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Polytope norms and related algorithms for the computation of the joint spectral radius

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***Abstract*— We address the problem of the computation of the spectral radius of a family of matrices. We briefly describe the extension of the concept of polytope to the complex space and outline the main geometric properties of such an object. Then we consider the norms determined by the complex polytopes and illustrate a possible algorithm for the approximation of the joint spectral radius of a family of matrices which is based on these complex polytope norms. As an example for our technique we consider the set of two matrices recently analyzed by Blondel, Nesterov and Theys to disprove the finiteness conjecture.**

I. INTRODUCTION

We consider a bounded family $\mathscr{F} = \{A^{(i)}\}_{i \in \mathscr{I}}$ of complex $n \times n$-matrices, where $\mathscr{I}$ is a set of indices, possibly infinite. For such a family $\mathscr{F}$, the following definitions are given in the literature. Let $\|\cdot\|$ be a given norm on the vector space $\mathbb{C}^n$ and let the same symbol $\|\cdot\|$ denote also the corresponding induced $n \times n$-matrix norm. Then, for each $k = 0, 1, \ldots$, consider the set $\Sigma_k(\mathscr{F})$ of all possible products of length k whose factors are elements of $\mathscr{F}$, that is

$$\Sigma_k(\mathscr{F}) = \{A^{(i_1)} \ldots A^{(i_k)} \mid i_1, \ldots, i_k \in \mathscr{I}\},$$

with the convention that $\Sigma_0(\mathscr{F}) = \{I\}$, I the identity matrix. For each $k \geq 0$ set

$$\hat{\rho}_k(\mathscr{F}) = \sup_{P \in \Sigma_k(\mathscr{F})} \|P\| \tag{1}$$

and define the *joint spectral radius* of $\mathscr{F}$ as

$$\hat{\rho}(\mathscr{F}) = \limsup_{k \to \infty} \hat{\rho}_k(\mathscr{F})^{1/k} \tag{2}$$

(see [18]). Note that the numbers $\hat{\rho}_k(\mathscr{F})$ depend on the particular norm $\|\cdot\|$ used in (1) whereas, by the equivalence of all the norms in finite dimensional spaces, it turns out that $\hat{\rho}(\mathscr{F})$ is independent of it.

Analogously, let $\rho(\cdot)$ denote the spectral radius of an $n \times n$-matrix and then, for each $k = 0, 1, \ldots$, consider

$$\bar{\rho}_k(\mathscr{F}) = \sup_{P \in \Sigma_k(\mathscr{F})} \rho(P)$$

and define the *generalized spectral radius* of $\mathscr{F}$ as

$$\bar{\rho}(\mathscr{F}) = \limsup_{k \to \infty} \bar{\rho}_k(\mathscr{F})^{1/k}$$

(see [5]).

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Recently it has been shown that

$$\hat{\rho}(\mathscr{F}) = \bar{\rho}(\mathscr{F})$$

(see [4], [6], [20] and [19]). This means that the joint and the generalized spectral radius of $\mathscr{F}$ are the same number, which we shall simply call the *spectral radius* of the family of matrices $\mathscr{F}$ and denote by $\rho(\mathscr{F})$. Such result generalizes the well-known Gelfand theorem for a *single* matrix.

We introduce now a further characterization of the joint spectral radius. Given a norm $\|\cdot\|$ on the vector space $\mathbb{C}^n$ and the corresponding induced $n \times n$-matrix norm, we shall still use the same notation to define

$$\|\mathscr{F}\| = \hat{\rho}_1(\mathscr{F}) = \sup_{i \in I} \|A^{(i)}\|.$$

The following result can be found, for example, in [18] and [6].

Theorem 1.1: The spectral radius of a bounded family $\mathscr{F}$ of complex $n \times n$-matrices is characterized by the equality

$$\rho(\mathscr{F}) = \inf_{\|\cdot\| \in \mathscr{N}} \|\mathscr{F}\|, \tag{3}$$

where $\mathscr{N}$ denotes the set of all possible induced $n \times n$-matrix norms.

Given a family $\mathscr{F}$, an important question to answer is whether or not the inf in (3) is actually attained by some induced matrix norm. To this purpose, we give the following definition.

Definition 1.1: We shall say that a norm $\|\cdot\|_*$ satisfying the condition

$$\|\mathscr{F}\|_* = \rho(\mathscr{F}) \tag{4}$$

is *extremal* for the family $\mathscr{F}$.

A family of matrices which admits an extremal norm is said *non-defective* (see, for example, [9]).

The actual computation of $\rho(\mathscr{F})$ is an important problem in several applications (see e.g. [10], [12], [16], [17], [1]) The problem, however, appears quite difficult in general (see e.g. [21]). Based on the inequalities (see [5])

$$\bar{\rho}_k(\mathscr{F}) \leq \rho(\mathscr{F})^k \leq \hat{\rho}_k(\mathscr{F}) \quad \text{for all } k \geq 0,$$

an algorithm for efficiently computing lower bounds and upper bounds to $\rho(\mathscr{F})$ is proposed in [7].

In the the recent paper [8] we have given a contribute in the direction of the computation of $\rho(\mathscr{F})$ considering special classes of families. In particular, we have determined sufficient conditions on the family which are sufficient to guarantee the existence of a *complex polytope* extremal norm, that is a norm whose unit ball is a *balanced complex polytope* with a finite *essential system of vertices* (we shall clarify these concepts in the next section). Such a finiteness property is very useful in view of the construction of algorithms aimed at the actual computation (or approximation) of $\rho(\mathscr{F})$.

The paper is organized as follows. In Section II we introduce a special kind of norms, the so-called polytope norms, which play a fundamental role in our theoretical results and in the algorithm we devise for computing the joint spectral radius. In Section III we give a polytope extremality result which has been recently proved in [8]; such result establishes a set of assumptions which guarantees the existence and the computability of an extremal complex polytope norm for a family of matrices. Based on this result, in Section IV we propose an algorithm for the computation of the joint spectral radius through the construction of a polytope norm. As an example, in Section V, we apply this algorithm to a family of two matrices recently analyzed in [3] to disprove the finiteness conjecture. As a result we are able to refine the result proved in [3].

II. Polytope norms

In this section we define the *complex polytopes* as generalizations of real polytopes (see, e.g. [23]) to the complex case.

Let $\mathscr{X}$ be a set in $\mathbb{C}^n$. It is well-known that $\text{absco}(\mathscr{X})$ is the set of all the finite absolutely convex linear combinations of vectors of $\mathscr{X}$, i.e. $x \in \text{absco}(\mathscr{X})$ if and only if there exist $x_1, \ldots, x_k \in \mathscr{X}$ with $k \geq 1$ such that

$$x = \sum_{i=1}^{k} \lambda_i x_i \quad \text{with} \quad \lambda_i \in \mathbb{C} \quad \text{and} \quad \sum_{i=1}^{k} |\lambda_i| \leq 1.$$

In particular, if $\mathscr{X} = \{x_i\}_{1 \leq i \leq m}$ is a finite set of vectors,

$$\text{absco}(\mathscr{X}) = \Big\{ x \in \mathbb{C}^n \;\Big|\; x = \sum_{i=1}^{m} \lambda_i x_i \text{ with } \sum_{i=1}^{m} |\lambda_i| \leq 1 \Big\}. \quad (5)$$

The forthcoming definition extends the usual definition of symmetric polytope in the real space $\mathbb{R}^n$.

Definition 2.1: We shall say that a bounded set $\mathscr{P} \subset \mathbb{C}^n$ is a *balanced complex polytope* (b.c.p.) if there exists a finite set of vectors $\mathscr{X} = \{x_i\}_{1 \leq i \leq m}$ such that $\text{span}(\mathscr{X}) = \mathbb{C}^n$ and

$$\mathscr{P} = \text{absco}(\mathscr{X}). \quad (6)$$

Moreover, if $\text{absco}(\mathscr{X}') \subsetneq \text{absco}(\mathscr{X})$ for all $\mathscr{X}' \subsetneq \mathscr{X}$, then $\mathscr{X}$ will be called an *essential system of vertices* for $\mathscr{P}$, whereas any vector $u x_i$ with $u \in \mathbb{C}$, $|u| = 1$, will be called a *vertex* of $\mathscr{P}$.

From a geometrical point of view, a b.c.p. $\mathscr{P}$ is not a classical polytope. In fact, if we identify the complex space $\mathbb{C}^n$ with the real space $\mathbb{R}^{2n}$, we see that $\mathscr{P}$ is not bounded by hyperplanes. In general, even the intersection $\mathscr{P} \bigcap \mathbb{R}^n$ is not a classical polytope. However, if the b.c.p. $\mathscr{P}$ admits an essential system of real vertices, then $\mathscr{P} \bigcap \mathbb{R}^n$ is a classical polytope.

Now we extend the concept of *polytope norm* to the complex case in a straightforward way.

Lemma 2.1: Any b.c.p. $\mathscr{P}$ is the unit ball of a norm $\|\cdot\|_{\mathscr{P}}$ on $\mathbb{C}^n$.

Proof: Since $\text{span}(\mathscr{X}) = \mathbb{C}^n$, the set $\mathscr{P}$ is *absorbing*. Therefore, since it is absolutely convex and bounded, the Minkowski functional associated to $\mathscr{P}$, defined for all $z \in \mathbb{C}^n$ by

$$\|z\|_{\mathscr{P}} = \inf\{\rho > 0 \mid z \in \rho \mathscr{P}\}, \quad (7)$$

is indeed a norm on $\mathbb{C}^n$ (see [14]). ■

Definition 2.2: We shall call *complex polytope norm* any norm $\|\cdot\|_{\mathscr{P}}$ whose unit ball is a b.c.p. $\mathscr{P}$.

The corresponding vector norm is characterized by the following Lemma.

Lemma 2.2: Let $\mathscr{P}$ be a b.c.p. and let $\|\cdot\|_{\mathscr{P}}$ be the corresponding complex polytope norm. Then, for any $z \in \mathbb{C}^n$, it holds that

$$\|z\|_{\mathscr{P}} = \min \Big\{ \sum_{i=1}^{m} |\lambda_i| \;\Big|\; z = \sum_{i=1}^{m} \lambda_i x_i \Big\}, \quad (8)$$

where $\mathscr{X} = \{x_i\}_{1 \leq i \leq m}$ is an essential system of vertices for $\mathscr{P}$.

Proof: The equality in (8) is got just by rewriting (7) taking Definition 2.1 into account. ■

The next theorem shows that the set of the complex polytope norms is *dense* in the set of all norms defined on $\mathbb{C}^n$ and that, consequently, the corresponding set of induced matrix complex polytope norms is *dense* in the set of all induced $n \times n$-matrix norms (see [13]).

Theorem 2.1: Let $\|\cdot\|$ be a norm on $\mathbb{C}^n$. Then for any $\varepsilon > 0$ there exists a b.c.p. $\mathscr{P}_\varepsilon$ whose corresponding complex polytope norm $\|\cdot\|_\varepsilon$ satisfies the inequalities

$$\|x\| \leq \|x\|_\varepsilon \leq (1+\varepsilon)\|x\| \quad \text{for all } x \in \mathbb{C}^n.$$

Moreover, denoting by $\|\cdot\|$ and $\|\cdot\|_\varepsilon$ also the corresponding induced matrix norms, it holds that

$$(1+\varepsilon)^{-1}\|A\| \leq \|A\|_\varepsilon \leq (1+\varepsilon)\|A\| \quad \text{for all } A \in \mathbb{C}^{n \times n}.$$

III. Polytope extremality results

Complex polytope norms play a particular role. In fact, Theorem 2.1 implies the following refinement of Theorem 1.1.

Theorem 3.1: The spectral radius of a bounded family $\mathscr{F}$ of complex $n \times n$-matrices is characterized by the equality

$$\rho(\mathscr{F}) = \inf_{\|\cdot\| \in \mathscr{N}_{pol}} \|\mathscr{F}\|,$$

where $\mathscr{N}_{pol}$ denotes the set of all possible induced $n \times n$-matrix complex polytope norms.

A first fundamental question concerns the construction of an extremal norm for a non-defective family. Now we give a very useful result in this direction.

We start with the following definition.

Definition 3.1 (s.m.p.): If $\mathscr{F}$ is a bounded family of complex $n \times n$-matrices, any matrix $\bar{P} \in \Sigma_k(\mathscr{F})$ satisfying

$$\rho(\mathscr{F}) = \bar{\rho}_k(\mathscr{F})^{1/k} = \rho(\bar{P})^{1/k} \quad (9)$$

for some $k \geq 1$ will be called a *spectrum-maximizing product* (in short, an *s.m.p.*) for $\mathscr{F}$. An s.m.p. is said *minimal* if it is not a power of another s.m.p. of $\mathscr{F}$. Any eigenvector $x \neq 0$ of $\bar{P}$ related to an eigenvalue λ with $|\lambda| = \rho(\bar{P})$ is said to be a *leading eigenvector* of $\mathscr{F}$.

Let us consider a family $\mathscr{F}^*$ with $\rho(\mathscr{F}^*) \geq 1$. Then, for any vector $x \in \mathbb{C}^n$, we define the set

$$\mathscr{T}[\mathscr{F}^*, x] = \{x\} \cup \{Px \mid P \in \Sigma(\mathscr{F}^*)\},$$

i.e. the *trajectory* obtained by applying all the products P of matrices of $\mathscr{F}^*$ to the vector x.

The following theorem illustrates the possible use of the trajectory in the determination of an extremal norm.

Theorem 3.2: Let $\mathscr{F}^*$ be a bounded family of complex $n \times n$-matrices s.t.

(i) $\rho(\mathscr{F}^*) \geq 1$

and, for a given vector $x \in \mathbb{C}^n$, let the trajectory $\mathscr{T}[\mathscr{F}^*, x]$ satisfy the following conditions:

(ii) $\operatorname{span}\Big(\mathscr{T}[\mathscr{F}^*, x]\Big) = \mathbb{C}^n$;

(iii) $\mathscr{T}[\mathscr{F}^*, x]$ is a bounded subset of $\mathbb{C}^n$.

Then we have that:

- $\mathscr{F}^*$ is non-defective and $\rho(\mathscr{F}^*) = 1$;
- the set $\mathscr{S}[\mathscr{F}^*, x] = \operatorname{absco}\Big(\mathscr{T}[\mathscr{F}^*, x]\Big)$ is the unit ball of an extremal norm $\|\cdot\|$ for $\mathscr{F}^*$ (that is $\|\mathscr{F}^*\| = 1$).

Proof: By (ii) and (iii) the absolutely convex set

$$\mathscr{S} = \mathscr{S}[\mathscr{F}^*, x] = \overline{\operatorname{absco}\Big(\mathscr{T}[\mathscr{F}^*, x]\Big)}$$

is bounded and absorbing. This means that we can define a vector norm

$$\|z\|_{\mathscr{S}} = \inf\{\rho > 0 \mid z \in \rho\mathscr{S}\}.$$

Now, by definition of $\mathscr{S}$,

$$A^{(i)}\mathscr{S} \subseteq \mathscr{S} \qquad \forall\, A^{(i)} \in \mathscr{F}^*,$$

which means that the family $\mathscr{F}^*$ maps the set $\mathscr{S}$ into itself. Therefore

$$\|\mathscr{F}^*\|_{\mathscr{S}} \leq 1 \implies \rho(\mathscr{F}^*) = 1.$$

■

When $\rho(\mathscr{F}^*) = 1$ the trajectory might play an important role in the construction of an extremal norm and, hence, in the computation of the spectral radius.

We remark that the typical way to fulfil assumption (i) is that of scaling the original family $\mathscr{F} = \{A^{(i)}\}_{i \in \mathscr{I}}$ by the scalar $\rho = \rho(Q_k)^{1/k}$, for some $Q_k \in \Sigma_k(\mathscr{F})$, that is setting

$$\mathscr{F}^* = \Big\{\rho^{-1} A^{(i)}\Big\}_{i \in \mathscr{I}}.$$

The next question is whether a non-defective family admits an extremal complex polytope norm or not.

Assume that the hypotheses of Theorem 3.2 hold. The possibility of actually determining an extremal polytope norm, if any, is based on the search for a *suitable initial vector x* to which it corresponds a trajectory s.t. the set $\mathscr{S}[\mathscr{F}, x]$ is a balanced complex polytope. Such choice is suggested by the forthcoming Theorem 3.3.

The following conjecture is partially related to the Extremality Conjecture in [15].

Conjecture 3.1 (CPE Conjecture): Assume that a finite family of complex $n \times n$-matrices $\mathscr{F} = \{A^{(i)}\}_{1 \leq i \leq m}$ is non-defective and has at least an s.m.p. $\bar{P}$. Then there exists an extremal complex polytope norm for $\mathscr{F}$.

We have proved (see [8]) a weaker version of the above conjecture, namely the Small CPE Theorem, by adding some hypotheses on the family $\mathscr{F}$.

In order to state the result we need to give the following definitions.

Definition 3.2: Let $\mathscr{F}^*$ be a family of complex $n \times n$-matrices and $\hat{\mathscr{F}} = (1/\rho(\mathscr{F}^*))\,\mathscr{F}^*$ the corresponding normalized family. A set $\mathscr{X} \subset \mathbb{C}^n$ is said to be *$\mathscr{F}^*$-cyclic* if for any pair $(x, y) \in \mathscr{X} \times \mathscr{X}$, there exist $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \cdot |\beta| = 1$, and two (finite) normalized products $\hat{P}, \hat{Q} \in \Sigma(\hat{\mathscr{F}})$ such that

$$y = \alpha\,\hat{P}x \quad \text{and} \quad x = \beta\,\hat{Q}y.$$

Definition 3.3: A non-defective bounded family $\mathscr{F}^*$ of complex $n \times n$-matrices is said to be *asymptotically simple* if the set $\mathscr{E}$ of its leading eigenvectors (see Definition 3.1) is finite (modulo scalar nonzero factors) and $\mathscr{F}^*$-cyclic.

Theorem 3.3 (Small CPE Theorem): Assume that a finite family $\mathscr{F}^*$ of complex $n \times n$-matrices fulfils the assumptions of Theorem 3.2. Furthermore assume that

(iv) $\mathscr{F}^*$ is asymptotically simple;

(v) x is a leading eigenvector of $\mathscr{F}^*$.

Then the set

$$\partial\mathscr{S}[\mathscr{F}^*, x] \bigcap \mathscr{T}[\mathscr{F}^*, x] \tag{10}$$

is finite modulo scalar factors of unitary modulus. As a consequence, there exist a finite number of products $\hat{P}^{(1)}, \ldots, \hat{P}^{(s)} \in \Sigma(\mathscr{F}^*)$ such that

$$\mathscr{S}[\mathscr{F}^*, x] = \operatorname{absco}\Big(\{x, \hat{P}^{(1)}x, \ldots, \hat{P}^{(s)}x\}\Big), \tag{11}$$

so that $\mathscr{S}[\mathscr{F}^*, x]$ is a b.c.p.

Then we have proved the following refinement of Theorem 3.3.

Theorem 3.4: Let the hypotheses of Theorem 3.3 hold and let $\mathscr{F}^*$ have a unique minimal s.m.p. (see Definition 3.1). Then all the leading eigenvectors of $\mathscr{F}^*$ (in the set $\Xi = \mathscr{E} \bigcap \partial\mathscr{S}[\mathscr{F}^*, x]$) are vertices of the b.c.p. $\mathscr{S}[\mathscr{F}^*, x]$.

IV. Algorithm and Computational Aspects

We present an algorithm based on the previous results.

Algorithm 4.1:

(0) Let $\mathscr{F} = \{A^{(i)}\}_{i \in \mathscr{I}}$; choose a candidate s.m.p. $Q_k \in \Sigma_k(\mathscr{F})$ (for some k).

(1) Set $\rho = \rho(Q_k)^{1/k}$ and define the scaled family

$$\mathscr{F}^* = \{\rho^{-1} A^{(i)}\}_{i \in \mathscr{I}} \quad \text{with } \rho(\mathscr{F}^*) \geq 1.$$

(2) Compute the leading eigenvector v_1 of Q_k and set $x_1 = v_1$.

(3) Set $s = 1$ and define $\mathscr{V}^{(1)} = \mathscr{X}^{(1)} = \{x_1\}$ and set $\mathscr{P}^{(1)} = \operatorname{absco}\Big(\mathscr{X}^{(1)}\Big)$.

(4) Compute the set of vectors

$$\mathscr{V}^{(s+1)} = \mathscr{F}^*\left(\mathscr{X}^{(s)}\right).$$

(5) If $\mathscr{V}^{(s+1)} \subseteq \mathscr{P}^{(s)}$ then
 Set $\mathscr{S}[\mathscr{F}^*, x] = \mathscr{P}^{(s)}$;
 Stop.

(6) Set $\mathscr{P}^{(s+1)} = \text{absco}\left(\mathscr{V}^{(s+1)} \bigcup \mathscr{X}^{(s)}\right)$ and compute an essential system of vertices $\mathscr{X}^{(s+1)}$ of $\mathscr{P}^{(s+1)}$.

(7) Set $s = s+1$ and Goto (4).

If the procedure halts (for finite s), $\mathscr{S}[\mathscr{F}^*, x]$ is a polytope. Furthermore, if $\text{span}\left(\mathscr{X}^{(s)}\right) = \mathbb{C}^n$, then $\mathscr{S}[\mathscr{F}^*, x]$ generates an extremal complex polytope norm.

This kind of algorithm has been successfully applied for analyzing the asymptotic stability of linear difference equations with variable coefficients arising from the discretization of differential equations (see e.g. [10], [12]) and seems to have a good potential in view of a large class of applications.

Now we briefly investigate how to compute the complex polytope norm of a vector and of a matrix. Given a b.c.p. $\mathscr{P}$ and an essential system of vertices $\mathscr{X} = \{x_i\}_{1 \le i \le m}$, define the *vertex matrix*

$$V = \left[x_1 \dots x_m\right].$$

The equality in (8) yields with $\lambda = [\lambda_1 \dots \lambda_m]^T$,

$$\|z\|_{\mathscr{P}} = \min_{V\lambda = z} \|\lambda\|_1. \tag{12}$$

Note that, if $m = n$, then (12) reduces to $\|z\|_{\mathscr{P}} = \|V^{-1}z\|_1$.

Then consider the case $m > n$. In order to compute $\|z\|_{\mathscr{P}}$, assume without any restriction that the first n columns of the vertex matrix V are linearly independent and define the matrices

$$V_1 = \left[x_1 \dots x_n\right] \quad \text{and} \quad V_2 = \left[x_{n+1} \dots x_m\right].$$

Then, if $\lambda \in \mathbb{C}^m$, define also the $(m-n)$-vector

$$\mu = [\lambda_{n+1} \dots \lambda_m]^T,$$

so that any solution of the equation $V\lambda = z$ may be written in the form

$$\lambda = \begin{bmatrix} V_1^{-1}(z - V_2\mu) \\ \mu \end{bmatrix}.$$

In conclusion, we obtain

$$\|z\|_{\mathscr{P}} = \min_{\mu \in \mathbb{C}^{m-n}} \left\| \begin{bmatrix} V_1^{-1}(z - V_2\mu) \\ \mu \end{bmatrix} \right\|_1, \tag{13}$$

that is the computation of $\|z\|_{\mathscr{P}}$ requires the solution of a minimization problem in $\mathbb{C}^{m-n}$.

Concerning the computation of the induced matrix norms $\|A\|_{\mathscr{P}}$, we have to deal (in parallel) with several optimization problems of the previous form.

First of all, observe that, by Definition 2.1, we immediately have

$$\|A\|_{\mathscr{P}} = \max_{z \in \partial\mathscr{P}} \|Az\|_{\mathscr{P}} = \max_{1 \le i \le m} \|Ax_i\|_{\mathscr{P}}. \tag{14}$$

Thus formula (13) yields

$$\|A\|_{\mathscr{P}} = \max_{1 \le i \le m} \min_{\mu \in \mathbb{C}^{m-n}} \left\| \begin{bmatrix} V_1^{-1}(Ax_i - V_2\mu) \\ \mu \end{bmatrix} \right\|_1. \tag{15}$$

In conclusion, by defining the $(m-n) \times m$-matrix

$$M = \left[\mu^{(1)} \dots \mu^{(m)}\right],$$

where, for each $i = 1, \dots, m$, $\mu^{(i)}$ minimizes the right-hand side in (15), we get the equality

$$\|A\|_{\mathscr{P}} = \left\| \begin{bmatrix} V_1^{-1}(AV - V_2M) \\ M \end{bmatrix} \right\|_1. \tag{16}$$

If the considered family is made of real matrices and the vertices of the polytope are real, we have to deal with the computation of real polytope norms. This turns out to be equivalent to a special linear programming problem (see [2]).

Efficient algorithms for the computation of a general complex polytope norm are presently being investigated.

V. Application to a model problem

Our starting point is the following result from [3], which gives an elementary counterexample to the well-known finiteness conjecture (see [15]) .

Theorem 5.1: There are uncountably many values of the parameter $b \in [0,1]$ such that the family $\mathscr{F} = \{A, B\}$, with

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \qquad B = b\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

does not satisfy the finiteness conjecture.

An explicit counterexample (that is a specific value of b) is unknown.

Theorem 5.1 can actually be refined by determining subintervals of $[0,1]$ such that an s.m.p. exists. This can be obtained by applying Algorithm 4.1.

When $b \ge \frac{4}{5}$ we observe (by a computational investigation) that $P = AB$ is a candidate s.m.p. for the family. In order to prove this rigorously we determine the leading eigen-pair of P, that is

$$\lambda = \frac{1}{4}\beta^2 b \tag{17}$$

$$v_1 = \begin{bmatrix} \frac{1}{2}\beta & 1 \end{bmatrix}^{\mathrm{T}}, \tag{18}$$

where $\beta = 1 + \sqrt{5}$.

Then we scale the family $\mathscr{F}$ by $\rho(P)^{1/2} = |\lambda|^{1/2}$ so as to obtain

$$\mathscr{F}^* = \{A^*, B^*\} = \left\{ \frac{A}{\rho(P)^{\frac{1}{2}}}, \frac{B}{\rho(P)^{\frac{1}{2}}} \right\},$$

that fulfils assumption (i) of Theorem 3.2.

Next we apply Algorithm 4.1 with $x_1 = v_1$. It can be shown by simple although rather technical algebraic manipulations that the algorithm ends successfully after 3 steps. Here are the steps in more detail.

S1. We set $\mathscr{X}^{(1)} = \{v_1\}$; by applying $\mathscr{F}^*$ to $\mathscr{X}^{(1)}$ we obtain $\mathscr{V}^{(2)} = \{v_2, v_3\}$, where

$$\begin{aligned} v_2 = B^* v_1 &= \sqrt{b} \begin{bmatrix} 1 & \dfrac{\beta}{2} \end{bmatrix}^{\mathrm{T}}, \\ v_3 = A^* v_1 &= \frac{1}{\sqrt{b}} \begin{bmatrix} \dfrac{\beta}{2} & \dfrac{2}{\beta} \end{bmatrix}^{\mathrm{T}}. \end{aligned}$$

We set $\mathscr{P}^{(2)} = \operatorname{absco}\left(\mathscr{V}^{(2)} \cup \mathscr{X}^{(1)}\right)$ and get

$$\mathscr{X}^{(2)} = \{v_1, v_2, v_3\}.$$

S2. We apply $\mathscr{F}^*$ to $\mathscr{X}^{(2)}$ and obtain $\mathscr{V}^{(3)} = \{v_4, v_5, v_6, v_7\}$, where

$$\begin{aligned} v_4 = A^* v_2 &= v_1, \\ v_5 = B^* v_2 &= b \begin{bmatrix} \dfrac{2}{\beta} & \dfrac{\beta}{2} \end{bmatrix}^{\mathrm{T}}, \\ v_6 = A^* v_3 &= \frac{1}{b} \begin{bmatrix} \dfrac{4+\beta^2}{\beta^2} & \dfrac{4}{\beta^2} \end{bmatrix}^{\mathrm{T}}, \\ v_7 = B^* v_3 &= \begin{bmatrix} 1 & \dfrac{4+\beta^2}{\beta^2} \end{bmatrix}^{\mathrm{T}}. \end{aligned}$$

We set $\mathscr{P}^{(3)} = \operatorname{absco}\left(\mathscr{V}^{(3)} \cup \mathscr{X}^{(2)}\right)$ and, since $v_7 \in \mathscr{P}^{(2)}$ and $v_5, v_6 \notin \mathscr{P}^{(2)}$, we get

$$\mathscr{X}^{(3)} = \{v_1, v_2, v_3, v_5, v_6\}.$$

S3. We apply $\mathscr{F}^*$ to $\mathscr{X}^{(3)}$ and obtain $\mathscr{V}^{(4)} = \{v_8, v_9, v_{10}, v_{11}\}$, where

$$\begin{aligned} v_8 = A^* v_5 &= \sqrt{b} \begin{bmatrix} \dfrac{4+\beta^2}{\beta^2} & 1 \end{bmatrix}^{\mathrm{T}}, \\ v_9 = B^* v_5 &= b^{3/2} \begin{bmatrix} \dfrac{4}{\beta^2} & \dfrac{4+\beta^2}{\beta^2} \end{bmatrix}^{\mathrm{T}}, \\ v_{10} = A^* v_6 &= \frac{1}{b^{3/2}} \begin{bmatrix} \dfrac{16+2\beta^2}{\beta^3} & \dfrac{8}{\beta^3} \end{bmatrix}^{\mathrm{T}}, \\ v_{11} = B^* v_6 &= \frac{1}{\sqrt{b}} \begin{bmatrix} \dfrac{8+2\beta^2}{\beta^3} & \dfrac{4}{\beta} - \dfrac{16}{\beta^4} \end{bmatrix}^{\mathrm{T}}. \end{aligned}$$

It turns out that $\mathscr{V}^{(4)} \subset \mathscr{P}^{(3)}$. Hence the algorithm halts. Since $\operatorname{span}\left(\mathscr{P}^{(3)}\right) = \mathbb{R}^2$, we can conclude that

$$\mathscr{P} = \mathscr{P}^{(3)} = \operatorname{absco}(\{v_1, v_2, v_3, v_5, v_6\})$$

is a (*real*) polytope inducing an extremal norm for $\mathscr{F}^*$. By Theorem 3.2, we have $\rho(\mathscr{F}^*) = 1$ and, consequently,

$$\rho(\mathscr{F}) = \frac{1+\sqrt{5}}{2}\sqrt{b}. \tag{19}$$

The above procedure works for all $b \in [\frac{4}{5}, 1]$. Figures 1 and 2 illustrate the case $b = \frac{9}{10}$.

Actually, for proving that (19) holds for $b \in [\frac{4}{5}, 1]$, it would be sufficient to positively check the boundary values $b = \frac{4}{5}$ and $b = 1$. In fact, in [3] it is proved that the set of b-values such that a product P is an s.m.p. is a closed interval.

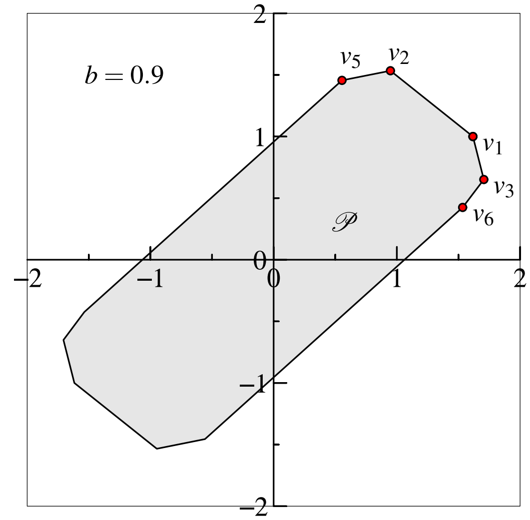

Fig. 1. Extremal polytope norm.

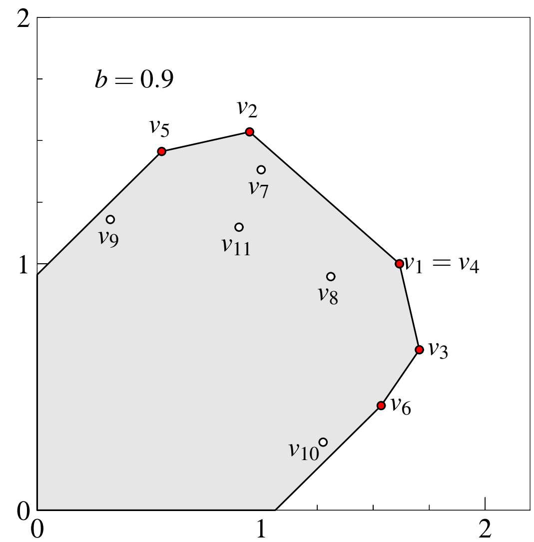

Fig. 2. The vectors computed by the algorithm.

When $b = \frac{4}{5}$ the vector v_7 lies on the boundary of $\mathscr{P}$ (see Figure 3); this choice of b coincides with the fact that a so-called limit spectrum maximizing product appears, that is a matrix $Q^* \in \overline{\Sigma(\mathscr{F}^*)}$ which is not an s.m.p. although being such that $\rho(Q^*) = 1$. In this case the limit spectrum maximizing product is $Q^* = \lim_{k\to\infty} A^* (B^* A^*)^k$, i.e.

$$Q^* = \begin{bmatrix} 1/2 & \beta/4 \\ 1/\beta & 1/2 \end{bmatrix},$$

the eigenvalues of which are 0 and 1. As soon as $b < \frac{4}{5}$ the vector v_7 lies outside the polytope $\mathscr{P}$, so that the algorithm would not halt at all since $\rho(\mathscr{F}^*) > 1$. Similarly, using the candidate s.m.p. $A^2 B$, we applied Algorithm 4.1 to show that

$$\rho(\mathscr{F}) = \left((2+\sqrt{3})\, b\right)^{1/3} \quad \text{for } b \in [0.5734..,0.7444..].$$

More generally, Algorithm 4.1 allows us to find closed subintervals of $[0,1]$ where the finiteness conjecture holds and also to find the corresponding s.m.p..

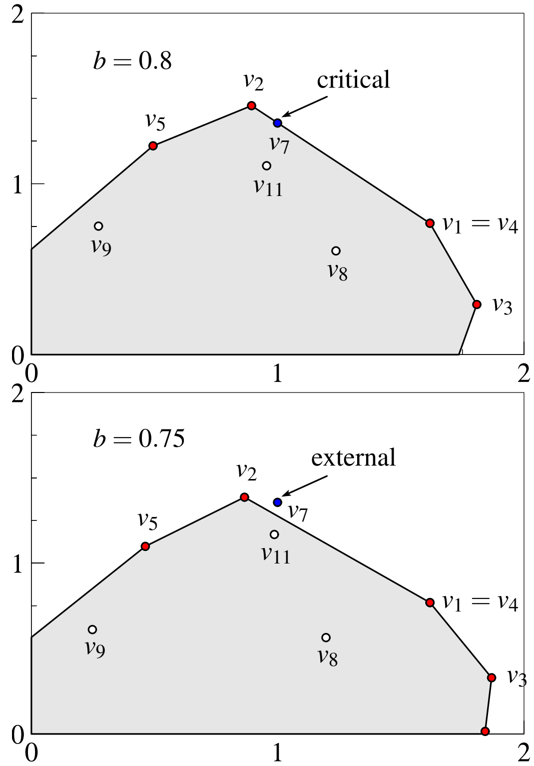

Fig. 3. The boundary case $b = 4/5$ and the case $b = 3/4$.

VI. Conclusions and future work

The main feature of our approach lies in the fact that we try to compute the joint spectral radius by constructing a polytope extremal norm, that is a norm that can be computed in a finite number of steps. However, the success is not guaranteed for all problems of this type. In fact, it is clear that, for each particular problem, a suitable guess has to be found for a spectrum-maximizing product and, moreover, a suitable choice of the initial vector has to be done in order to (possibly) construct a polytope extremal norm.

An important class of problems we plan to investigate by our algorithm is that of the robustness of a stable system with respect to a given class of uncertainties. Given a discrete time system $x(t+1) = A_0 x(t)$ which is asymptotically stable (which means that $\rho(A_0) < 1$), we consider the perturbed system

$$x(t+1) = \left(A_0 + \sum_{i=1}^{p} \delta_i(t) A_i \right) x(t), \qquad t \in \mathbb{N}.$$

The matrices $\{A_i\}_{i=1}^{p}$ are given and the perturbations $\{\delta_i(t)\}$ are unknown. For a robustness analysis we define the family

$$\mathscr{F}_\alpha = \left\{ A_0 + \sum_{i=1}^{p} \delta_i A_i \;\middle|\; \|\delta\| \le \alpha \right\},$$

where $\delta = (\delta_1 \; \delta_2 \; \dots \; \delta_p)^{\mathrm{T}}$, and focus our attention on

$$x(t+1) \in \{A\,x(t) \mid A \in \mathscr{F}_\alpha\}.$$

The goal of the analysis is that to determine the largest uncertainty level α^* such that for $\alpha < \alpha^*$ the system remains stable, that is $\alpha^* = \inf\{\alpha \ge 0 \mid \rho\,(\mathscr{F}_\alpha) \ge 1\}$ (see e.g. [22]).

We plan to develop a code implementing our algorithm in an efficient way and to apply it to this kind of examples and to real-life problems arising in many fields of control theory.

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