Proceedings of the
44th IEEE Conference on Decision and Control, and
the European Control Conference 2005
Seville, Spain, December 12-15, 2005

Distributed Consensus in Networks of Dynamic Agents

D. Bauso, L. Giarré and R. Pesenti

***Abstract*— Stationary and distributed consensus protocols for a network of n dynamic agents under local information is considered. Consensus must be reached on a group decision value returned by a function of the agents' initial state values. As a main contribution we show that the agents can reach consensus if the value of such a function computed over the agents' state trajectories is time invariant. We use this basic result to introduce a protocol design rule allowing consensus on a quite general set of values. Such a set includes, e.g., any generalized mean of order p of the agents' initial states. We demonstrate that the asymptotical consensus is reached via a Lyapunov approach. Finally we perform a simulation study concerning the alignment maneuver of a team of unmanned air vehicles.**

I. INTRODUCTION

Distributed *consensus protocols* are local control policies based on partial information that allow the coordination of multi-agent systems. Agents implement a consensus protocol to reach *consensus*, that is to (make their states) converge to a same value, called *consensus-value*, or *group decision value* [6].

Coordination of agents/vehicles is an important task in several applications including autonomous formation flight [7], [8], cooperative search of unmanned air-vehicles (UAVs) [9], [10], swarms of autonomous vehicles or robots [11], [12], multi-retailer inventory control [13], [14] and congestion/flow control in communication networks [15].

Actually, a central point in consensus problems is the connection between the graph structure, defined by the Laplacian Matrix, and delays or distortions in communication links [16]. Switching topology and directional communications are studied in [6], [17], [18], [19], [20], [21], while cooperation based on the notion of *coordination variable* and *coordination function* in [22], [23]. There, coordination variable is referred to as the *minimal amount of information needed to effect a specific coordination objective*, whereas a coordination function *parameterizes the effect of the coordination variable on the myopic objectives of each agent*.

In this paper, n dynamic agents reach consensus on a group decision value by implementing *optimal*, *distributed* and *stationary control policies* based on neighbors' state feedback. Here, neighborhood relations are defined by a time-invariant connected undirected communication topology. To generalize our results to systems with switching topology and directional communications, we introduce *dwell time* [17], [1] between switchings and exploit an analysis tool for switched systems known as *common Lyapunov function* [6], [2], [3], [4]. Similarly to [6], [7], [24], the dynamics of the agents is a simple first order one. We restrict the group decision value to be a *permutation invariant* function of the agents' state initial values. Permutation invariance means that the value of the function is independent of the agents indexes.

Our contribution to the study on consensus problems is to show that consensus can be reached if the agents' state trajectories satisfy a certain time invariancy property. On the basis of such a result, we prove that the group decision values considered are sufficiently general to include any mean of order p of the agents' state initial values, such as the arithmetic/min/max means usually dealt with in the literature (see, e.g., [25]). Finally we argue that agents reach asymptotically consensus on the desired group decision value by studying equilibrium properties and stability of the group decision value via Lyapunov theory.

This work was supported by MIUR-PRIN "Robust Techniques for uncertain systems control"

D. Bauso and R. Pesenti are with DINFO, Università di Palermo, Italy [dario.bauso,pesenti]@unipa.it

L. Giarré is with DIAS, Università di Palermo, Italy giarre@unipa.it

II. The Consensus Problem

We consider a system of n agents $\Gamma = \{1, \dots, n\}$ and model the interaction topology among agents through a time-invariant connected undirected network (graph) $G = (\Gamma, E)$. The network is undirected since we assume the existence of only bidirectional information exchange links between pairs of agents. Then, each edge (i, j) in the edgeset E means that agent i can receive information from agent j and, vice versa, agent j can receive information from agent i. The network is connected since we assume that for any agent $i \in \Gamma$ there exists a path, i.e., a sequence of edges in E, $(i, k_1)(k_1, k_2)\dots(k_r, j)$, that connects it with any other agent $j \in \Gamma$. Finally, the network G is not complete since each agent i exchanges information only with a subset of other agents $N_i = \{j : (i, j) \in E\}$ called *neighborhood of* i. Each agent i has a (simplified) *first-order dynamics* controlled by a *distributed* and *stationary* control policy

$$\dot{x}_i = u_i(x_i, x^{(i)}) \quad \forall i \in \Gamma, \tag{1}$$

where $x^{(i)}$ is the state vector of the agents in N_i with generic component j defined as follows, $x_j^{(i)} = \begin{cases} x_j & \text{if } j \in N_i, \\ 0 & \text{otherwise.} \end{cases}$

and such that (1) has unique solutions. The policy is distributed since, for each agent i, it depends only on the local information available to it, which is x_i and $x^{(i)}$. No other information on the current or past system state is available to agent i. The policy is stationary since it does not depend explicitly on time t. In other words, the policy is a time invariant and memoryless function of the state. Define the vector $x(t) = \{x_i(t), i \in \Gamma\}$ as the system state and $u(.) = \{u_i(.) : i \in \Gamma\}$ as a *distributed stationary protocol* or simply *a protocol*. Let $\hat{\chi} : \mathbb{R}^n \rightarrow \mathbb{R}$ be a generic continuous function of n variables $x_1, \ldots, x_n$ which is permutation invariant, i.e., $\hat{\chi}(x_1, x_2, \ldots, x_n) = \hat{\chi}(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)})$ for any one to one (permutation) mapping $\sigma(.)$ from the set Γ to the set Γ. Henceforth $\hat{\chi}$ is also called *agreement function*. Putting together slightly different definitions in [6], [5], [25], we can say that a protocol $u(.)$ makes the agents asymptotically reach consensus on a *group decision value* $\hat{\chi}(x(0))$ if $\|x_i - \hat{\chi}(x(0))\| \longrightarrow 0$ as $t \longrightarrow \infty$. When this happens we also say that the system converges to $\hat{\chi}(x(0))\mathbf{1}$. Here and in the following, $\mathbf{1}$ stands for the vector $(1, 1, \ldots, 1)^T$.

Notwithstanding each agent i has only a local information $(x_i, x^{(i)})$ about the system state x, we are interested in making the agents reach consensus on group decision values that are functions of the whole system initial state $x(0)$. In particular, we are interested in agreement functions verifying

$$\min_{i\in\Gamma}\{y_i\} \leq \hat{\chi}(y) \leq \max_{i\in\Gamma}\{y_i\}, \quad \text{for all } y \in \mathbb{R}^n. \tag{2}$$

The above condition means that the group decision value must be confined between the minimum and the maximum agents' initial state values.

Finally, we define an *individual objective* for an agent i, i.e.,

$$J_i(x_i, x^{(i)}, u_i) = \lim_{T\longrightarrow\infty} \int_0^T \left(F(x_i, x^{(i)}) + \rho u_i^2\right) dt \tag{3}$$

where $\rho > 0$ and $F : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$ is a nonnegative *penalty function* that measures the deviation of x_i from neighbors' states. We say that a protocol is *optimal* if each u_i optimizes the corresponding individual objective.

In the above context, we face the following problem.

Problem 1: (Consensus Problem) Consider a network $G = (\Gamma, E)$ of dynamic agents with first-order dynamics. For any function $\hat{\chi}(.)$ determine a (distributed stationary) protocol, whose components have the feedback form (1), that makes the agents asymptotically reach consensus on $\hat{\chi}(x(0))$ for any initial condition $x(0)$.

In the following, a protocol that solves the consensus problem is also referred to as a *consensus protocol*.

III. Time Invariancy of $\hat{\chi}(x(t))$

Initially, we show that if a protocol, which solves a consensus problem, is distributed and stationary then the system state trajectory enjoys the property that $\hat{\chi}(x(t))$ is time invariant. Then, we find a family of non trivial protocols that guarantee such a property. We prove that some of such protocols are consensus protocols with respect to $\hat{\chi}(x(0))$ in the next section.

Lemma 1: (Time invariancy) *Consider a network $G = (\Gamma, E)$ of dynamic agents with first-order dynamics. For any function $\hat{\chi}(.)$ implement a distributed stationary protocol $u(t)$, whose components have the feedback form (1), that makes the agents converge to value $\hat{\chi}(x(0))$ for any initial state $x(0)$. Then the value of $\hat{\chi}(x(t))$ is time invariant, i.e., $\hat{\chi}(x(t)) = \hat{\chi}(x(0))$ for all $t > 0$.*

In the above proof the key idea is that time-invariance of the feedback protocol implies the time-invariance of the decision values. Nevertheless, observe that there may exist consensus protocols not implying the time invariancy of $\hat{\chi}(x(t))$. However, such protocols must rely on additional information about the whole system initial state $x(0)$ or the value of $\hat{\chi}(x(0))$. Unfortunately, in the case under study the local information alone is in general not sufficient to reconstruct either $x(0)$ or $\hat{\chi}(x(0))$.

From continuity of function $\hat{\chi}$ and supposing that the state trajectory reaches the point $\hat{\chi}(x(0))\mathbf{1}$, the time invariancy property stated in Lemma 1 implies also that $\hat{\chi}(x(0)) = \hat{\chi}(\hat{\chi}(x(0))\mathbf{1})$. Note that the last condition satisfies (2). Actually, (2) imposes that function $\hat{\chi}(.)$ must be chosen such that any point $\lambda\mathbf{1}$, for all $\lambda \in \mathbb{R}$, is a *fixed point*, i.e., $\hat{\chi}(\lambda\mathbf{1}) = \lambda$, as it can be trivially derived assuming $y = \lambda\mathbf{1}$.

With this consideration in mind, let us impose the time invariancy of $\hat{\chi}(x(t))$. It holds $\hat{\chi}(x(t)) = const$ when

$$\frac{d\hat{\chi}(x(t))}{dt} = \nabla_x \hat{\chi}(x) \cdot \dot{x} = \sum_{i\in\Gamma} \frac{\partial \hat{\chi}(x)}{\partial x_i} \dot{x}_i = \sum_{i\in\Gamma} \frac{\partial \hat{\chi}(x)}{\partial x_i} u_i = 0. \tag{4}$$

The trivial protocol constantly equal to 0 leaves any value $\hat{\chi}(x(t))$ time invariant, for any possible $\hat{\chi}(.)$, but obviously does not make the system converge. Consequently, it is no longer considered hereafter.

Some other solutions of equation (4) can be obtained easily when $\hat{\chi}(x)$ presents a particular structure. A first possibility is when the following condition holds

$$\frac{\partial \hat{\chi}(x)}{\partial x_i} u_i = 0 \quad \forall i \in \Gamma. \tag{5}$$

For example, $\hat{\chi}(x) = \min\{x_i\}$ and $u_i = h(x_i, \min_{j\in N_i}\{x_j\})$ satisfy the above condition, for any $h(x, y) : \Re^2 \rightarrow \Re$ such that $h(x, y) = 0$ when $x = y$. Actually, $\frac{\partial \hat{\chi}(x)}{\partial x_i} \neq 0$ only for i such that $x_i = \min_{j\in\Gamma}\{x_j\}$, then, by definition of function $h(.)$, it holds $u_i(x_i) = 0$ and hence (5). The system converges to $\hat{\chi}(x(0))$ if we impose the additional condition $h(x, y) < 0$ when $x > y$. Trivially, analogous argument applies to $\hat{\chi}(x) = \max\{x_i\}$.

We specialize our study considering a more general family of function $\hat{\chi}(x)$.

Assumption 1: (Structure of $\hat{\chi}(.)$) Assume that the generic agreement function $\hat{\chi}(.)$ satisfies condition (2) and is such that $\hat{\chi}(x) = f(\sum_{i\in\Gamma} g(x_i))$, for some $f, g : \Re \rightarrow \Re$ with $\frac{dg(x_i)}{dx_i} \neq 0$ for all x_i.

A point of interest is that the above family of agreement function is more general than the arithmetic/min/max means already reported in the literature (see, e.g., Tab. I). In this sense, observe that the structure of the agreement function is general to the extent that any value in the range between the minimum and the maximum agents' initial state values can be chosen as a group decision value. To see this, it is

mean	$\hat{\chi}(x)$	$f(y)$	$g(z)$
arithmetic	$\sum_{i\in\Gamma}\frac{1}{n}x_i$	$\frac{1}{n}y$	z
geometric	$\sqrt[n]{\prod_{i\in\Gamma}x_i}$	$e^{\frac{1}{n}y}$	$\log z$
harmonic	$\frac{1}{\sum_{i\in\Gamma}\frac{n}{x_i}}$	$\frac{n}{y}$	$\frac{1}{z}$
mean of order p	$\sqrt[p]{\sum_{i\in\Gamma}\frac{1}{n}x_i^p}$	$\sqrt[q]{\frac{1}{n}y}$	z^p

TABLE I

COMMONLY USED MEANS

sufficient to consider mean of order p with p varying between $-\infty$ and ∞.

Theorem 1: (Protocol design rule) For any agreement function $\hat{\chi}(.)$ as in Assumption 1, the non trivial protocol

$$u_i(x_i, x^{(i)}) = \frac{1}{\frac{dg(x_i)}{dx_i}} \sum_{j\in N_i} \phi(x_j, x_i), \quad \textit{for all } i \in \Gamma \qquad (6)$$

lets the value $\hat{\chi}(x(t))$ be time invariant, if $\phi : \Re^2 \to \Re$ is an antisymmetric function, i.e., $\phi(x_j, x_i) = -\phi(x_i, x_j)$.

Consider the linear function $\phi(x_j, x_i) = \alpha(x_j - x_i)$ and the different means introduced in Tab. I. The arithmetic mean is time invariant under protocol $u(x_i, x^{(i)}) = \alpha\sum_{j\in N_i}(x_j - x_i)$; the geometric mean under protocol $u(x_i, x^{(i)}) = \alpha x_i \sum_{j\in N_i}(x_j - x_i)$; the harmonic mean under protocol $u(x_i, x^{(i)}) = -\alpha x_i^2 \sum_{j\in N_i}(x_j - x_i)$; the mean of order p under protocol $u(x_i, x^{(i)}) = \alpha\frac{x_i^{1-p}}{p}\sum_{j\in N_i}(x_j - x_i)$. Obviously, due to the time invariancy of $\hat{\chi}(x(t))$ if the system converges, it will converge to $\hat{\chi}(x(0))\mathbf{1}$, but it does not necessarily converge. As it turns out at the end of the next section, for the cases in the example, the system converges to $\hat{\chi}(x(0))\mathbf{1}$ only if $\alpha > 0$ (for the harmonic mean only if $\alpha < 0$). In addition, we must also assume that $x_i(0) > 0$ for all $i \in \Gamma$, when we deal with means different from the arithmetic one.

IV. SUFFICIENT CONDITIONS FOR CONVERGENCE

In the previous section, we find a family of protocols as in (6) that guarantees the time invariancy of $\hat{\chi}(x(t))$. In this section, we determine sufficient conditions on the structure of functions $g(.)$ and $\phi(.)$ such that a protocol of type (6) makes the system converge to $\hat{\chi}(x(0))\mathbf{1}$ for any agreement function $\hat{\chi}(.)$ and initial state $x(0)$. In particular, we prove that the system converges when the function $g(.)$ is strictly increasing and the function $\phi(.)$ is defined as follows:

$$\phi(x_j, x_i) = \alpha\hat{\phi}(\vartheta(x_j) - \vartheta(x_i)), \qquad (7)$$

where $\alpha > 0$, function $\hat{\phi} : \Re \to \Re$ is continuous, locally Lipschitz, odd and strictly increasing, and function $\vartheta : \Re \to \Re$ is differentiable with $\frac{d\vartheta(x_i)}{dx_i}$ locally Lipschitz and strictly positive.

Putting together (6) and (7) the resulting protocol is

$$u_i(x_i, x^{(i)}) = \alpha\frac{1}{\frac{dg}{dx_i}} \sum_{j\in N_i} \hat{\phi}(\vartheta(x_j) - \vartheta(x_i)), \quad \text{for all } i \in \Gamma. \qquad (8)$$

Initially, we study the stability of the system under protocol (8).

Lemma 2: Consider a network $G = (\Gamma, E)$ of dynamic agents with first-order dynamics and implement a distributed and stationary protocol $u(.)$ whose components have the feedback form (8). Then, for any initial state $x(0)$, the system may not converge (asymptotically) to any equilibrium point different from $\hat{\chi}(x(0))\mathbf{1}$.

We are now ready to prove that the agents asymptotically reach consensus on $\hat{\chi}(x(0))\mathbf{1}$ when function $g(.)$ is strictly increasing, i.e., $\frac{dg(y)}{dy} > 0$ for all $y \in \mathbb{R}$.

Theorem 2: Consider a network $G = (\Gamma, E)$ of dynamic agents with first-order dynamics and implement a distributed and stationary protocol whose components have the feedback form (8). If function $g(.)$ is strictly increasing, the agents asymptotically reach consensus on $\hat{\chi}(x(0))\mathbf{1}$ for any initial state $x(0)$.

It is possible to partially relax the assumptions of Theorem 2 concerning the monotonicity of function $g(.)$. The reason is evident from the following theorem establishing that all agents' state trajectories are bounded.

Theorem 3: Assume all the conditions in Theorem 2 hold. Then, condition (7) implies that for all $i \in \Gamma$ and $t \geq 0$

$$\min_{j\in\Gamma}\{x_j(0)\} \leq x_i(t) \leq \max_{j\in\Gamma}\{x_j(0)\}. \qquad (9)$$

Trivially, condition (9) holds even if $g(y)$ is strictly increasing only in the subset of $\Re$ defined by $\min_{j\in\Gamma}\{x_j(0)\} \leq y \leq \max_{j\in\Gamma}\{x_j(0)\}$, since the agents' state trajectory values are bounded within the same set. The boundedness of the agents' state trajectories allows us to partially relax the assumptions of Theorem 2 concerning the monotonicity of function $g(.)$. Theorem 2 still holds if $g(.)$ is strictly increasing in only a subset $X \in \Re$. However, in this case, it must be true that $x_i(t) \in X$, for all $t \geq 0$ and for all $i \in \Gamma$. Theorem 3 proves that the latter condition is certainly satisfied if X is a connected subset and $x_i(0) \in X$ for all $i \in \Gamma$. Theorem 2 holds even if $g(.)$ is strictly decreasing. However, in this case, α in (7) must be strictly negative instead of positive.

An immediate consequence of the above considerations is the following. Since the means introduced in Tab. I have the component $g(.)$ strictly increasing except the harmonic mean, if we consider the linear function $\phi(x_j, x_i) = \alpha(x_j - x_i)$, the system converges to $\hat{\chi}(x(0))\mathbf{1}$ for $\alpha > 0$ except for the harmonic mean where we need $\alpha < 0$. Dealing with means different from the arithmetic one we also need that $x_i(0) > 0$ for all $i \in \Gamma$, since $g(y)$ is strictly monotone for $y > 0$ but not in $y = 0$.

V. SIMULATION STUDIES: ALIGNMENT MANEUVER FOR UAVS

We consider a team of 4 UAVs in longitudinal flight and initially at different heights. Each UAV controls the vertical rate without knowing the relative position of all UAVs but only of neighbors according to the communication graph topology depicted in Fig. 1. For instance, the 4th UAV knows the position of only the 1st UAV, and the 1st UAV

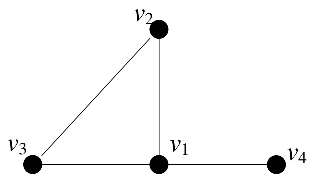

Fig. 1. The information flow in a network of 4 agents

knows the position of the 4th and 2nd UAV and so on. In the above partial information context, we are interested in determining a suitable distributed vertical rate control strategy that allows the UAVs to align their paths according to the path of a *virtual leader*, the formation center. In any of the four simulated alignment maneuvers, the position of the formation center is computed with respect to the positions of all UAVs respectively as the i) arithmetic mean, ii) geometric mean, iii) harmonic mean, iv) mean of order 2. The initial height is $x(0) = [5, 5, 10, 20]'$. We stress once again that the challenging aspect is that the UAVs know the heights of only their neighbors and are required to align their paths according to the path of the formation center, which in turns depend on the unknown position of all UAVs. In case i), (see e.g., [6], [7]) the UAVs implement the linear protocol

$$u(x_i, x^{(i)}) = \sum_{j \in N_i} (x_j - x_i) \tag{10}$$

to asymptotically align on the arithmetic mean of $x(0)$. Figure 2 shows the simulation of the longitudinal flight dynamics.

In case ii) the UAVs implement the protocol

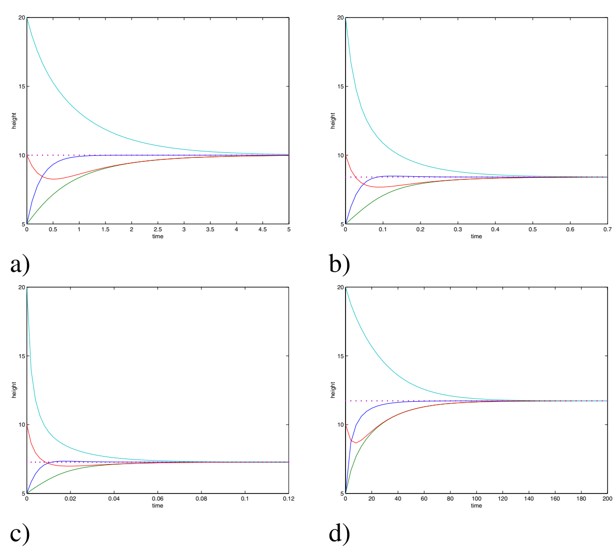

Fig. 2. Longitudinal flight dynamics converging to a) the arithmetic mean under protocol (10); b) the geometric mean under protocol (11); c) the harmonic mean under protocol (12); d) the mean of order 2 under protocol (13).

$$u(x_i, x^{(i)}) = x_i \sum_{j \in N_i} (x_j - x_i) \tag{11}$$

to asymptotically align on the geometric mean of $x(0)$. Figure 2(a)) shows the simulation of the longitudinal flight dynamics. In case iii) the UAVs implement the protocol

$$u(x_i, x^{(i)}) = -x_i^2 \sum_{j \in N_i} (x_j - x_i) \tag{12}$$

to asymptotically align on the harmonic mean of $x(0)$. Figure 2(b)) shows the simulation of the longitudinal flight dynamics. Finally, in case iv) the UAVs implement the protocol

$$u(x_i, x^{(i)}) = \frac{1}{2x_i} \sum_{j \in N_i} (x_j - x_i) \tag{13}$$

to asymptotically align on the mean of order 2 of $x(0)$. Figure 2(c)) shows the simulation of the longitudinal flight dynamics.

Protocols (10)-(13) are characterized by different converging times (see Figs. 2). These differences are due to the fact that the protocols multiply the common term $\sum_{j \in N_i}(x_j - x_i)$ for different powers of x_i, respectively 1, x_i, $-x_i^2$ and $\frac{1}{2}x_i^{-1}$. Being $x_i \geq 1$ for all $i \in \Gamma$ and $t \geq 0$, the lower the power, the higher the converging time. Consider the alignment to the mean of power 2. To obtain a converging time comparable with the one of the alignment to the arithmetic mean, we modify the protocol so that it turns to be a ratio between polynomials whose numerator is of an order greater than the denominator as in the arithmetic mean case. As an example, in Fig. 3(a)) results are reported with the protocol (13) modified as

$$u(x_i, x^{(i)}) = \frac{1}{2x_i} \sum_{j \in N_i} (x_j^2 - x_i^2). \tag{14}$$

An analogous result can be obtained if we multiply the protocol (13) by twice an upper bound of $\max_{i \in \Gamma}\{x_i(0)\}$. The resulting scaled protocol is

$$u(x_i, x^{(i)}) = \frac{\max_{i \in \Gamma}\{x_i(0)\}}{2x_i} \sum_{j \in N_i} (x_j - x_i) \tag{15}$$

and the corresponding longitudinal dynamics is displayed in Fig. 3 (b)). Observe that to implement protocol (15) the UAVs must have an a-priori knowledge or at least a bound of $\max_{i \in \Gamma}\{x_i(0)\}$.

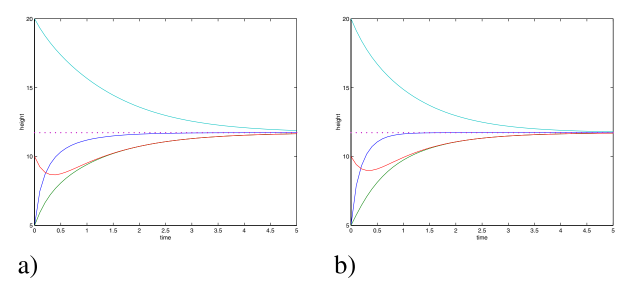

Fig. 3. Longitudinal flight dynamics converging to the mean of order 2: a) under protocol (14); b) under protocol (15).

An example of alignment maneuver under protocol (15) is displayed in Fig. 4.

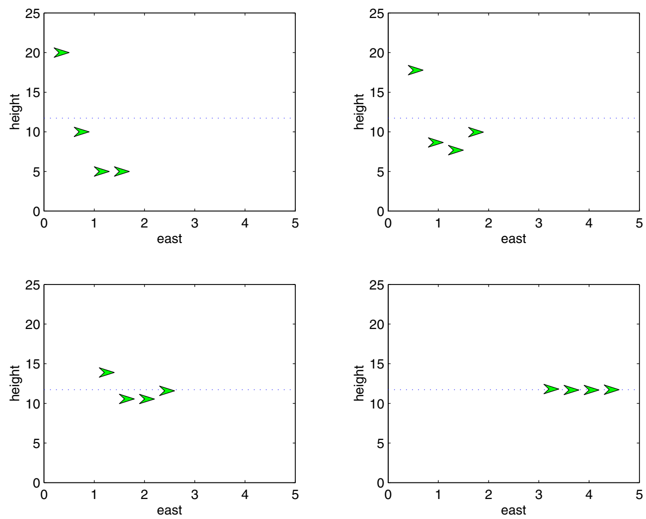

Fig. 4. Alignment to the mean of order 2 on the vertical plane.

VI. Conclusions

We have studied stationary and distributed consensus protocols for a network of n dynamic agents under local information. Consensus is reached when the agreement function, computed over the agents' state trajectories, is time invariant. We use this basic result to introduce a protocol design rule allowing consensus on a quite general set of values. Such a set includes, e.g., any generalized mean of order p of the agents' initial states. In future works, we will cast the above consensus protocols within a game theoretic framework. Under this perspective, consensus will be the result of a mechanism design, where a supervisor imposes individual objectives. Given the objective functions, the agents optimize on a local basis and reach asymptotically consensus on the desired group decision value.

References

[1] , J.C. Geromel and P. Colaneri, "Stabilization of continuous-time switched systems", *Proc. of the 16th IFAC World Congress*, Prague, Jul 2005.

[2] , D. Liberzon, "Switching in Systems and Control", *Volume in series Systems and Control: Foundations and Applications.*, Birkhauser, Boston, MA,Jun 2003.

[3] , L. Vu, and D. Liberzon, "Common Lyapunov functions for families of commuting non linear systems", *Systems and Control Letters*, vol. 54, 405–416, 2005.

[4] , M. S. Branicky, "Multiple Lyapunov Function and Other Analysis Tools for Switched and Hybrid Systems", *IEEE Trans. on Automatic Control*, vol. 43, 475–482, 1998.

[5] , W. Ren and R. Beard and E. M. Atkins, "A Survey of Consensus Problems in Multi-Agent Coordination", *Proc. of the American Control Conference*, 1859-1864,2005.

[6] R. Olfati-Saber, R. Murray, Consensus problems in networks of agents with switching topology and time-delays, IEEE Trans. on Automatic Control 49 (9) (2004) 1520–1533.

[7] D. Bauso, L. Giarrè, R. Pesenti, Attitude Alignment of a Team of UAVs under Decentralized Information Structure, in: Proc. of the IEEE Conference on Controls and Applications, Instanbul, Turkey, 2003, pp. 486–491.

[8] F. Giulietti, L. Pollini, M. Innocenti, Autonomous Formation Flight, IEEE Control Systems Magazine 20 (2000) 34–44.

[9] R. W. Beard, T. W. McLain, M. A. Goodrich, E. P. Anderson, Coordinated Target Assignment and Intercept for Unmanned Air Vehicles, IEEE Trans. on Robotics and Automation 18 (6) (2002) 911–922.

[10] H. G. Tanner, A. Jadbabaie, G. J. Pappas, Coordination of Multiple Autonomous Vehicles, in: Proc. of the 11th Mediterranean Conference on Control and Automation, also available at http://www.unm.edu/ tanner/, Rhodes, Greece, 2003.

[11] V. Gazi, K. Passino, Stability analysis of social foraging swarms, IEEE Trans. on Systems, Man, and Cybernetics 34 (1) (2004) 539–557.

[12] Y. Liu, K. Passino, Stable social foraging swarms in a noisy environment, IEEE Trans. on Automatic Control 49 (1) (2004) 30–44.

[13] D. Bauso, L. Giarré, R. Pesenti, Distribuited consensus protocols for coordinated buyers, in: Proc. of the 42nd IEEE Conference on Decision and Control, Vol. 1, Maui,Hawaii, 2003, pp. 588–592.

[14] A. J. Kleywegt, V. S. Nori, M. W. P. Savelsbergh, The Stochastic Inventory Routing Problem with Direct Deliveries, Transportation Science 36 (1) (2002) 94–118.

[15] S. H. Low, F. Paganini, J. C. Doyle, Internet Congestion Control, IEEE Control System Magazine 22 (1) (2002) 28–43.

[16] A. Fax, R. M. Murray, Information flow and cooperative control of vehicle formations, in: Proc of the 15th IFAC World Congress, Barcellona,S, 2002.

[17] A. Jadbabaie, J. Lin, A. Morse, Coordination of Groups of mobile autonomous agents using nearest neighbor rules, IEEE Trans. on Automatic Control 48 (6) (2003) 988–1001.

[18] L. Moreau, Leaderless coordination via bidirectional and unidirectional time-dependent communication, in: Proc. of the 42nd IEEE Conference on Decision and Control, Maui, Hawaii, 2003, pp. 3070–3075.

[19] W. Ren, R. Beard, Consensus of information under dynamically changing interaction topologies, in: Proc. of the American Control Conference, Boston, Massachussets, 2004.

[20] W. Ren, R. Beard, Consensus seeking in multi-agent systems under dynamically changing interaction topologies, in: IEEE Trans. on Automatic Control. To appear.

[21] H. G. Tanner, A. Jadbabaie, G. J. Pappas, Stable flocking of mobile agents, part ii: Dynamic topology, in: Proc. of the 42th IEEE Conference on Decision and Control, Maui, Hawaii, 2003, pp. 2016–2021.

[22] T. W. McLain, R. W. Beard, Coordination Variables Coordination Functions and Coooperative Timing Missions, in: Proc. of the IEEE American Control Conference, Denver, Colorado, 2003, pp. 296–301.

[23] W. Ren, R. Beard, T. McLain, Coordination variables and consensus building in multiple vehicle systems, in: Proc. of the Block Island Workshop on Cooperative Control, Springer-Verlag, 2004.

[24] R. Olfati-Saber, R. Murray, Consensus protocols for networks of dynamic agents, in: Proc. of the American Control Conference, Vol. 2, Denver, Colorado, 2003, pp. 951–956.

[25] L. Xiao, S. Boyd, Fast linear iterations for distributed averaging, Systems and Control Letters 53 (1) (2004) 65–78.

VII. Appendix

Proof of Lemma 1 The key idea is that the protocols used are time invariant (so the system of differential equations (1) is autonomous) and such that the system has unique solutions. Assume by contradiction that $\hat{\chi}(x(t))$ is not time invariant under protocol $u(.)$ and initial state $x(0) = a$. The consensus protocol makes the system converge to $\hat{\chi}(a)$. Let be $x(t_0) = b$ the state at the first time instant $t_0 > 0$ with $\hat{\chi}(a) \neq \hat{\chi}(b)$. Denote as $\bar{u}(t)$ the realization of the protocol for all $t \geq 0$. Now consider the different situation in which the protocol $u(.)$ is implemented starting from the initial state $x(0) = b$. In this case, the consensus protocol makes the system converge to $\hat{\chi}(b)$. Denote as $\tilde{u}(t)$ the realization of the protocol for all $t \geq 0$ in this second situation. As for each agent i, the control depends only on $x_i(t)$ and $x^{(i)}(t)$ and no other information on the current or past system state, it holds $\tilde{u}(t) = \bar{u}(t + t_0)$, for all $t \geq 0$. Then, we obtain the following contradictory result. In the two above situations, the same controls applied starting from state b make the system converge to two different group decision values, $\hat{\chi}(a)$ and $\hat{\chi}(b)$.

Proof of Theorem 1 A sufficient condition for $\hat{\chi}(x(t))$ being time invariant is that its argument $\sum_{i\in\Gamma} g(x_i(t))$ is time invariant, too. The latter condition means

$$\sum_{i\in\Gamma} \frac{dg(x_i(t))}{dt} = \sum_{i\in\Gamma} \frac{dg(x_i)}{dx_i}\dot{x}_i = \sum_{i\in\Gamma} \frac{dg(x_i)}{dx_i} u_i = 0.$$

It is immediate to verify that protocol (6) satisfies condition $\sum_{i\in\Gamma} \frac{dg(x_i)}{dx_i} u_i = 0$ since the antisymmetry of ϕ guarantees that $\sum_{i\in\Gamma}\sum_{j\in N_i} \phi(x_j, x_i) = 0$.

Proof of Lemma 2 First, we show that any equilibrium point x^* must have all its component equal, i.e., $x^* = \lambda\mathbf{1}$ where λ is a constant value. Then we prove by contradiction that λ cannot be different from $\hat{\chi}(x(0))$.

Sufficiency. Since $\hat{\phi}(.)$ is odd and strictly increasing, then $\hat{\phi}(\vartheta(x_j) - \vartheta(x_i)) = 0$ if and only if $\vartheta(x_j) - \vartheta(x_i) = 0$. In turn, since $\vartheta(.)$ is strictly increasing, $\vartheta(x_j) = \vartheta(x_i)$ if and only if $x_j = x_j$. Then, $x_i = \lambda$, for all $i \in \Gamma$ implies that the control u_i is null for all $i \in \Gamma$. Thus, the point $x^* = \lambda\mathbf{1}$ is an equilibrium point.

Necessity. Assume that there exists an equilibrium point $x^* \neq \lambda\mathbf{1}$. We prove that such an assumption implies the existence of at least one agent i with $u_i < 0$, and this last result contradicts the definition of equilibrium for x^*. Define $I = \{i \in \Gamma : x_i^* \geq x_j^*,\ \forall j \in \Gamma\}$ as the set of agents with maximum state value. Trivially, I is included but not equal to Γ, as $x^* \neq \lambda\mathbf{1}$. Then, i and j with $(i,j) \in E$ such that $x_i^* \neq x_j^*$ exist, since the network G is connected. In particular, we can always choose $i \in I$ such that there exists $j \in N_i$ with $x_j^* < x_i^*$. Now observe that, since $\hat{\phi}(.)$ is an odd and strictly increasing and $\vartheta(.)$ is strictly increasing, then $\sum_{j\in N_i} \hat{\phi}(\vartheta(x_j^*) - \vartheta(x_i^*)) < 0$. Actually, all the terms of the sum are non positive and at least one is strictly negative. Since it also holds that $\alpha\frac{1}{\frac{dg}{dx_i}} \neq 0$, the contradiction is proved.

Non convergence. The above arguments show that $\hat{\chi}(x(0))\mathbf{1}$ is an equilibrium point. Now, we prove, by contradiction, that the system may not converge to equilibrium points different from the $\hat{\chi}(x(0))\mathbf{1}$. To do this, let us assume that the system actually converges to a different equilibrium point $x^* = \lambda\mathbf{1} \neq \hat{\chi}(x(0))\mathbf{1}$. As $\hat{\chi}(.)$ enjoys the fixed point property, we have $\hat{\chi}(x^*) = \lambda \neq \hat{\chi}(x(0))$. Then, we obtain a contradiction with the time invariancy of $\hat{\chi}(x)$ under protocol (8).

Proof of Theorem 2 We follow a line of reasoning similar to the one in [6]. First, observe that consensus reaching corresponds to asymptotic stability of a new variable $\eta = \{\eta_i, i \in \Gamma\}$, where $\eta_i = g(x_i) - g(\hat{\chi}(x(0)))$. The vector η is a bijective function of the system state, since η_i is as strictly increasing as $g(.)$, and $\eta = 0$ corresponds to $x = \hat{\chi}(x(0))\mathbf{1}$. We prove the asymptotical stability (in the quotient space $\Re^n/span\{\mathbf{1}\}$) of the equilibrium point $\eta = 0$ by introducing a candidate Lyapunov function $V(\eta) = \frac{1}{2}\sum_{i\in\Gamma}\eta_i^2$. Trivially, $V(\eta) = 0$ if and only if $\eta = 0$; $V(\eta) > 0$ for all $\eta \neq 0$. It remains to prove that $\dot{V}(\eta) < 0$ for all $\eta \neq 0$.

$$\begin{aligned}
&\dot{V}(\eta) = \\
&= \tfrac{1}{2}\textstyle\sum_{(i,j)\in E}(\hat{\phi}(\eta_{ji})\dot{\eta}_{ji} + \hat{\phi}(\eta_{ij})\dot{\eta}_{ij}) = \\
&= \tfrac{1}{2}\textstyle\sum_{(i,j)\in E}(\hat{\phi}(\eta_{ji})(\frac{d\vartheta(x_j)}{dx_j}\dot{x}_j - \frac{d\vartheta(x_i)}{dx_i}\dot{x}_i) \\
&+\hat{\phi}(\eta_{ij})(\tfrac{d\vartheta(x_i)}{dx_i}\dot{x}_i - \tfrac{d\vartheta(x_j)}{dx_j}\dot{x}_j)) = \\
&= -\textstyle\sum_{i\in\Gamma}\frac{d\vartheta(x_i)}{dx_i}\dot{x}_i\sum_{j\in N_i}\hat{\phi}(\eta_{ji}) = \\
&= -\textstyle\sum_{i\in\Gamma}\frac{1}{\alpha}\frac{dg(x_i)}{dx_i}\frac{d\vartheta(x_i)}{dx_i}\dot{x}_i\frac{\alpha}{\frac{dg(x_i)}{dx_i}}\sum_{j\in N_i}\hat{\phi}(\eta_{ji}) = \\
&= -\textstyle\sum_{i\in\Gamma}\frac{1}{\alpha}\frac{dg(x_i)}{dx_i}\frac{d\vartheta(x_i)}{dx_i}u_i^2
\end{aligned}$$

In the above expression we simply express η_i and $\dot{\eta}_i$ in terms of the state variables and their derivatives and we reorder the terms and exploit the fact that $j \in N_i$ if and only if $i \in N_j$ for each $i, j \in \Gamma$. We have $\dot{V}(\eta) \leq 0$ for all η and, in particular, $\dot{V}(\eta) = 0$ only for $\eta = 0$. Actually, as $\alpha > 0$ and $g(.)$, $\hat{\phi}(.)$, and $\vartheta(.)$ are strictly increasing, we have that, for any $(i,j) \in E$, $x_j > x_i$ implies $g(x_j) - g(x_i) > 0$, $\vartheta(x_j) - \vartheta(x_i) > 0$ and $\hat{\phi}(\vartheta(x_j) - \vartheta(x_i)) > 0$. Hence, we obtain $\alpha(g(x_j) - g(x_i))\hat{\phi}(\vartheta(x_j) - \vartheta(x_i)) > 0$ if $x_j > x_i$. Trivially, a symmetrical argument holds if $x_j < x_i$.

Proof of Theorem 3. Let $\alpha = \min_{j\in\Gamma}\{x_j(0)\}$ and $\beta = \max_{j\in\Gamma}\{x_j(0)\}$. The aim is to prove that all solutions stay inside the hypercube $[\alpha,\beta]^n$. This can be shown by noticing that, for each generic agent i, the state $x_i(t)$ is a continuous. Observe, that on the faces of the polyhedron $[\alpha,\beta]^n$ the corresponding vector field does not point outwards. We prove this briefly for the case in which there exists $\hat{\jmath} \in N_i$, such that, $\min_{j\in\Gamma}\{x_j(0)\} < x_{\hat{\jmath}} < \max_{j\in\Gamma}\{x_j(0)\}$. As the inequalities are strict, we have that $u_i(\alpha, x^{(i)}) > 0$ (and $u_i(\beta, x^{(i)}) < 0$) which follows immediately from the definition of u_i. When either one or both of the above inequalities are weak it may occur that either $u_i(\alpha, x^{(i)}) = 0$ or $u_i(\beta, x^{(i)}) = 0$, but in any case, by the definition of u_i, it is not possible that $u_i(\alpha, x^{(i)}) < 0$ or $u_i(\beta, x^{(i)}) > 0$.