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The Role of Metric Regularity in State Constrained Optimal Control

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Abstract— **The aim of this paper is to show that a common set of analytical tools (conditions for metric regularity of the state constraint) resolve a number of important questions in state constrained optimal control. On the one hand, they lead to simple, directly verifiable criteria for non-degeneracy of the state constrained Maximum Principle. On the other hand, they make possible the characterization of the value function as a unique solution to the Hamilton-Jacobi equation, in an appropriate generalized sense, for problems with state and end-point constraints.**

I. Introduction

Consider the following optimal control problem with state constraint

$$(P)\qquad \begin{cases} \text{Minimize } g(x(1)) \\ \text{subject to} \\ \dot{x}(t) \in f(x(t), u(t)) \text{ a.e. } t \in [0,1] \\ u(t) \in \Omega \\ x(0) = x_0 \\ h(x(t)) \le 0 \text{ for all } t \in [0,1]. \end{cases}$$

Here, $g : R^n \to R$, $f : R^n \times R^m \to R^n$ and $h : R^n \to R$ are given functions, $\Omega \subset R^m$ is a given subset and x_0 is a given n-vector.

A measurable function $u : [0,1] \to R^m$ such that $u(t) \in \Omega$, a.e., is called a control function. A function $x \in W^{1,1}$ ($W^{1,1}$ denotes the space of absolutely continuous R^n valued functions on the interval $[0,1]$) is called a state trajectory corresponding to the control function $u(.)$, if it satisfies $\dot{x}(t) = f(x(t), u(t))$, a.e., and $x(0) = x_0$. A couple (x, u), comprising a control function u and a state trajectory corresponding to it, is called a process. If the state trajectory satisfies the state constraint, then the process is termed admissible. Finally, an admissible process is said to be minimizing, if it minimizes the cost over all admissible processes.

The purpose of this paper is to draw attention to the diverse roles of a common set of analytical tools, conditions for 'metric regularity' of the state constraint, in resolving some important issues relating to the above problem. Metric regularity of a constraint is terminology from nonlinear analysis, describing situations where, given a point x violating the constraint there exists a neighbouring point satisfying the constraint, whose closeness to x is estimated by the extent of constraint violation at x.

In the present context, the x's are state trajectories, and the extent of the constraint violation is measured by a scaled version of

$$\max_{t\in[0,1]} h^+(x(t))\,,$$

in which $h^+(z) := \max\,\{0, h(z)\}$, and 'closeness' of state trajectories is measured by the supremum norm.

We show how conditions for metric regularity contribute to the theory of necessary conditions of optimality, by supplying simple, verifiable hypotheses for the state constrained Maximum Principle to be valid in normal form. We also show how examination of conditions for metric regularity leads to the formulation of the correct boundary conditions on the Hamilton-Jacobi equation for an arbitrary generalized solution to be interpreted as the value function for (P). It is assumed throughout that

(H1): f is continuous, $f(., u)$ is of class C^1 for each u and $\nabla_x f(.,.)$ is continuous. Furthermore, there exists $k > 0$ such that

$$|f(x,u)| \;\le\; k(1+|x|) \quad \text{for all } x \in R^n \text{ and } u \in \Omega\,.$$

(H2): Ω is a compact set and $f(x, \Omega) := \cup_{u\in\Omega}\{f(x,u)\}$ is a convex set for each $x \in R^n$

(H3): h is of class C^2 and g is of class C^1.

These hypotheses are imposed for clarity of exposition. They can be considerably weakened in the metric regularity analysis and applications to follow, to allow for multiple state constraints, time dependent state constraints and nondifferentiable data.

We shall write

$$A := \{R^n \,|\, h(x) \le 0\}, \quad \partial A := \{R^n \,|\, h(x) = 0\}$$
$$\text{and} \quad \text{int}\, A := \{R^n \,|\, h(x) < 0\}$$

(the state constraint set, its boundary and its interior respectively).

II. Conditions for Metric Regularity of the State Constraint

The following theorem summarizes the key analytical tools of the paper. It asserts metric regularity of the state constraint of problem (P) under the following 'inward pointing' hypothesis.

(I): There exists $\alpha > 0$ such that

$$\min_{u\in\Omega} \nabla h(x)\cdot f(x,u) < -\alpha \quad \text{for all } x \in \partial A$$

Theorem 2.1 Assume hypotheses (H1)-(H3). Assume also (I). Then there exists $K > 0$ with the following property: given any process (x, u) (that possibly violates the state constraint), there exists an admissible process (x', u') such that

$$||x - x'||_C \;\le\; K \max_{t\in[0,1]} h^+(x(t))\,,$$

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where $||.||_C$ denotes the supremum norm.

An early version of this theorem is due to Soner [14]. Subsequent refinements, to allow for multiple state constraints, 'wedged-shaped' implicit state constraints , time-varying state constraints, more general dynamic models, to dispense with the convexity hypothesis (H2) and to provide a sharper inequality in which $||.||_C$ is replaced $||.||_{W^{1,1}}$ are reported in [13], [12], [11], [10], [7], [9], [4].

III. The Degeneracy Issue for the State Constrained Maximum Principle

The following necessary condition of optimality for problem (P) results from specializing the standard Maximum Principle (see, e.g. [15]) to the problem at hand.

Theorem 3.1 Assume (H1)–(H3). Let $(\bar{x}, \bar{u})$ be a minimizing process. Then there exists $p \in W^{1,1}$, a non-negative measure μ and $\lambda \geq 0$ such that

$$(p(.), \mu, \lambda) \neq 0$$
$$-\dot{p}(t) =$$
$$(p(t) + \int_{[0,t)} \nabla h(\bar{x}(s))\mu(ds)) \cdot \nabla_x f(\bar{x}(t), \bar{u}(t)) \quad \text{a.e.}$$
$$u \to (p(t) + \int_{[0,t)} \nabla h(\bar{x}(s))\mu(ds)) \cdot f(\bar{x}(t), u)$$
$$\text{is maximized over } \Omega \text{ at } u = \bar{u}(t).$$
$$\text{supp } \{\mu\} \subset \{t : h(\bar{x}(t)) = 0\}$$
$$-(p(1) + \int_{[0,1]} \nabla h(\bar{x}(s))\mu(ds)) = \lambda \nabla g(\bar{x}(1)).$$

For some examples of (P) of interest, the above condition yields no useful information about minimizers. Notable is the case when the fixed initial state lies in the state constrained boundary, i.e. x_0 satisfies the condition:

$$h(x_0) = 0 .$$

Indeed, take any admissible process $(\bar{x}, \bar{u})$. Then the conditions of the state constrained Maximum Principle are satisfied for the following choice of 'multiplier set':

$$p(.) \equiv -\nabla h(x_0),\ \mu = \delta_{\{0\}} \text{ and } \lambda = 0 . \qquad (1)$$

($\delta_{\{0\}}$ denotes the unit measure concentrated at $\{0\}$.) This observation confirms that the above conditions do not discriminate minimizing admissible processes in any way.

It should be mentioned that degeneracy of the problem in these circumstances is associated with the fact that, if the state constraint is formulated as

$$\phi(x(.)) \leq 0 ,$$

where $\phi : W^{1,1} \to R$ is the function

$$\phi(x(.)) = \max \{h^+(x(t)) \,|\, t \in [0, 1]\} ,$$

then there are no $x(.)$'s such that $\phi(x(.)) < 0$. Such situations, in which not admissible points exist strictly satisfying an ineqality constraint, are well know to give rise to multiplier degeneracy in nonlinear programming. The following theorem establishes that, besides the degenerate multiplier set (1), there are other possible choices of $(p(.), \mu, \lambda)$, for which $\lambda = 1$, when the inward pointing hypothesis is satisfied. Problems for which such multiplier sets exist are called 'normal'.

Theorem 3.2 (Conditions for non-degeneracy of the state constrained Maximum Principle) Assume in addition to (H1)-(H3) that (I) is satisfied. Then the assertions of Thm. 2.1 are satisfied for some multiplier set $(p(.), \mu, \lambda)$ such that $\lambda = 1$.

There is now a substantial literature on 'non-degenerate' versions of the state constrained Maximum Principle. See [1] and references therein. Versions of the approaches described in this paper were developed in [13], [12]. See also [6]. We provide a sketch of the proof, in which emphasis is given to the point where metric regularity intervenes.

Sketch of Proof. Let $(\bar{x}, \bar{u})$ be a minimizer. We claim that $((\bar{x}, \bar{z} \equiv 0), \bar{u})$ is a minimizer for the modified problem:

$$(Q) \begin{cases} \text{Minimize } g(x(1)) + Kk_g z(1) \\ \text{subject to} \\ \dot{x} = f(x, u),\ \dot{z} = 0 \\ x(0) = x_0, z(1) \geq 0 \\ h(x(t)) - z(t) \leq 0 \quad \text{for all } t \in [0, 1] . \end{cases}$$

Here K is the constant of Thm. 2.1 and k_g is the Lipschitz constant of g on some suitably large region in state space. Indeed, suppose to the contrary that there exists a process (x', z', u') with lower cost. Then

$$g(x'(1)) + Kk_g \max_{t\in[0,1]} h^+(x'(t)) < g(\bar{x}(1)) .$$

But according to the metric regularity theorem Thm. 2.1, applied to (x', u'), there exists (x, u), admissible for the original problem (P), such that

$$||x - x'||_C \leq K \max_{t\in[0,1]} h^+(x'(t)) .$$

But then (x, u) is admissible for (P) and satisfies

$$\begin{aligned} g(x(1)) &\leq g(x'(1)) + k_g ||x - x'||_C \\ &\leq g(x'(1)) + Kk_g \max_{t\in[0,1]} h^+(x'(t)) \\ &< g(\bar{x}(1)) . \end{aligned}$$

This contradicts the optimality of $(\bar{x}, \bar{u})$.

Now apply the standard state constrained Maximum Principle to (Q). Notice that this results in an extra costate component (to match the extra state component z). The conditions reproduce the conditions of the Maximum Principle applied directly to (P), but now supplemented with the extra condition:

$$\int_{[0,1]} d\mu(s) \leq \lambda . \qquad (2)$$

Notice that (2) excludes the possibility '$\lambda = 0$'. Indeed this would imply that $\mu = 0$ (from (2)), that $p(.) \equiv 0$, since the co-state equation is homogeneous and (when $\mu = 0$) has zero boundary condition, and that the new costate component associate with the extra state component is zero. But the multipliers cannot all be zero, so $\lambda > 0$. We can, finally, scale all multipliers to arrange that $\lambda = 1$.

IV. Metric Regularity and State Constrained Dynamic Programming

Take a closed set $C \subset R^n$ and consider a refinement of (P), to which has been added an endpoint constraint '$x(T) \in C$'.

$$(P^C) \qquad \begin{cases} \text{Minimize } g(x(1)) \\ \text{subject to} \\ \dot{x}(t) \in f(x(t), u(t)) \ \text{ a.e. } t \in [0,1] \\ u(t) \in \Omega \\ x(0) = x_0 \text{ and } x(1) \in C \\ h(x(t)) \leq 0 \text{ for all } t \in [0,1]. \end{cases}$$

The value function $V : [0,1] \times R^n \to R \cup \{+\infty\}$ for (P^C) is

$$V(t,x) \ := \ \inf(P^C_{t,x})$$

where 'inf' denotes infimum cost (inf $= +\infty$ if no admissible processes exist) and $(P^C_{t,x})$ is the variant on (P^C), in which (t,x) replaces the initial data $(0,x_0)$. Write

$$\Psi_C(x) \ = \ \begin{cases} 0 & \text{if } x \in C \\ +\infty & \text{otherwise}. \end{cases}$$

Dynamic programming aims to provide a characterization of the value function V in terms of the Hamilton-Jacobi equation

$$V_t(t,x) - H(x, -V_x(t,x)) \ = \ 0 \qquad (3)$$

where H is the Hamiltonian

$$H(x,p) \ = \ \max \ \{p \cdot f(x,u) \mid u \in \Omega\}.$$

V is not in general differentiable, or even finite valued, so solutions to (3) must be interpreted in a generalized sense. We follow the approach involving 'proximal subgradients' [5]:

Definition Take a function $\phi : R^k \to R \cup \{+\infty\}$ and a point $z \in R^k$. The proximal subgradient $\partial\phi(z)$ of ϕ at z, $\partial\phi(z)$, is the set

$$\{\xi \mid \exists M > 0 \text{ s.t. } \phi(z') - \phi(z) \ \geq \ \xi \cdot (z' - z) - M|z' - z|^2 \quad \forall z \in R^k\}$$

if $\phi(z) < +\infty$, and the empty set, if $\phi(z) = +\infty$.

Theorem 4.1 Assume (H1)-(H3). Assume also the 'outward pointing' condition

$$\max_{u \in \Omega} \nabla_x h(x) \cdot f(x,u) \ > 0 \quad \text{ for all } (t,x) \in [0,1] \times \partial A\,.$$

Then the value function is the unique lower semicontinuous function $V : [0,1] \times A \to R \cup \{+\infty\}$ such that

(a): for every $(t,x) \in (0,1) \times \text{int}\, A$

$$\eta^0 - H(x, -\eta^1) = 0 \quad \text{for all } (\eta^0, \eta^1) \in \partial^P V(t,x);$$

(b:) for every $(t,x) \in (0,1) \times \partial A$

$$\eta^0 - H(x, -\eta^1) \geq 0 \quad \text{for all } (\eta^0, \eta^1) \in \partial^P V(t,x);$$

(c): for all $x \in R^n$

$$\liminf_{t' \uparrow 1, x' \overset{\text{int}\, A}{\to} x} V(t', x') = g(x(1)) + \Psi_C(x(1))$$

$$\text{and} \ \ \liminf_{t' \downarrow 0, x' \to x} V(t', x') = V(0,x)\,.$$

Notice that the introduction of the state constraint results in a boundary condition on the boundary of A (condition (b) above).

This, and related characterizations, are well-known when $A = R^n$ ('no state constraints'). They have been developed both within a viscosity solutions framework (see the extensive bibliography in [2]) and within 'the construction of trajectories' framework. (See [8], [5], [15].) We sketch the proof of the extension to allow for pathwise and endpoint state constraints, placing emphasis on the key role of metric regularity, first identified in [11]. See also [3] [10].

Sketch of Proof. Establishing that the value function V satisfies conditions (a)–(c) is straightforward. So take any function V satisfying (a)–(c). We must show that it is the value function. This will follow if we can show that, for arbitrary (τ, ξ),

(i): $V(\tau,\xi) \ \leq \ g(x(1)) + \Psi_C(x(1))$ for all admissible processes (x,u) for $(P^C_{\tau,\xi})$

(ii): $V(\tau,\xi) \ \geq \ g(\bar{x}(1)) + \Psi_C(\bar{x}(1))$ for some admissible process $(\bar{x}, \bar{u})$ for $(P^C_{\tau,\xi})$.

(ii) follows from (b) and a standard viability/weak invariance analysis (see [15], [8]), so we concentrate on (i). Take an admissible process (x,u) satisfying $x(\tau) = \xi$. If $x(.)$ is interior to A, then $x(.)$ evolves in the interior of a tube where V is a solution to the Hamilton-Jacobi equation. Under such circumstances, a standard strong invariance analysis in 'reverse time', that uses condition (b), gives the desired lower bound on the cost of (x,u).

It remains then to consider the case when $x(.)$ does not evolve in the interior of A. In this case we can use Thm. 2.1 ('metric regularity'), the outward pointing hypothesis and the boundary condition (c) to deduce the following: there exist $t_i \uparrow 1$ and a sequence of processes (x_i, u_i) on $[\tau, 1]$ such that $x_i(.)$ is interior on $[\tau, t_i]$ and $x_i(.) \to x(.)$ uniformly. Furthermore, $V(t_i, x_i(t_i)) \to g(x(1)) + \Psi_C(x(1))$, as $i \to \infty$. Since $x_i(.)$ is interior on $[\tau, t_i]$, we have $V(\tau, x_i(\tau)) \leq V(t_i, x_i(t_i))$ for each i. In view of the semicontinuity of V, we deduce

$$V(\tau,\xi) \leq \liminf_{i \to \infty} V(\tau, x_i(\tau)) \leq \lim_{i \to \infty} V(t_i, x_i(t_i)) = V(1, x(1)) + \Psi_C(x(1))\,.$$

This is the required relationship.

REFERENCES

[1] A. V. ARUTYANOV and S. M. ASEEV,"Investigation of the Degeneracy Phenomenon of the Maximum Principle for Optimal Control Problems with State Constraints", *SIAM J. Control and Opt.*, **35**, 1997, pp. 930-952.

[2] M. BARDI and I. CAPUZZO-DOLCETTA, " Optimal Control and Viscosity Solutions of Hamilton–Jacobi Equations", Birkhäuser, Boston, 1997.

[3] F. H. CLARKE, L. RIFFORD and R. J. STERN, "Feedback in State Constrained Optimal Control", *ESAIM Control Optim. and Calc. Var.*, **7**, (2002), pp. 97-134.

[4] F. H. CLARKE and R. J. SSTERN, "State Constrained Feedback Stabilization", *SIAM J. Control and Optim.*, **42**, (2003), pp. 442-441.

[5] F. H. CLARKE, Y. S. LEDYAEV, R. J. STERN and P. R. WOLENSKI, "Nonsmooth Analysis and Control Theory", Graduate Texts in Mathematics vol. 178, Springer Verlag, New York, 1998.

[6] M. M. A. FERREIRA and R. B. VINTER,"When is the Maximum Principle for State-Constrained Problems Degenerate?" , *J. Math. An. and Applic.*, **187**, (1994), pp.432-467.

[7] F. FORCELLINI and F. RAMPAZZO, "On Convex Differential Inclusions whose State is Constrained in the Closure of an Open set" , *Differential and Integral Equations*, **12**, (1999), pp.49-478.

[8] H. FRANKOWSKA, "Lower Semicontinuous Solutions of the Hamilton–Jacobi Equation", *SIAM J. Control Optim.*, **31**, 1993, pp. 257–272.

[9] H. FRANKOWSKA and F. RAMPAZZO, "Filippov's and Filippov-Wazewski's theorems on closed domains", *J. Diff. Eq.*, **161**, (2000), pp. 449-478.

[10] H. FRANKOWSKA and S. PLASKACZ, "Semicontinuous solutions of Hamilton Jacobi Equations with State Constraints", in "Differential Inclusions and Optimal Control", Lecture Notes on Nonlinear Analysis, Vol 2, J. Schauder Center for Nonlinear Studies, Eds. J. Andres, L. Gorniewicz and P. Nistri, (1998), pp. 145-161.

[11] H. FRANKOWSKA and R. B. VINTER, "Existence of Neighbouring Feasible Trajectories: Applications to Dynamic Programming for State Constrained Optimal Control Problems", *J. Optim. Theory and Applic.*, **104**, (2000), 21-40.

[12] F. RAMPAZZO and R. B. VINTER, "Degenerate Optimal Control Problems with State Constraints", *SIAM J. Control Optim.*, **39**, (2000), pp. 989-1007.

[13] F. RAMPAZZO and R. B. VINTER "A Theorem on the Existence of Neighbouring Feasible Trajectories With Applications to Optimal Control", *IMA J. Mathematical Control and Information*, **16**, (1999), pp. 335-351.

[14] H.M. SONER, "Optimal Control with State-Space Constraints", *SIAM J. Control and Optimization* **24**, (1986), pp. 552-561.

[15] R. B. VINTER, Optimal Control, Birkhaüser, Boston, 2000.