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Feedforward control of nonlinear systems using fictitious inputs

J. Deutscher†, F. Antritter and K. Schmidt

Abstract— **This paper presents the differential parameterization of nonlinear systems using a parameterizing output with differentially dependent elements, that results from introducing fictitious inputs in the original system description. The proposed differential parameterization is used to design feedforward controllers for output tracking. The results of the paper are illustrated by a non-flat helicopter model and a three-phase ac/dc voltage-source converter with unstable tracking dynamics.**

I. INTRODUCTION

The flatness based approach to the analysis and control of nonlinear systems is an important design strategy for nonlinear control systems. In [1] the flatness based approach is presented in a differential algebraic setting and the differential geometric setting can be found in [2]. Nonlinear flat systems

$$\dot{x} = f(x, u) \tag{1}$$

with a smooth vector function f and p smooth inputs u are characterized by the existence of a flat output

$$y_f = \Phi(x, u, \dot{u}, \ldots, u^{(\alpha)}) \tag{2}$$

with $\dim(y_f) = p$ such that the system variables x and u can be expressed by the output (2) and a finite number of its time derivatives according to

$$x = \psi_x(y_f, \dot{y}_f, \ldots, y_f^{(\beta)}) \tag{3}$$

$$u = \psi_u(y_f, \dot{y}_f, \ldots, y_f^{(\beta+1)}) \tag{4}$$

Since the elements of the flat output are differentially independent, reference trajectories can be assigned to each element of the flat output independently. Thus, one can assign arbitrary but sufficiently smooth reference trajectories in the coordinates of the flat output to solve a given trajectory planning problem. The corresponding feedforward controller assuring that the plant tracks the desired trajectory is simply obtained by inserting the reference trajectory in (4).

The aim of this paper is to determine a differential parameterization for systems that are not flat or where a flat output is not known. In this case the differential parameterization (3)–(4) cannot be determined. However, a parameterization of system (1) can always be computed, where the parameterizing output contains differentially dependent elements. This is achieved by introducing fictitious inputs u_f in (1) to obtain an artificial flat system, which admits a differential parameterization (3)–(4). In order to use this result for the original system (1), the condition $u_f \equiv 0$ has to be satisfied. Consequently, the differential parameterization for the artificial system also applies to the original system (1) with the additional constraint $u_f \equiv 0$. However, due to this constraint elements of the parameterizing output for the original system become differentially dependent. As a consequence, only a part of the elements of the parameterizing output can be assigned freely, whereas the remaining part is obtained by solving differential equations.

After introducing this differential parameterization in Section II, the problem of computing feedforward controllers for output tracking is considered in Section III. The non-flat model of a helicopter and the nonlinear model of a three-phase ac/dc converter with unstable tracking dynamics is used in Section IV to demonstrate the results of the paper.

Joachim Deutscher, Klaus Schmidt and Felix Antritter are with the Lehrstuhl für Regelungstechnik, Universität Erlangen-Nürnberg, Cauerstraße 7, D-91058 Erlangen, Germany
†joachim.deutscher@rt.eei.uni-erlangen.de

II. DIFFERENTIAL PARAMETERIZATION OF NONLINEAR SYSTEMS USING FICTITIOUS INPUTS

A. Derivation of the differential parameterization

Consider the following nth order nonlinear system

$$\dot{x} = f(x, u) \tag{5}$$

with a smooth vector function f and p smooth inputs u. Assume that

$$\operatorname{rank} \frac{\partial f(x, u)}{\partial u} = p \tag{6}$$

holds locally such that the inputs are independent. In order to obtain a differential parameterization for system (5) an artificial flat system is considered by augmenting system (5) with p_f fictitious inputs u_f yielding

$$\dot{x} = f(x, u) + G_f(x)u_f \tag{7}$$

In (7) the (n,p_f) matrix $G_f(x)$ has to be chosen such that the inputs u and u_f are independent, i.e.

$$\operatorname{rank} \begin{bmatrix} \frac{\partial f(x,u)}{\partial u} & G_f(x) \end{bmatrix} = p + p_f \le n \tag{8}$$

has to be satisfied locally.

Remark 1: In most cases it suffices to choose $G_f(x)$ constant, i.e. $G_f(x) = G_f$.

According to the definition of flatness in the introduction, one has to find a flat output

$$y_f = \Phi(x, u, u_f, \dot{u}, \dot{u}_f, \ldots, u^{(\alpha)}, u_f^{(\alpha_f)}) \tag{9}$$

for the extended system (7) with dimension $\dim(y_f) = m_f = p + p_f$ such that

$$x = \psi_x(y_f, \dot{y}_f, \ldots, y_f^{(\beta)}) \tag{10}$$

$$u = \psi_u(y_f, \dot{y}_f, \ldots, y_f^{(\beta+1)}) \tag{11}$$

$$u_f = \psi_{u_f}(y_f, \dot{y}_f, \ldots, y_f^{(\beta+1)}) \tag{12}$$

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Since the dimension of y_f in (9) is increased by the number of fictitious inputs u_f, it is compared to system (5) in general easier to find a flat output for system (7).

Remark 2: Note, that by introducing $n - p$ fictitious inputs a flat system (7) can always be obtained, since then $y_f = x$ qualifies as a flat output according to (9)–(12).

In order to relate the differential parameterization (9)–(12) to the original system (5) $u_f \equiv 0$ has to be satisfied. As a consequence the components of y_f become differentially dependent in view of

$$0 = \psi_{u_f}(y_f, \dot{y}_f, \ldots, y_f^{(\beta+1)}) \tag{13}$$

(see (12)).

Remark 3: It should be noted in light of (13), that y_f in (9) is not a flat output for system (5). Therefore y_f is called a *parameterizing output* for system (5).

Remark 4: The introduction of fictitious inputs does not change the properties of system (5), since the additional inputs (which lead to a controllable system if system (5) is not controllable) are set to zero according to (13).

B. Reduction of the parameterizing output

In most cases the differential parameterization (10)–(11) and (13) is not minimal in the sense that elements of the parameterizing output y_f can be eliminated in the differential parameterization. Since a reduction of the number of elements of the parameterizing output also decreases the number of differential equations (13), it is desirable to reduce its dimension as much as possible. To this end consider the following partition of the elements y_{fi}, $i = 1(1)m_f$, of the parameterizing output

$$\zeta = \big(\zeta_1, \ldots, \zeta_{m_f - l}\big) = \big(y_{f1}, \ldots, y_{f(m_f-l)}\big) \tag{14}$$
$$\xi = \big(\xi_1, \ldots, \xi_l\big) = \big(y_{f(m_f-l+1)}, \ldots, y_{fm_f}\big) \tag{15}$$

where $0 < l \leq p_f$. Assume that the p_f differential equations (13) can be written in the form

$$\varphi_1(\xi, \zeta, \dot{\zeta}, \ldots, \zeta^{(\beta+1)}) = 0 \tag{16}$$
$$\varphi_2(\xi, \zeta, \dot{\xi}, \dot{\zeta}, \ldots, \xi^{(\beta+1)}, \zeta^{(\beta+1)}) = 0 \tag{17}$$

with $\dim(\varphi_1) = l$ and $\dim(\varphi_2) = p_f - l$. If (16) is solvable for ξ, i.e.

$$\xi = \varphi_1^{-1}(\zeta, \dot{\zeta}, \ldots, \zeta^{(\beta+1)}) \tag{18}$$

then by substituting (18) in the differential parameterization (10)–(11) one obtains

$$x = \bar{\psi}_x(\zeta, \dot{\zeta}, \ldots, \zeta^{(\gamma)}) \tag{19}$$
$$u = \bar{\psi}_u(\zeta, \dot{\zeta}, \ldots, \zeta^{(\gamma+1)}) \tag{20}$$

The remaining differential equations (17) can be expressed as

$$\bar{\varphi}_2(\zeta, \dot{\zeta}, \ldots, \zeta^{(\gamma+1)}) = 0 \tag{21}$$

if (18) is substituted in (17). Relation (21) denotes the remaining conditions on ζ in form of $p_f - l$ differential equations such that a new parameterization (19)–(21) is obtained. Thus, the dimension of the parameterizing output ζ has been reduced to $p_f - l$.

Remark 5: If p_f elements ξ of the parameterizing output y_f can be eliminated from the differential (10)–(11) and (13), i.e. (13) is solvable for ξ giving

$$\xi = \psi_{u_f}^{-1}(\zeta, \dot{\zeta}, \ldots, \zeta^{(\beta+1)}) \tag{22}$$

then system (5) is obviously flat with flat output $y_f = \zeta$.

III. FEEDFORWARD CONTROL

A. Output tracking problem

In this section the problem of tracking a given sufficiently smooth reference trajectory

$$y_d(t), \quad t \in [0, T] \tag{23}$$

for p system outputs

$$y = h(x) \tag{24}$$

is solved using the differential parameterization (10)–(11) and (13). Since y_f parametrizes the inputs u (see (11)) the first step in solving the posed tracking problem is to compute the reference trajectory $y_{f,d}$ for the parameterizing output y_f from the given y_d. The computation of $y_{f,d}$ depends on the fact, whether the outputs y are elements of the parameterizing output or not.

B. Computation of the reference trajectory $y_{f,d}$, where the outputs y are elements of y_f

In this case there exists a partition of the parameterizing output y_f in the form

$$y = [y_{f1} \ \ldots \ y_{fp}]^T \tag{25}$$
$$\bar{y}_f = [y_{f(p+1)} \ \ldots \ y_{fm_f}]^T \tag{26}$$

where $\bar{y}_f$ denotes the remaining $m_f - p$ elements of the parameterizing output. Since (13) constitutes p_f conditions on the parameterizing output y_f, $m_f - p_f = p$ elements of the m_f-dimensional parameterizing output can be assigned freely. Consequently, it is possible to assign arbitrary but sufficiently smooth reference trajectories y_d for the output y, since y is part of the parameterizing output and has p elements. In order to obtain the complete reference trajectory $y_{f,d}$ for the parameterizing output, the reference trajectory for the remaining elements $\bar{y}_f$ has to be computed. This trajectory is obtained as a solution $\bar{y}_{f,d}$ of the differential equation

$$\psi_{u_f}(y_d, \ldots, y_d^{(\beta+1)}, \bar{y}_{f,d}, \ldots, \bar{y}_{f,d}^{(\beta+1)}) = 0 \tag{27}$$

that follows from substituting (25) and (26) in (13).

C. Computation of the reference trajectory $y_{f,d}$, where the outputs y are not elements of y_f

In order to obtain the reference trajectory $y_{f,d}$ from the assignment y_d in (23) the differential parameterization

$$y = \psi_y(y_f, \dot{y}_f, \ldots, y_f^{(\beta)}) \tag{28}$$

of (24) is determined by inserting (10) in (24). Thus, by (28) the reference trajectory $y_{f,d}$ has to be a solution of

$$\psi_y(y_{f,d}, \dot{y}_{f,d}, \ldots, y_{f,d}^{(\beta)}) = y_d \tag{29}$$

Since the parameterizing output y_f also has to satisfy (13), the reference trajectory $y_{f,d}$ can be computed by solving the set of m_f differential equations

$$\psi_y(y_{f,d}, \dot{y}_{f,d}, \ldots, y_{f,d}^{(\beta)}) = y_d \tag{30}$$

$$\psi_{u_f}(y_{f,d}, \dot{y}_{f,d}, \ldots, y_{f,d}^{(\beta+1)}) = 0 \tag{31}$$

Remark 6: A prerequisite for the proposed approach is that the reference trajectory y_d is chosen such that (30)–(31) admit a sufficiently smooth solution $y_{f,d}$.

D. Tracking dynamics

The differential equations (27) and (30)–(31) respectively represent the *tracking dynamics*, which have to be solved in order to generate the reference trajectory $y_{f,d}$ for the feedforward controller. Since the tracking dynamics are given by the internal dynamics (i.e. the driven zero dynamics) of the system with respect to y (see [3]), a prerequisite for a bounded solution of (27) and (30)–(31) respectively is that the system has a locally stable zero dynamics. However, if the zero dynamics of the system is unstable, then the solutions $\bar{y}_{f,d}$ and $y_{f,d}$ respectively of the differential equations are in general not bounded. In order to compute bounded trajectories the approach in [4] is applied. For a nonlinear system in Byrnes Isidori normal form this algorithm can be used to construct bounded trajectories in the presence of nonminimum phase (hyperbolic) zero dynamics (i.e. the Jacobian matrix of the zero dynamics has no eigenvalues on the imaginary axis). In the following it is shown that this algorithm can also be applied to the tracking dynamics (27) and (30)–(31), which were derived without transforming the system (5) with output (24) to Byrnes Isidori normal form. To this end assume that (27) can be solved for the highest time derivatives of $\bar{y}_{f,d}$ giving

$$\bar{y}_{f,d}^{(\beta+1)} = \psi_{u_f}^{-1}(y_d, \ldots, y_d^{(\beta+1)}, \bar{y}_{f,d}, \ldots, \bar{y}_{f,d}^{(\beta)}) \tag{32}$$

By introducing the states

$$\eta = [\bar{y}_{f,d} \; \dot{\bar{y}}_{f,d} \; \ldots \; \bar{y}_{f,d}^{(\beta)}]^T \tag{33}$$

the state space representation

$$\dot{\eta} = q(\eta, y_d^{\beta+1}) \tag{34}$$

for (32) is obtained, which is driven by the output reference trajectory y_d and by its time derivatives represented by

$$y_d^{\beta+1} = [y_d \; \dot{y}_d \; \ldots \; y_d^{(\beta+1)}]^T \tag{35}$$

Without loss of generality it can be assumed that system (34) has the equilibrium point $\eta^0 = 0$ for $y_d^{\beta+1} = 0$, i.e. $q(0,0) = 0$ holds. For the tracking dynamics formulated as in (34) the algorithm in [4] can be used to generate bounded η-trajectories for given trajectories y_d (in this context bounded means bounded with respect to the $||\cdot||_{1+\infty} := ||\cdot||_1 + ||\cdot||_\infty$ norm). If the function $q(\eta, y_d^{\beta+1})$ in (34) satisfies certain Lipschitz conditions and $||y_d^{\beta+1}(\cdot)||_{1+\infty}$ is sufficiently small, then the unique and bounded solution of (34) with the boundary condition

$$\eta(\pm\infty) = 0 \tag{36}$$

can be constructed by solving a sequence of boundary value problems for the differential equation

$$\dot{\eta}_{k+1} = A\eta_{k+1} + \underbrace{q(\eta_k, y_d^{\beta+1}) - A\eta_k}_{\tilde{q}(\eta_k, y_d^{\beta+1})} \tag{37}$$

with boundary conditions

$$\eta_{k+1}(\pm\infty) = 0 \tag{38}$$

starting from $\eta_0 \equiv 0$. In (37) the matrix A denotes the Jacobian matrix of the tracking dynamics at $\eta^0 = 0$ for $y_d^{\beta+1} = 0$. The term $\tilde{q}(\eta_k, y_d^{\beta+1})$ in (37), which can be viewed as a perturbation to the linearized system, takes the nonlinear dynamics of (34) into account. The matrix A can (possibly after a linear transformation) be assumed to be in block diagonal form

$$A = \begin{bmatrix} A_- & 0 \\ 0 & A_+ \end{bmatrix} \tag{39}$$

with $\mathrm{Re}\,\lambda_i(A_-) < 0$ $(\forall i)$ and $\mathrm{Re}\,\lambda_j(A_+) > 0$ $(\forall j)$. This corresponds to a partition

$$\eta_{k+1} = [\eta_{k+1,s} \; \eta_{k+1,a}]^T \tag{40}$$

where the dynamics of the stable and the antistable part of the tracking dynamics (34) are decoupled in the first approximation. Thus, the boundary condition (38) can be reformulated as

$$\eta_{k+1,s}(-\infty) = 0 \tag{41}$$

$$\eta_{k+1,a}(\infty) = 0 \tag{42}$$

such that the stable subsystem can be integrated forward in time and the antistable subsystem backward in time using the Green's function. In [4] it has been shown that the solutions η_k of (37) and (38) converge to the bounded solution of (34) and (36) as $k \to \infty$. The situation, when the tracking dynamics are given by (30) and (31) can be treated similarly. In this case the states in (33) are defined by

$$\eta = [y_{f,d} \; \dot{y}_{f,d} \; \ldots \; y_{f,d}^{(\beta)}]^T \tag{43}$$

E. Computation of the feedforward controller

If the initial conditions of the differential equations (27) and (30)–(31) respectively are chosen such that

$$x(0) = \psi_x(y_{f,d}(0), \dot{y}_{f,d}(0), \ldots, y_{f,d}^{(\beta)}(0)) \tag{44}$$

is satisfied (see (10)), the feedforward controller assuring exact tracking of the reference trajectory y_d is given by

$$u_d = \psi_u(y_{f,d}, \dot{y}_{f,d}, \ldots, y_{f,d}^{(\beta+1)}) \tag{45}$$

This feedforward controller results from inserting the reference trajectory $y_{f,d}$ in (11).

If condition (44) is not satisfied, then only asymptotic tracking of y_d is possible. This can be achieved by using a linear time-varying tracking controller, that is designed on the basis of the linearization of system (5) about the reference trajectory. Note, that the related linearization of system (5) is easily attainable, since the reference trajectory x_d in the state space can be obtained by inserting $y_{f,d}$ in (10), i.e.

$$x_d = \psi_x(y_{f,d}, \dot{y}_{f,d}, \ldots, y_{f,d}^{(\beta)}) \tag{46}$$

IV. Examples

A. Non-flat helicopter model

In the sequel the application of the proposed approach is illustrated by means of a model describing the one dimensional forward motion of a helicopter. This system is described by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} x_2 \\ -g \tan x_3 \\ x_4 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -\frac{1}{M \cos x_3} \\ 0 \\ L \end{bmatrix} u \tag{47}$$

Where x_1 denotes the forward position of the helicopter, x_3 denotes the attitude angular position of the main rotor and u is the control input. In [5] this model has been shown to be non-flat. To obtain a flat system a fictitious input is introduced to system (47) with input vector

$$G_f = [0\ 1\ 0\ 0]^T \tag{48}$$

Thus the extended system

$$\dot{x} = f(x) + G_f^\star(x) u_f^\star \tag{49}$$

is obtained, where $u_f^\star = [u\ u_f]^T$ and

$$G_f^\star(x) = \begin{bmatrix} 0 & 0 \\ -\frac{1}{M \cos x_3} & 1 \\ 0 & 0 \\ L & 0 \end{bmatrix} \tag{50}$$

obviously satisfies condition (8). A flat output of the extended system (49) is given by $y_f = [y_{f1}\ y_{f2}]^T = [x_1\ x_3]^T$ with the differential parameterization

$$[x_1\ x_2\ x_3\ x_4]^T = \psi_x(y_f, \dot{y}_f) = [y_{f1}\ \dot{y}_{f1}\ y_{f2}\ \dot{y}_{f2}]^T \tag{51}$$

$$u = \psi_u(\ddot{y}_f) = \frac{\ddot{y}_{f2}}{L} \tag{52}$$

$$u_f = \psi_u(y_f, \ddot{y}_f) = \ddot{y}_{f1} + g \tan y_{f2} + \frac{1}{ML \cos y_{f2}} \ddot{y}_{f2} \tag{53}$$

Setting $u_f = 0$ yields the tracking dynamics

$$0 = \ddot{y}_{f1} + g \tan y_{f2} + \frac{\ddot{y}_{f2}}{ML \cos y_{f2}} \tag{54}$$

as introduced in Section III-D. Equation (54) can be solved for $\ddot{y}_{f1}$

$$\ddot{y}_{f1} = \psi_{\ddot{y}_{f1}}(y_{f2}, \ddot{y}_{f2}) = -g \tan y_{f2} - \frac{\ddot{y}_{f2}}{ML \cos y_{f2}} \tag{55}$$

This equation can be integrated with respect to time, yielding

$$\dot{y}_{f1}(t) = \int_0^t -g \tan y_{f2}(\tau) - \frac{\ddot{y}_{f2}(\tau)}{ML \cos y_{f2}(\tau)} d\tau + \dot{y}_{f1}(0) \tag{56}$$

$$y_{f1}(t) = \int_0^t \int_0^\sigma -g \tan y_{f2}(\tau) - \frac{\ddot{y}_{f2}(\tau)}{ML \cos y_{f2}(\tau)} d\tau d\sigma + \dot{y}_{f1}(0) t + y_{f1}(0) \tag{57}$$

The result (56)–(57) can be substituted into (51) showing that (47) is a *Liouvillian system* (see [6], [7]) since

$$x = \psi_x(y_{f2}, \dot{y}_{f2}, \int_0^t \psi_{\ddot{y}_{f1}}(\tau) d\tau, \int_0^t \int_0^\sigma \psi_{\ddot{y}_{f1}}(\tau) d\tau d\sigma) \tag{58}$$

This can be seen as a special case of the simplification elaborated in Section II-B. In this case (16) (see also (54)) is of the form

$$\varphi_1(\xi^{(\beta+1)}, \zeta, \dot{\zeta}, \ldots, \zeta^{(\beta+1)}) = 0 \tag{59}$$

and is solvable for $\xi^{(\beta+1)}$. For the control of the helicopter it is more convenient to assign a reference trajectory $y_{f1,d}$ for the forward position y_{f1}. Solving (54) for $\ddot{y}_{f2}$ yields the tracking dynamics

$$\ddot{y}_{f2} = -ML \cos y_{f2} \ddot{y}_{f1,d} - gML \sin y_{f2} \tag{60}$$

Remark 7: From the system model (47) it is obvious that $y_{f1} = x_1$ has relative degree $r = 2$ (see [8]). Thus the order of the tracking dynamics (60) coincides with the dimension $n - r = 2$ of the zero dynamics of the output $y_{f1} = x_1$.

In [5] rest-to-rest maneuvers have been investigated and it has been shown that for such maneuvers with initial conditions $y_{f1}(0) = y_{f1,i}$, $\dot{y}_{f1}(0) = y_{f2}(0) = \dot{y}_{f2}(0) = 0$ and final conditions $y_{f1}(T) = y_{f1,f}$, $\dot{y}_{f1}(0) = 0$ also y_{f2} and $\dot{y}_{f2}$ approach again the equilibrium point $y_{f2} = \dot{y}_{f2} = 0$ of (60), when $\ddot{y}_{f1,d} = 0$. The simulation results for such a maneuver with a forward position change from $100\,\mathrm{m}$ to $150\,\mathrm{m}$ are shown in Figure (1). The parameters have been taken as $M = 4313\,\mathrm{kg}$, $g = 9.81\,\frac{\mathrm{m}}{\mathrm{sec}^2}$ and $L = 1.0456 \cdot 10^{-4}\,\frac{\mathrm{rad}}{\mathrm{N \cdot sec}^2}$ (see [5]). To ensure a smooth start and end of the trajectory a ninth order polynomial has been assigned for $y_{f1,d}$ having four time derivatives at $t = 0$ and $t = T$ respectively equal to zero.

B. AC-DC-converter

In this example a three-phase ac/dc voltage-source converter taken from [9] is considered. The system behavior can be modeled in the form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -\frac{R}{L} x_1 - \omega x_2 + \frac{E_m}{L} \\ \omega x_1 - \frac{R}{L} x_2 \\ -\frac{1}{C} i_L \end{bmatrix} + \begin{bmatrix} -\frac{1}{2L} x_3 & 0 \\ 0 & -\frac{1}{2L} x_3 \\ \frac{3}{4C} x_1 & \frac{3}{4C} x_2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{61}$$

with the outputs

$$y = [x_2\ x_3]^T \tag{62}$$

The state variables x_1 and x_2 are currents and x_3 is a voltage in a two-phase stationary reference frame d-q model. In view of

$$D(x) = \begin{bmatrix} L_{g_1} h_1(x) & L_{g_2} h_1(x) \\ L_{g_1} h_2(x) & L_{g_2} h_2(x) \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{2L} x_3 \\ \frac{3}{4C} x_1 & \frac{3}{4C} x_2 \end{bmatrix} \tag{63}$$

(see [8]) system (61)–(62) has vector relative degree $\{1, 1\}$ for $x_1 x_3 \neq 0$.

In the following a feedforward controller for tracking output trajectories is computed by using the results of Section III. To this end a fictitious input u_f is introduced in (61) with input vector

$$G_f = [1\ 0\ 0]^T \tag{64}$$

It is easy to check that G_f satifies condition (8), as the input matrix

$$G_f^\star(x) = \begin{bmatrix} -\frac{1}{2L}x_3 & 0 & 1 \\ 0 & -\frac{1}{2L}x_3 & 0 \\ \frac{3}{4C}x_1 & \frac{3}{4C}x_2 & 0 \end{bmatrix}$$

of the fictitious system defined by

$$\dot{x} = f(x) + G_f^\star(x)u_f^\star \tag{65}$$

with $u_f^\star = [u_1\ u_2\ u_f]^T$ has the property

$$\text{rank } G_f^\star(x) = p + p_f = m_f = 3 \tag{66}$$

for $x_1 x_3 \neq 0$. A possible flat output y_f of dimension $m_f = 3$ for system (65) is given by

$$y_f = [y_{f1}\ y_{f2}\ y_{f3}]^T = [x_1\ x_2\ x_3]^T \tag{67}$$

since (65) is solvable for $u_f^\star$ in view of (66). The parameterization of the system inputs in terms of the flat output y_f reads

$$u_1 = \frac{6Ly_{f2}\dot{y}_{f2} - 6\omega Ly_{f1}y_{f2} + 6Ry_{f2}^2 + 4Cy_{f3}\dot{y}_{f3} + 4i_L y_{f3}}{3y_{f1}y_{f3}} \tag{68}$$

$$u_2 = \frac{2\omega Ly_{f1} - 2L\dot{y}_{f2} - 2Ry_{f2}}{y_{f3}} \tag{69}$$

$$u_f = \frac{3L\dot{y}_{f2}y_{f2} + 3Ry_{f2}^2 + 3Ry_{f1}^2 + 2C\dot{y}_{f3}y_{f3}}{3Ly_{f1}} + \frac{3L\dot{y}_{f1}y_{f1} + 2i_L y_{f3} - 3E_m y_{f1}}{3Ly_{f1}} \tag{70}$$

The equations (68) and (69) represent the feedforward controller for tracking a given reference trajectory $y_{f,d}$. Since the system has two inputs only two elements of y_f can be assigned freely, that are given by y (see (62) and (67)).

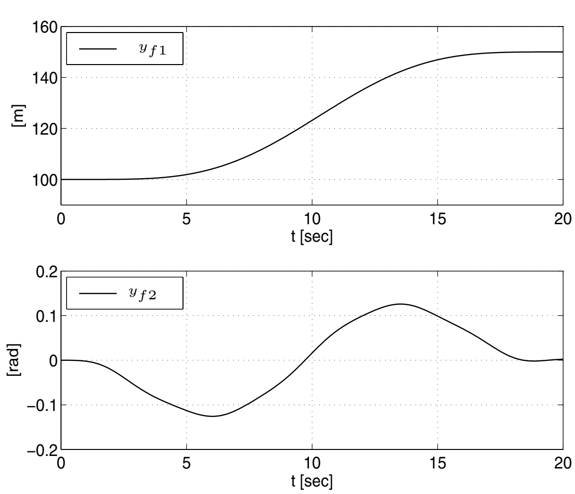

Fig. 1. Forward position y_{f1} and attitude angular position y_{f2} for the rest-to-rest maneuver

The third one has to be determined by solving the tracking dynamics. By setting equation (70) identically to zero the tracking dynamics (27) is obtained. Solving the result for $\dot{y}_{f1}$ yields for $y_{f1} \neq 0$ the state space representation

$$\dot{\eta} = q(\eta, y_d^1) = -\frac{-3E_m\eta + 3R\eta^2 + 2i_L y_{2,d} + 2Cy_{2,d}\dot{y}_{2,d}}{3L\eta} - \frac{3Ly_{1,d}\dot{y}_{1,d} + 3Ry_{1,d}^2}{3L\eta} \tag{71}$$

of the internal dynamics with $\eta = y_{f1}$ and

$$y_d^1 = [y_{1,d}\ \dot{y}_{1,d}\ y_{2,d}\ \dot{y}_{2,d}]^T \tag{72}$$

Remark 8: Since (61)–(62) has vector relative degree $\{1, 1\}$ for $x_1 x_3 \neq 0$ (see (63)) and the order of the system is 3, it has a first order internal dynamics (71) with respect to the outputs y.

The derivation of (68)–(69) and (71) shows that the proposed procedure needs only algebraic manipulations to determine the feedforward controller and the tracking dynamics and is strictly systematic.

In the sequel a trajectory for a temporary change of the dc output voltage x_3 (i.e. $y_{2,d}$) from 200 V to 140 V and back is designed. To maintain a unity power factor $x_2 \equiv 0$ A (i.e. $y_{1,d} \equiv 0$ A) should be assured during the whole trajectory. The reference trajectory y_d is assigned such that $y_d^1(0) = y_d^1(T) = y_{d0}^1 = [0\text{ A } 0\text{ A/sec } 200\text{ V } 0\text{ V/sec}]^T$ (see (72)). A transition phase of 5 sec for the dc voltage output changes was used and the output voltage of 140 V is held constant for 10 sec. Thus, the total duration of the trajectory is $T = 20$ sec. The desired output trajectory y_d is shown in Figure 2. The parameter values for the system differential

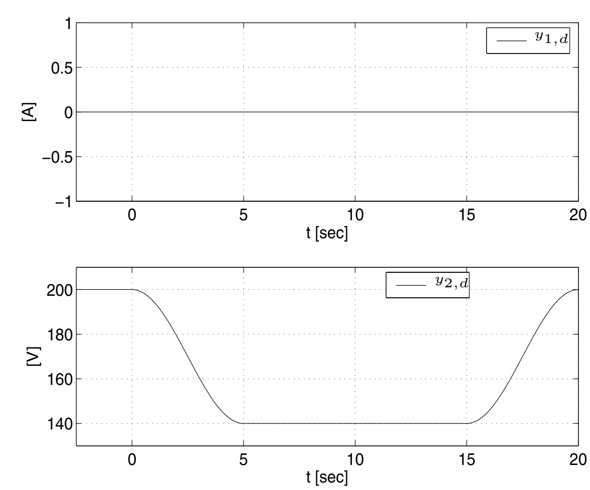

Fig. 2. Output reference trajectory y_d

equations (61) are $E_m = 80$ V, $V_r = 200$ V, $L = 5$ mH, $R = 0.1\,\Omega$, $C = 2200\,\mu$F, $\omega = 120\pi$ rad/sec and $R_L = 200\,\Omega$ such that $i_L = V_r/R_L = 1$ A. Investigating the tracking dynamics (71) for the initial value y_{d0}^1 yields the equilibrium points $\eta^0 = 1.67$ A and $\eta^0 = 798.3$ A, where the only technically realisable value is $\eta^0 = 1.67$ A (see [9]). Computing the Jacobian matrix A of the tracking dynamics at this equilibrium point results in

$$A = 9539.96\ 1/\text{sec} \tag{73}$$

Thus, the tracking dynamics are unstable and therefore suitable trajectory planning is necessary to achieve a bounded η-trajectory. By introducing the new coordinates

$$\Delta\eta = \eta - \eta^0 \quad (74)$$

$$\Delta y_d^1 = y_d^1 - y_{d0}^1 \quad (75)$$

the tracking dynamics (71) become

$$\Delta\dot{\eta} = q(\eta^0 + \Delta\eta, y_{d0}^1 + \Delta y_d^1) = \bar{q}(\Delta\eta, \Delta y_d^1) \quad (76)$$

having the equilibrium point $\Delta\eta^0 = 0$ since $\bar{q}(0,0) = 0$. With the Jacobian matrix (73), the iterations (37) can be formulated as

$$\Delta\dot{\eta}_{k+1} = A\Delta\eta_{k+1} + \bar{q}(\Delta\eta_k, \Delta y_d^1) - A\Delta\eta_k \quad (77)$$

with boundary condition

$$\Delta\eta_{k+1}(+\infty) = 0 \quad (78)$$

Using the Green's function

$$G(t) = \begin{cases} 0 & \text{if} \quad t \geq 0 \\ -e^{At} & \text{if} \quad t < 0 \end{cases} \quad (79)$$

for this boundary value problem the solution $\Delta\eta_{k+1}$ can be computed as

$$\Delta\eta_{k+1}(t) = \int_{-\infty}^{+\infty} G(t-\tau)F(\tau)d\tau \quad (80)$$

with

$$F(t) = \bar{q}(\Delta\eta_k(t), \Delta y_d^1(t)) - A\Delta\eta_k(t) \quad (81)$$

Performing the iterations, one ends up with the bounded trajectory η for the tracking dynamics (71), shown in Figure 3.

Remark 9: As the Jacobian matrix of the tracking dynamics (71) exhibits a large positive eigenvalue so called preactuation (i.e. noncausal feedforward inputs before $t = 0$ sec) can be neglected (see Figure 3).

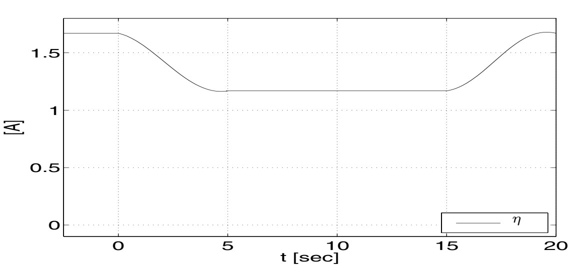

Fig. 3. Reference trajectory η for the tracking dynamics

Thus, the complete reference trajectory $y_{f,d} = [\eta \; y_{1,d} \; y_{2,d}]^T$ has been determined such that the feedforward controller is given by (68)–(69). The simulation results for the application of the computed feedforward controller are shown in Figures 4 and 5

It should be mentioned that system (61) is static feedback linearizable and therefore flat. A flat output of system (61) is given by $y_f = [x_2 \; 3L(x_1^2 + x_2^2) + 2Cx_3^2]^T$. However, since the flat output does not coincide with the output y which has to be controlled, the internal dynamics would have to be condidered also when designing a flatness based feedforward controller.

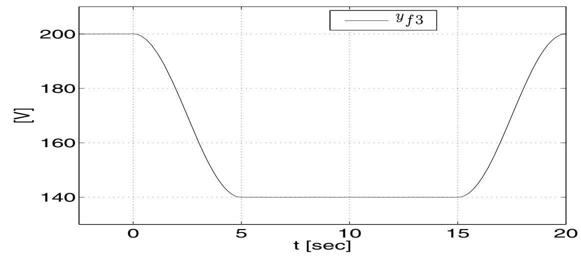

Fig. 4. DC output voltage y_{f3}

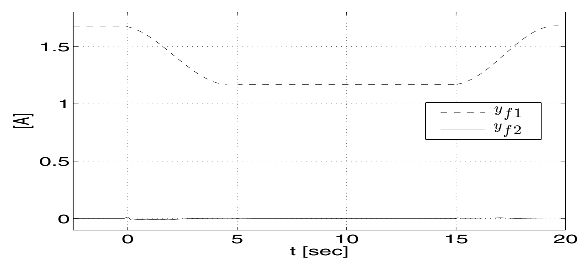

Fig. 5. Currents y_{f1} and y_{f2}

V. CONCLUSIONS

In this contribution a parameterization of nonlinear systems with an output, that contains differentially dependent elements is presented. Based on this parameterization a feedforward controller for tracking a desired output trajectory can be designed without transforming the system to Byrnes Isidori normal form. The proposed approach is applicable to systems with stable and unstable tracking dynamics. A more detailed comparison with the approach in [8] for trajectory design based on input-output linearization is done in [10].

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