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Further results on the observability of quantum systems under general measurement

Domenico D'Alessandro and Raffaele Romano

Abstract— **In this paper, we present a collection of results on the observability of quantum mechanical systems, in the case the output is the result of a discrete nonselective measurement. By defining an *effective observable*, we extend previous results, on the Lie algebraic characterization of observable systems, to general measurements. Further results include the characterization of a 'best probe' (i.e. a minimally disturbing probe) in indirect measurement and a study of the relation between disturbance and observability in this case. We also discuss how the observability properties of a quantum system relate to the problem of state reconstruction. Extensions of the formalism to the case of selective measurements are also given.**

I. Introduction

The *structural properties* of controllability and observability have been studied in depth for deterministic control systems of the form

$$\dot{x} = f(t, x, u), \tag{1}$$

with output

$$y = y(x). \tag{2}$$

In (1) (2), x is the state of the system varying on a given manifold M, u is the control, f a smooth vector field and y a smooth map $M \to \mathbf{R}$ which models how observations on the system depend on the state. For quantum systems, the study of controllability has received greater attention (see e.g. [1], [2], [3], [4]). A study of the observability for quantum systems is complicated by the fact that, in general, the output has a probabilistic nature and the associated probability distribution depends on the current state. Moreover, different types of measurements can be considered according to the specific experimental situation at hand. In the standard text-book selective Von Neumann-Luders measurement (see e.g. [5]), the measured quantity is represented by a Hermitian operator S and the result of the measurement is given by an eigenvalue of S with probability depending on the current state. However several different scenarios and mathematical models of quantum measurements can be considered in different situations (see e.g. [6]). Therefore different definitions of observability may be appropriate and of physical interest in different cases. Nevertheless, there are several reasons to study observability for quantum mechanical control systems. From the viewpoint of the fundamental development of the theory, observability is one of the main concepts to be extended to quantum systems. It is related to the notion of input-output equivalence and therefore to the general question of modeling time varying Hamiltonians[1]. The problem of determining the state from the observation of a quorum of observables is an important one in quantum mechanics [9]. Techniques to find a set of observables which would determine the state without ambiguity have been extensively studied in quantum physics (see e.g. [10], [11]). Observability of quantum systems is also particularly important in view of the recent interest in implementing feedback at the quantum level (see e.g. [12], [13], [14], [15], [16]). A feedback controller uses the knowledge on the current state to update the value of the control, i.e. it is of the form $u = u(t, x)$. The knowledge of the state is obtained through the output and therefore an a priori knowledge of the extent to which information on the state can be obtained from the output is essential in the design of state feedback control scheme.

In a recent paper [17], a study was presented on the observability properties of quantum systems subject to *nonselective measurement* i.e. a measurement where either the result is not read or it is given by the expectation value of a given observable. The latter case is of interest in several experimental scenarios such as nuclear magnetic resonance where the output signal is averaged over a large number of quantum systems. In these cases, the definition and treatment of observability is simplified by the fact that one does not have to consider probabilities explicitly and natural definitions of observability can be given. In this paper we expand upon the treatment of [17] for general measurements. A unified treatment for the various types of measurements is presented using notions of generalized measurement theory [6].

We shall be interested in the dynamics of finite dimensional quantum systems whose state is described by a density matrix ρ. We shall consider measurements occurring at discrete instants of time. In between two measurements, the evolution of ρ is governed by Liouville's equation (see e.g. [5])

$$i\dot{\rho} = [H(u(t)), \rho], \tag{3}$$

where the Hamiltonian H explicitly depends on a control $u = u(t)$. In general, for nonselective measurement the

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The authors are with the Department of Mathematics at Iowa State University, Ames IA-50011 daless@iastate.edu rromano@iastate.edu

[1] Two models are input output equivalent if they produce the same output function for any input. Two input-output equivalent models cannot be distinguished by applying control inputs and observing the output and therefore modeling via input-output experiments may only be made up to equivalence classes of input-output equivalent models. This question is explored for networks of particles with spin in [7], [8]

result can be assumed to be a linear function of the current state ρ. This is the case when one performs a Von Neumann-Luders measurement of the expectation value of a given observable S in which case the output y associated to a system (3) is given by

$$y = Tr(S\rho). \tag{4}$$

Another example is the indirect measurement discussed in detail in Section III. We shall treat the nonselective case in greater detail and then present some extensions to the selective case in Section VI.

The effect of nonselective measurements on the state ρ of the system can be described in general using the formalism of operations [6], [18]. In particular, if $\mathcal{M}$ is a measurable set of possible outcomes, upon measurement the state ρ is modified as $\rho \to \mathcal{F}(\rho)$, where

$$\mathcal{F}(\rho) := \int_{\mathcal{M}} \Phi_m(\rho)\, dm \quad \text{or} \quad \mathcal{F}(\rho) := \sum_{m\in\mathcal{M}} \Phi_m(\rho), \tag{5}$$

according to whether $\mathcal{M}$ is a continuous or discrete set respectively. The super-operators Φ_m are called *operations* and, according to Kraus representation theorem [18], can be expressed as

$$\Phi_m(\rho) := \sum_k \Omega_{mk}\rho\,\Omega^*_{mk}, \tag{6}$$

for a countable set of operators Ω_{mk}.

This paper presents the results of a study on the observability of quantum systems under general nonselective measurement (we refer to [19] for an extensive discussion including the proofs). After introducing the basic definitions and results concerning the observability of quantum systems by means of *effective* observables (Section II), in Section III we present some results for the special case of *indirect* measurement. These include an expression for the effective observable and the derivation of the optimal measurement in terms of minimal disturbance on the state. In Section IV we show that there is not a conflict between observability and low disturbance of the system. In Section V the design of quantum state reconstruction is discussed and related to observability. Section VI presents an extension of the formalism to the case of selective measurement.

II. Observability under general nonselective measurement

If the output y of system (3) is a linear function of the current state, as we assume here, it is always possible to express y as

$$y(t) = Tr\big(S_{eff}\rho(t)\big), \tag{7}$$

for some Hermitian matrix S_{eff}, which represents an *effective observable*. Without loss of generality, we can assume that S_{eff} has zero trace since a trace different from zero would only introduce a constant shift in the value of the output which does not play any role in our treatment. Alternatively, we could quotient all the subspaces (the observability spaces defined in (9) below) by $span\,\{i\mathbf{1}\}$.

Denote by $\rho_k(t,u,\bar{\rho})$ the solution of (3) with initial condition $\bar{\rho}$, control u at time t after $k-1$ measurements, where, at every measurement, the state is modified as in (5)-(6). Then, two states $\bar{\rho}_1$ and $\bar{\rho}_2$ are called *indistinguishable in k steps* (or after k measurements) if, for every control u and time t

$$Tr\big(S_{eff}\rho_k(t,u,\bar{\rho}_1)\big) = Tr\big(S_{eff}\rho_k(t,u,\bar{\rho}_2)\big). \tag{8}$$

A system is called *observable in k steps* if indistinguishability in k steps of $\bar{\rho}_1$ and $\bar{\rho}_2$ implies $\bar{\rho}_1 = \bar{\rho}_2$. A system is called *observable* if it is observable in k steps for some k.

As in the study of controllability (cf. [1], [4], [2]) the *dynamical Lie algebra* associated to the quantum system (3) plays a prominent role. The dynamical Lie algebra $\mathcal{L}$ is defined as the Lie algebra generated by $span_{u\in\mathcal{U}}\{-iH(u)\}$, where $\mathcal{U}$ is the set of possible values for the control u. In order to express the conditions for observability in an arbitrary number of steps, under general nonselective measurement, we associate to the super-operator $\mathcal{F}$ a *dual* super-operator $\mathcal{F}^*$ acting on observables S and defined from the requirement that, for every S and ρ, $Tr(\mathcal{F}^*(S)\rho) = Tr(S\mathcal{F}(\rho))$. Then, we define *generalized observability spaces* $\mathcal{V}_k$, $k = 0, 1, ...$, recursively as

$$\begin{gathered}\mathcal{V}_0 := span\{iS_{eff}\}, \qquad \mathcal{V}_1 := \textstyle\bigoplus_{j=0}^{\infty} ad^j_{\mathcal{L}}\mathcal{V}_0, \\ \mathcal{V}_k := \textstyle\bigoplus_{j=0}^{\infty} ad^j_{\mathcal{L}}\mathcal{F}^*(\mathcal{V}_{k-1}),\end{gathered} \tag{9}$$

where $ad^j_{\mathcal{L}}\mathcal{V}$ is defined as spanned by all the repeated Lie brackets $[R_1,[R_2,\ldots,[R_j,iA]\ldots]]$, and the Lie bracket is taken j times, $R_1,\ldots,R_j \in \mathcal{L}$ and $iA \in \mathcal{V}$. With these definitions, the main results of [17] can be summarized as follows.

Theorem 1: System (3) with output y in (7) is observable in k steps if and only if

$$\mathcal{V}_k = su(n). \tag{10}$$

More in general, write $\rho = \rho_1 + \rho_2$ where ρ_1 is the component of ρ in $i\mathcal{V}_k$[2] and ρ_2 is the component along $i\mathcal{V}_k^{\perp}$ where $\mathcal{V}_k^{\perp}$ is the orthogonal complement of $\mathcal{V}_k$ in $u(n)$. Then, we have the following decomposition of the dynamics

$$\dot{\rho}_1 = -i[H(u),\rho_1], \qquad \dot{\rho}_2 = -i[H(u),\rho_2], \tag{11}$$

and we have

$$y(t) := Tr\big(S_{eff}\rho(t)\big) = Tr\big(S_{eff}\rho_1(t)\big). \tag{12}$$

Initial states are indistinguishable in k steps if and only if they differ by an element in $i\mathcal{V}_k^{\perp}$.

In several interesting scenarios, the measurement scheme has a '*repetition property*' which can be defined by imposing that the operators Ω_{mk} in (6) satisfy $\Omega_{mk}\Omega_{rl} = \delta_{mr}\delta_{kl}\Omega_{mk}$, $\forall\, m, r \in \mathcal{M}$ and $\forall\, k, l$. In these cases $\Phi_m(\Phi_m(\rho)) = \Phi_m(\rho)$ $\forall\rho$, $\mathcal{F}^2 = \mathcal{F}$, and $\mathcal{F}^{*2} = \mathcal{F}^*$.

[2] vector space of Hermitian matrices obtained by multiplying by i the skew-Hermitian matrices in $\mathcal{V}_k$

Physically this means that a second measurement does not modify the state more than the first one. In these cases, it is easy to show that $\mathcal{V}_{k-1} \subseteq \mathcal{V}_k$ so that states that are indistinguishable in k steps are also indistinguishable in $k-1$ steps [3] Moreover, because of the assumption of finite dimensionality, there exists a k such that $\mathcal{V}_k = \mathcal{V}_{\bar{k}}$ for all $\bar{k} > k$. An example is the standard Von Neumann-Luders measurement of the observable S. In this case $S_{eff} = S$. Expressing S as

$$S = \sum_j \lambda_j \Pi_j, \tag{15}$$

where the λ_j's are the eigenvalues of S and Π_j are the orthogonal projections onto the corresponding eigenspaces which play the role of Ω_{mk}'s. $\mathcal{F}$ is given by

$$\mathcal{F}(\rho) := \sum_j \Pi_j \rho \Pi_j. \tag{16}$$

In order to use the results of Theorem 1 we need to find an expression for $\mathcal{F}$ and S_{eff} which describe the particular measurement considered. In the following section we treat in detail the case of *indirect measurement*.

III. Observability under indirect nonselective measurement

In indirect measurement, the system evolves as in (3) until it is in a state ρ_S and it is put in contact with a *probe* system whose initial state we denote by ρ_P. The total system of system and probe at the beginning of the measurement process is in the state

$$\rho_{TOT} := \rho_S \otimes \rho_P. \tag{17}$$

During the measurement process, of duration τ, the total system evolves according to an Hamiltonian

$$H_{TOT} := H(u) \otimes \mathbf{1} + g(t) A \otimes B + \mathbf{1} \otimes H_P. \tag{18}$$

The term H_P describes the dynamics of the probe system alone. The term $g(t) A \otimes B$ gives the interaction between probe and system, where $g(t)$ is nonzero only during the interval $[0, \tau]$; $\mathbf{1}$ is the identity operator. It is usually assumed that, when the interaction is active, it represents the dominant term in the Hamiltonian H_{TOT}. Therefore we shall first assume

$$H_{TOT} := g(t) A \otimes B. \tag{19}$$

[3] The proof uses an expression of S_{eff} in terms of effects F_m defined in Section VI. When the output is an expectation value, then

$$S_{eff} = \sum_{m \in \mathcal{M}} m F_m. \tag{13}$$

Moreover using the expression for the effects

$$F_m = \sum_k \Omega_{mk}^* \Omega_{mk}, \tag{14}$$

and the repetition property, one has $\mathcal{F}^*(S_{eff}) = S_{eff}$ and therefore $\mathcal{V}_1 = \mathcal{F}^*(\mathcal{V}_0)$. $\mathcal{V}_0 \subseteq \mathcal{V}_1$ and by induction one obtains $\mathcal{V}_{k-1} \subseteq \mathcal{V}_k$.

At the end of the interval $[0, \tau]$, an observable S is measured on the probe system, or equivalently an observable $\mathbf{1} \otimes S$ is measured on the total system. In the following proposition we calculate an expression for S_{eff}.

Proposition 3.1: With the above definitions and notations, for indirect measurement

$$S_{eff} = \sum_{k=0}^{\infty} A^k Tr\Big(\big(ad_{-iB}^k \rho_P \big) S \Big) \frac{G^k}{k!}, \tag{20}$$

where

$$G := \int_0^\tau g(t) dt. \tag{21}$$

We notice some features of the expression of S_{eff} (20).

Remark 3.2: Assume we retain only the terms up to first order in G. This is reasonable if the interaction is very quick and of small magnitude. Then we have

$$S_{eff} \approx Tr_P(\rho_P S) \mathbf{1} + Tr_P([-iB, \rho_P] S) G A, \tag{22}$$

so that, if $Tr([-iB, \rho_P] S) \neq 0$ there is a one to one correspondence, in first approximation, between the values of the output and the value of the observable A, and therefore we can say that we are measuring A indirectly.

Remark 3.3: In the special case where S and B are *canonically conjugate* observables on the probe, i.e. $[B, S] = i\gamma \mathbf{1}$ with $\gamma \in \mathbf{R}$, the above correspondence between mean values of S_{eff} and A is exact. This is the case treated in [6]. For more details see [19].

Remark 3.4: In some cases, it is not appropriate to neglect the term containing $H(u)$ in (18). In these cases, it is not possible, in general, to obtain a simple expression of S_{eff} as in (20). However Remark 3.2 above still holds true, assuming the $g(t)$ is a simple square function in $[0, \tau]$ so that $G = \tau$, and u is constant in $[0, \tau]$ (see [19]).

Remark 3.5: The expression of S_{eff} does not depend on the probe being finite dimensional.

In (20), there is a dependence of S_{eff} on the initial state of the probe. As a consequence, it could be possible to modify the observability property for the system by suitably choosing ρ_P. However, the disturbance induced on the system depends on ρ_P as well, and it is interesting to investigate whether there is a conflict between observability and low disturbance of the system. We provide here an analysis of the disturbance on the state while performing an indirect non selective measurement and show how to find the initial state of the probe which gives the (worst case) minimal disturbance. Using this result, we shall show that there is in general no conflict between observability and minimal disturbance.

We consider, as a measure of the disturbance on the state ρ_S, the trace norm

$$d := \|\mathcal{F}(\rho_S) - \rho_S(0)\| = \big[Tr \big(\mathcal{F}(\rho_S) - \rho_S \big)^2 \big]^{\frac{1}{2}}, \tag{23}$$

expressing the distance between the initial state ρ_S and the final one, $\mathcal{F}(\rho_S)$. If we fix all the parameters of the measurement process, the disturbance d will in general be a convex function of ρ_S. Since ρ_S varies on a convex and compact set, the set of all the density matrices, the maximum will in general be achieved on the boundary i.e. it will be a pure state. We shall now show how it is possible to find this worst case pure state in the small time approximation in the case where all the terms in (18) are possibly different from zero (and u is constant). After that, we will derive the corresponding distance d, depending on ρ_P. Then, it will be immediate to find the initial state of the probe which gives the minimum for d. In the above situation, neglecting higher order terms in τ, d^2 can be written as

$$d^2 = -\tau^2\, Tr\big([H(u) + Tr_P(B\rho_P)A, \rho_S(0)]\big)^2. \qquad (24)$$

If we set $X := H(u) + Tr_P(B\rho_P)A$, we have

$$d^2 = 2\tau^2 Tr(X^2\rho_S^2 - X\rho_S X\rho_S), \qquad (25)$$

where we write ρ_S for $\rho_S(0)$ as there is no possibility of confusion. As an orthonormal basis for the Hilbert space of the system, we choose the eigenvectors of the Hermitian operator X, $|\phi_k\rangle$, $k = 1, \ldots, n$, and x_k are the real eigenvalues of X. Since the worst case ρ_S is a pure state, we can write $\rho_S = |\psi\rangle\langle\psi|$ for some $|\psi\rangle = \sum_k r_k|\phi_k\rangle$ where the n coefficients r_k completely specify ρ_S. They can be assumed real by suitably redefining the eigenvectors $|\phi_k\rangle$. We have the further constraint $\sum_k r_k^2 = 1$ since $Tr\rho_S = 1$. To determine the worst case ρ_S, we rewrite (25) as a function of the r_k coefficients

$$d^2 = 2\tau^2\Big(\sum_{k>j}(x_k - x_j)^2(r_k r_j)^2\Big). \qquad (26)$$

We can maximize d^2 with respect to the n parameters r_k using the Lagrange method. The system that we obtain always admits a solution since the function d^2 is continuous over the compact set of pure density matrices. In the next section we will explicitly compute ρ_S in a particular case. We summarize our discussion in the following theorem.

Theorem 2: The worst case disturbance in a small time approximation is given by d^2 in (26), where $(r_1, \ldots, r_n)$ are obtained using the Lagrange method. Therefore given u, A and B in the definition of X, the initial state of the probe which minimizes the worst case error has to be chosen so as to minimize this d^2.

IV. Observability and minimal disturbance

As a concrete example of observability under an indirect measurement, we consider the simple case of two-dimensional system and probe. The system is a qubit with external control u affecting a two-components magnetic field, for example

$$H(u) = E_x(u)\sigma_x + E_y(u)\sigma_y. \qquad (27)$$

We assume a piecewise constant control $u \in \{u_1, u_2\}$ that flips the magnetic field directions x and y, that is $E_x(u_1) = E$, $E_y(u_1) = 0$ and $E_x(u_2) = 0$, $E_y(u_2) = E$. We use a second qubit as probe and we let it interact with the system for a short time τ in which the free evolution (27) can be neglected. To get information about the initial state ρ_S we measure $S = \sigma_z$ on the probe. Assuming a simple Ising model of interaction, $A = \sigma_y$ and $B = \sigma_x$, the effective observable S_{eff} can be explicitly computed [19]:

$$S_{eff} = Tr_P(\sigma_z\rho_P)\cos 2G\,\mathbf{1} + Tr_P(\sigma_y\rho_P)\sin(2G)\,\sigma_y. \qquad (28)$$

Remark 4.1: The observability properties of our system strongly depend on the initial state of the probe ρ_P. Suppose that $Tr(\sigma_y\rho_P) = 0$, then $S_{eff} = 0$ and the observability spaces $\mathcal{V}_k$ contain only the null vector. Then the system is not observable and the states are all indistinguishable $\forall\, k$. On the other hand, suppose $Tr(\sigma_y\rho_P) \neq 0$. In such a case $S_{eff} = Tr(\sigma_y\rho_P)\sin 2G\,\sigma_y$ and $\mathcal{V}_k = su(2)$ for all k, and the system is observable in k steps $\forall\, k$.

We now determine the minimal disturbing probe described in Theorem 2. We assumed that during the time interval τ the control does not change, and its actual value is relevant in order to find the minimal disturbing probe. In our example, the Lagrange method gives

$$\begin{cases} r_1\big((x_2 - x_1)^2 r_2^2 + \lambda\big) = 0 \\ r_2\big((x_2 - x_1)^2 r_1^2 + \lambda\big) = 0 \\ r_1^2 + r_2^2 = 1. \end{cases} \qquad (29)$$

where x_1, x_2 are the eigenvectors of X and they depend on u. Solving (29) we find the worst case ρ_S:

$$\rho_S = \frac{1}{2}\left(|\phi_1\rangle\langle\phi_1| + |\phi_2\rangle\langle\phi_2| \pm |\phi_1\rangle\langle\phi_2| \pm |\phi_2\rangle\langle\phi_1|\right) \qquad (30)$$

leading to $d^2 = (x_2 - x_1)^2/4$. For $u = u_1$, $x_2 - x_1 = 2[E^2 + (Tr_P(\sigma_x\rho_P))^2]$, for $u = u_2$, $x_2 - x_1 = 2(E + Tr_P(\sigma_x\rho_P))$. Then, the minimally disturbing probe must satisfy

$$\begin{cases} Tr_P(\sigma_x\rho_P) = 0 & \text{for } u = u_1, \\ Tr_P(\sigma_x\rho_P) = \max\{-E, -1\} & \text{for } u = u_2. \end{cases} \qquad (31)$$

In both cases there is not a conflict between observability and minimal disturbance (see Remark 4.1).

V. Observability and state reconstruction

We present in this section a system theoretic treatment of the problem of state determination for the system (3) with output (4). In systems and control theory, for a continuous time system such as (3), under observability conditions, the (initial) state is determined from a continuous reading of the output. From a physics point of view, a continuous monitoring of the output will introduce a back action on the state of the quantum system and therefore it will render invalid the model (3). However, this scheme is of interest for quantum systems in situations like the following. Assume we want to determine the unknown (initial) state and we have many copies of the same system. We perform a nonselective measurement on each copy at slightly different times so as to simulate a continuous measurement. The data so obtained can then be used by the observer to reconstruct the state of the system (without measurement back-action).

With this motivation in mind, a method for reconstructing the initial state can be obtained by adapting to our case techniques for time varying linear systems [20]. Observability (in one step) is a necessary and sufficient condition for reconstructing the initial state from a reading of the output. In fact, if the system is not observable, then it is not possible to discern between two indistinguishable initial states. Viceversa, assume the system is observable. Then, we have that [17]

$$\{X^* iSX | X \in e^{\mathcal{L}}\} = su(n). \tag{32}$$

This means that we can choose a control u, so that, for the corresponding solution X_u of Schrödinger operator equation

$$\dot{X} = -iH(u)X, \quad X(0) = I, \tag{33}$$

the $n^2 - 1$ elements of the matrix $X_u^* S X_u$ (namely the real functions composing the matrix modulo the fact that this matrix is Hermitian) are linearly independent. $e^{\mathcal{L}}$ is the Lie group of all the matrices X_m for which there exists a control steering X in (33) from the identity to X_m. We can select $n^2 - 1$ matrices $X_1, ..., X_{n^2-1}$ so that $X_1^* S X_1$,...,$X_{n^2-1}^* S X_{n^2-1}$ are linearly independent and then concatenate the controls steering the matrix X in (33) to X_1, $X_2 X_1^*$, $X_3 X_2^*$,...,$X_{n^2-1} X_{n^2-2}^*$. Now assume that, in the control interval $[0, T]$, the (significant) real entries of $X_u^* S X_u$ are linearly independent and define the linear operator $\mathcal{W}$ which maps $n \times n$ Hermitian matrices with zero trace into $n \times n$ Hermitian matrices with zero trace as follows

$$\mathcal{W}_u(\hat{\rho_0}) := \int_0^T X_u^*(t) S X_u(t) Tr\big(X_u^*(t) S X_u(t) \hat{\rho_0}\big) dt. \tag{34}$$

The operator $\mathcal{W}_u$ has the following property.

Proposition 5.1: If the $n^2 - 1$ real functions composing $X_u^* S X_u$ are linearly independent then $\mathcal{W}_u$ has rank $n^2 - 1$ and therefore it has an inverse $\mathcal{W}_u^{-1}$.

Now, from formula (4), we obtain

$$y(t) = Tr\big(X_u^* S X_u (\rho_0 - \frac{1}{n} I_{n \times n})\big). \tag{35}$$

Therefore, using the definition of $\mathcal{W}_u$ (34), we have the following formula for the reconstruction of the initial state ρ_0,

$$\rho_0 = \frac{1}{n} I_{n \times n} + \mathcal{W}_u^{-1} \left(\int_0^T X_u^*(t) S X_u(t) dt \right). \tag{36}$$

Formula (36) represents a system theoretic alternative to methods for quantum state tomography. We summarize the discussion in the following theorem.

Theorem 3: Consider system (3) with output (4). If the system is observable (in one step), then there exists a control such that formula (36) gives the initial state.

An alternative to the '*static*' state reconstruction formula (36) is the design of an *asymptotic observer* namely a dynamical system which uses only a reading of the output and whose state asymptotically converges to the actual state of the system. A proposal for such an asymptotic observer which is inspired the treatment for linear time varying systems in [21] is presented in [19].

VI. Some extensions to selective measurement

In this section, we discuss how the theory described above for nonselective measurement extends to selective measurement. There is no difficulty in doing this in the most general case namely in the context of the generalized measurement theory of operations and effects [6]. According to this theory, given a measurement scheme, to every result m is associated a positive operator F_m, called an *effect*. If ρ is the current state of the system, the probability of obtaining the result m (or of an event m to occur) is

$$P(m) = Tr(F_m \rho). \tag{37}$$

After a result m (or, more generally an event m) has occurred, the state is modified according to

$$\rho \rightarrow P(m)^{-1} \Phi_m(\rho), \tag{38}$$

where the positive super-operators Φ_m are the same *operations* as in (6) and $Tr(\Phi_m(\rho)) = P(m) = Tr(F_m \rho)$. Two initial states $\bar{\rho}_1$ and $\bar{\rho}_2$ are said to be indistinguishable in k steps, in selective measurement, if they give every possible result with the same probability at the $k-$th measurement, for every choice of the control u. In formulas (cf. (8))

$$Tr(F_m \rho_k(t, u, \bar{\rho}_1)) = Tr\big(F_m \rho_k(t, u, \bar{\rho}_2)\big) \qquad \forall m \in \mathcal{M}, \tag{39}$$

where $\mathcal{M}$ is the set of possible results (events). Let $P_k(m)$ be the probability of having the result m at the $k-$th measurement and let $P(m_1, ..., m_k)$ be the joint probability of having result m_1 at the first step, m_2 at the second step and so on. Also, indicate by $P_k(m_k | m_1, \ldots m_{k-1})$ the conditional probability of having m_k at the $k-$th measurement, given m_1,...m_k as ordered results of the previous measurements. By use of the formula

$$P_k(m) = \sum_{m_1 \ldots m_{k-1}} P_k(m | m_1, ..., m_{k-1}) P(m_1, ..., m_{k-1}), \tag{40}$$

and repeated use of Bayes' formula we can write $P_k(m)$ starting from an initial condition ρ_0 as

$$P_k(m) = \sum_{m_1 \ldots m_{k-1}} Tr\Big(F_m X_k (\Phi_{m_{k-1}} (X_{k-1} (\Phi_{m_{k-2}} \ldots (\Phi_{m_1} (X_1 \rho_0 X_1^*)) \ldots)) X_{k-1}^*)) X_k^*\Big) \tag{41}$$

where X_j, $j = 1, \ldots, k$ is the evolution solution of the Schrödinger operator equation (33) in the interval between the $(j-1)$-th measurement and the $j-$th measurement. Using (41) and using the linearity of the operators Φ_m, we can rewrite $P_k(m)$ as

$$P_k(m) = Tr\Big(F_m X_k \mathcal{F}(X_{k-1} ... \mathcal{F}(X_1 \rho_0 X_1^*) ... X_{k-1}^*) X_k^*\Big). \tag{42}$$

From this point on the theory goes as in [17] and the result is an extension of Theorem 1. In particular, one defines the 'selective' observability spaces (cf. (9))

$$\mathcal{V}_0^{sel} := span_{m\in\mathcal{M}}\{iF_m\}, \qquad \mathcal{V}_1^{sel} := \bigoplus_{j=0}^{\infty} ad_{\mathcal{L}}^{j}(\mathcal{V}_0^{sel}),$$

$$\mathcal{V}_k^{sel} := \bigoplus_{j=0}^{\infty} ad_{\mathcal{L}}^{j}\mathcal{F}^*(\mathcal{V}_{k-1}^{sel}), \tag{43}$$

and Theorem 1 extends by replacing nonselective observability with selective observability and the spaces $\mathcal{V}$ with the spaces $\mathcal{V}^{sel}$ [4].

The remarks following Theorem 1 on the implications of the repetition property also extend with only minor formal modifications. In the particular case of the standard Von Neumann-Luders measurement, the observable S is written in terms of the projectors Π_λ and the eigenvalues λ as

$$S = \sum_{\lambda\in\mathcal{M}} \lambda \Pi_\lambda, \tag{44}$$

and the above theory holds with Π_λ playing the role of the effects F_m.

Remark 6.1: The observability space $\mathcal{V}_0^{sel}$ does, in general, include the observability space $\mathcal{V}_0$ and therefore the same is true for the observability spaces $\mathcal{V}_k^{sel}$ and $\mathcal{V}_k$. This implies that nonselective observability implies selective observability, as it is intuitive but not viceversa. A specific example (a spin $1/2$ particle for which the z component of the spin is measured with a Von Neumann-Lüders measurement) is described in [19].

VII. Conclusions

This paper has presented a collection of results on the observability of quantum systems with emphasis on the case of nonselective measurement. In particular

1. Using the formalism of generalized measurement and of effects and operations we have extended the basic definitions and criteria of observability to the case of general measurement by introducing an effective observable.
2. We have derived a general expression for the effective observable for a Von Neumann indirect measurement.
3. In the case of indirect measurement, we have derived an expression for the state of the probe which would introduce the minimum disturbance in the state to be measured. We have showed that the requirement of a minimal disturbing probe does not in general compromise the observability properties of the resulting system and therefore the amount of information obtained on the state by the measurement of the output.
4. We have presented two system theoretic methods to reconstruct the state by a measurement of the expectation value of an appropriate observable. One of them is through an integral formula and uses readings over a finite interval of time. The other is through an asymptotic observer whose state converges to the state of the measured system.
5. We have extended the basic definitions and observability criteria to selective measurements.

We believe that the system theoretic approach to quantum state determination is worth being further investigated. Extensions of our definitions and results to continuous measurements, optimization of the methods for state determination in specific settings, applications of observer design in closed loop quantum systems are only few possible subjects for future research.

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[4]Condition (14) of Theorem 1 needs to be slightly modified as the effects F_m do not necessarily have zero trace, by replacing $\mathcal{V}_k$ with $\mathcal{V}_k/span\{i\mathbf{1}\}$ or by making all the effects traceless

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