1$ (the case $u = 1$ is symmetric) and has an even number n of switchings. Then it must be

$$S = e^{s_f X^-} \underbrace{e^{v(s_i)X^+} \cdots\cdots e^{v(s_i)X^+}}_{n-1\ times} e^{s_i X^-} N \tag{16}$$

where N and S denote respectively the north and the south pole, and we have that

$$e^{s_i X^-} S = e^{v(s_i)X^-} e^{v(s_i)X^+} \cdots\cdots e^{v(s_i)X^+} e^{s_i X^-} N =$$

$$= e^{\frac{1}{2}n\theta(s_i)Z(s_i)} e^{s_i X^-} N$$

from which we deduce that s_i must satisfy

$$\frac{1}{2}n\theta(s_i) = \pi + 2p\pi \quad \text{for some integer } p.$$

It is easy to see that a value of s_i which satisfies previous equation with $p > 0$ doesn't give rise to an optimal trajectory (since, roughly speaking, the corresponding number of switchings is larger than the number of switchings needed to cover the whole sphere). Therefore in previous equation it must be $p = 0$.

If n is odd the relation (16) becomes

$$S = e^{s_f X^+} \underbrace{e^{v(s_i)X^-} \cdots\cdots e^{v(s_i)X^+}}_{n-1\ times} e^{s_i X^-} N \tag{17}$$

and, moreover, by symmetry:

$$N = e^{s_f X^-} e^{v(s_i)X^+} \cdots\cdots e^{v(s_i)X^-} e^{s_i X^+} S.$$

Then, combining with (17) and using the relation (15), we find:

$$N = e^{-s_i X^-} \underbrace{e^{v(s_i)X^-} \cdots\cdots e^{v(s_i)X^+}}_{2n\ times} e^{s_i X^-} N =$$

$$= e^{-s_i X^-} e^{n\theta(s_i)Z(s_i)} e^{s_i X^-} N.$$

Since $e^{s_i X^-} N$ is orthogonal to the rotation axis $R(s_i)$ corresponding to $Z(s_i)$, previous identity is satisfied if and only if $n\theta(s_i) = 2m\pi$ with m positive integer. As in the previous case, for an optimal trajectory, it must be $m = 1$, and therefore the proof of Proposition 4 is complete.

(♠) For simplicity call $s_i = s_f = s$. Assume, as before, that the optimal trajectory starts with $u = -1$. If this trajectory has $n = 2q + 1$ switchings then it must be

$$S = e^{sX^+} e^{q\theta(s)Z(s)} e^{sX^-} N.$$

In particular the points $e^{-sX^+}S$ and $e^{sX^-}N$ must belong to a plane invariant with respect to rotations generated by $Z(s)$ and therefore the difference $e^{sX^-}N - e^{-sX^+}S$ must be orthogonal to the rotation axis $R(s)$.

Actually it is easy to see that this is true for every value $s \in [0, \pi]$, since both $e^{-sX^+}S$ and $e^{sX^-}N$ are orthogonal to $R(s)$. Moreover, since the circle passing through $e^{sX^-}N$ and $e^{-sX^+}S$ corresponding to the rotations around $R(s)$ has radius 1, it is easy to compute the angle $\beta(s)$ between these points. In particular the distance between $e^{sX^-}N$ and $e^{-sX^+}S$ coincides with $2\sin(\frac{\beta(s)}{2})$, and so one can easily get the expression $\beta(s) = 2\arccos(\sin(\alpha)\cos(\alpha)(1 - \cos(s)))$. Then Proposition 5 is proved when n is odd.

Suppose now that the optimal trajectory has $n = 2q + 2$ switchings, then we can assume without loss of generality that $S = e^{sX^-} e^{v(s)X^+} e^{q\theta(s)Z(s)} e^{sX^-} N$. First of all it is possible to see that $e^{-v(s)X^+} e^{-sX^-} S$ is orthogonal to $R(s)$. So it remains to compute the angle $\tilde{\beta}(s)$ between the point $e^{sX^-}N$ and the point $e^{-v(s)X^+}e^{-sX^-}S$ on the plane orthogonal to $R(s)$. As before the distance between these points coincides with $2\sin(\frac{\tilde{\beta}(s)}{2})$.

Instead of computing directly $\tilde{\beta}(s)$ we compute the difference between the angles $\tilde{\beta}(s)$ and the angle $\beta(s)$. We know that

$$2\sin(\frac{\tilde{\beta}(s) - \beta(s)}{2}) = |e^{-v(s)X^+} e^{-sX^-} S - e^{-sX^+} S| =$$

$$= |e^{-sX^-} S - e^{v(s)X^+} e^{-sX^+} S| = |e^{-sX^-} S - e^{s'(s)X^+} S|.$$

Using the fact that s and $s'(s)$ satisfy the relation $v(s) = v(s'(s))$ one can easily find that

$$|e^{-sX^-} S - e^{s'(s)X^+} S| = 2\sqrt{1 - \cos^2(\alpha)\sin^2\left(\frac{1}{2}v(s)\right)}.$$

Therefore $\tilde{\beta}(s) = \beta(s) - 2\arccos\left(\cos(\alpha)\sin\left(\frac{1}{2}v(s)\right)\right)$.

This leads to $\beta(s) - \tilde{\beta}(s) = \theta(s)/2$ and the proposition is proved also in the case n is even.

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