Proceedings of the
44th IEEE Conference on Decision and Control, and
the European Control Conference 2005
Seville, Spain, December 12-15, 2005

Reachable Set of Bilinear Control Systems With Time Varying Drift

Haidong Yuan and Navin Khaneja

Abstract— **In this article, we give a complete characterization of the reachable set at all times for a class of bilinear control systems with time varying drift and unbounded control amplitude. These results are of fundamental interest in geometric control theory and have important applications to control of coupled spins in solid state NMR spectroscopy.**

I. INTRODUCTION

The main result presented in this paper implies the following:
Consider the control system

$$\dot{Y}(t) = (A(t) + \sum_j v_j B_j)Y(t) \tag{1}$$

on $n \times n$ matrices, such that the Lie algebra generated by B_j is all $n \times n$ skew symmetric matrices. Let $\lambda^{\downarrow}(t)$ be the eigenvalues of

$$\frac{A(t) + A^T(t)}{2} - Tr(A(t))I$$

arranged in the decreasing order and let

$$\mu(T) = \int_0^T \lambda^{\downarrow}(t)\mathrm{d}t$$

where the integration is performed for each entry of the vector. Then the closure of all the points that can be reached at time T ($T > 0$) is given by the set

$$\exp(\int_0^T Tr(A(\tau))\mathrm{d}\tau)\ K_1\ \exp(D)\ K_2$$

where K_1,$K_2 \in SO(n)$, D is a diagonal matrix whose diagonal entries $(d_{11}, d_{22}, \ ..., d_{nn})$ lies in the convex hull of $\mu(T)$ and all its permutations, i.e., the diagonal of D is majorized by $\mu(T)$ ($D \prec \mu(T)$).

There is a generalization of this result for arbitrary semi-simple Lie groups, which is the main result of this paper (Theorem 3). These problems are motivated by time optimal control of quantum systems([1]). In non-relativistic quantum mechanics, the time evolution of the quantum system is given by

$$\dot{U}(t) = -iH(t)U(t), \qquad U(0) = I,$$

here $H(t)$ and $U(t)$ are the Hamiltonian and the unitary evolution operators respectively. We can split the Hamiltonian as follows:

$$-iH(t) = X_d(t) + \sum_{j=1}^{m} v_j(t)X_j,$$

where $X_d(t)$ is the part of the Hamiltonian that is internal to the system and we call it the drift or free Hamiltonian and $\sum_{j=1}^{m} v_j(t)X_j$ is the part of the Hamiltonian that can be externally changed, we call it the control Hamiltonian. A fundamental problem in quantum control and spectroscopy is to find all the unitary operators that can be reached in a specified time T through appropriate choices of $v_j(t)$. This helps in computing the minimum time required to synthesize a desired unitary evolution. The choice of $v_j(t)$ in practice corresponds to design of a pulse sequence.

In [1], these systems have been extensively studied under the hypothesis that the Lie algebra generated by $\{X_j\}$ is a special part of a Cartan decomposition of the unitary group and the drift Hamiltonian is time independent. In this paper, we generalize these results to the case of time varying drift Hamiltonian. This is motivated by control of coupled spin-1/2 in solid state NMR spectroscopy, where the interaction between the spins are made to vary with time. In quantum information, these results lead to the time optimal implementation of quantum gates in the presence of time varying coupling Hamiltonian.

The paper is organized as follows. In Section II, we recall some basic definitions and properties of semi-simple Lie algebras required to understand the main result of this paper. Section III gives the main result of this paper. In Section IV, we give some interesting examples which are special cases of the main result of this paper.

II. PRELIMINARIES

Consider a Lie group G and its corresponding Lie algebra $\mathfrak{g}$. The adjoint representation Ad_g is a map from the Lie algebra $\mathfrak{g}$ to $\mathfrak{g}$ which is the differential of the conjugation map a_g from the Lie group G to G given by $a_g(h) = ghg^{-1}$. For matrix Lie algebras, $\mathrm{Ad}_g(Y) = gYg^{-1}$, where g and Y are both represented as matrices of compatible dimensions. The differential of the adjoint representation is denoted by ad, and ad_X is a map from the Lie algebra $\mathfrak{g}$ to $\mathfrak{g}$ given by the Lie bracket with X, that is, $\mathrm{ad}_X(Y) = [X, Y]$.

We define a bilinear form on $\mathfrak{g}$ by the Killing form $B(X, Y) = tr(\mathrm{ad}_X\, \mathrm{ad}_Y)$. Let K be a compact subgroup of G, and $\mathfrak{k}$ the Lie algebra of K. Assume that $\mathfrak{g}$ admits a

Haidong Yuan is with Division of Engineering and Applied Science, Harvard University, Cambridge, MA 02138, USA hyuan@deas.harvard.edu
Navin Khaneja is with Division of Engineering and Applied Science, Harvard University, Cambridge, MA 02138, USA navin@eecs.harvard.edu

direct sum decomposition $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$, such that $\mathfrak{p} = \mathfrak{k}^{\perp}$ with respect to Killing form.

Definition 1 (Cartan decomposition of the Lie algebra) Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{R}$ and let the decomposition $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$, $\mathfrak{p} = \mathfrak{k}^{\perp}$ satisfy the commutation relations

$$[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}, \quad [\mathfrak{p}, \mathfrak{k}] \subset \mathfrak{p}, \quad [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}. \tag{2}$$

This decomposition is called a Cartan decomposition of $\mathfrak{g}$, and the pair $(\mathfrak{g}, \mathfrak{k})$ is called an orthogonal symmetric Lie algebra pair.

Remark 1 Each semisimple Lie algebra $\mathfrak{g}$ over $\mathbb{R}$ has a Cartan decomposition[2]P.183.

Example 1
Let $\mathfrak{g} = \mathfrak{su}(n)$, $\mathfrak{k} = \mathfrak{so}(n)$ and $\mathfrak{p} = \{iS|S \text{ is } n \times n \text{ traceless real symmetric matrix}\}$. Then $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ is a Cartan decomposition.

Proposition 1 (Decomposition of the Lie group [2]P.249) Given a semisimple Lie algebra $\mathfrak{g}$ over $\mathbb{R}$, $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ a Cartan decomposition, let $\mathfrak{h}$ be a maximal abelian subalgebra contained in $\mathfrak{p}$, then $G = K \exp(\mathfrak{h}) K$.

Remark 2 In Example 1, $\mathfrak{h} = \{iD| \text{ is } n \times n \text{ traceless diagnal matrix}\}$ is a maximal abelian subalgebra contained in $\mathfrak{p}$, so $SU(n) = \{K_1 \exp(a) K_2 | K_1, K_2 \in SO(n), a \in \mathfrak{h}\}$

Let M and M' denote the centralizer and normalizer of $\mathfrak{h}$ in K, respectively. In other words,

$$M = \{k \in K | Ad_k(X) = X \text{ for each } X \in \mathfrak{h}\},$$

$$M' = \{k \in K | Ad_k(\mathfrak{h}) \subset \mathfrak{h}\}.$$

Definition 2 (Weyl group) The quotient group M'/M is called the *Weyl group* of the pair (G, K). It is denoted by $W(G, K)$.

$W(G, K)$ is a finite group [2].

Let $\mathfrak{h}'$ be a maximal abelian subalgebra of $\mathfrak{g}$ containing $\mathfrak{h}$, for $X \in \mathfrak{h}'$, let $W \in \mathfrak{g}$ be an eigenvector of ad_X and $\alpha(X)$ the corresponding eigenvalue, *i.e.*,

$$[X, W] = \alpha(X) W. \tag{3}$$

The linear function α is called a *root* of $\mathfrak{g}$ with respect to $\mathfrak{h}'$. Let $\Delta_{\mathfrak{p}}$ denote the set of roots which do not vanish identically on $\mathfrak{h}$. Each $\alpha \in \Delta_p$ defines a hyperplane $\alpha(X) = 0$ in the vector space $\mathfrak{h}$. These hyperplanes divide the space $\mathfrak{h}$ into finitely many connected components, called the *Weyl chambers*.

For each $\alpha \in \Delta_p$, let s_α denote the reflection with respect to the hyperplane $\alpha(X) = 0$ in $\mathfrak{h}$.

Proposition 2 (Generation of the Weyl group[2]P.289) The Weyl group is generated by the reflections s_α, $\alpha \in \Delta_p$.

Remark 3 In Example 1, the set of roots are given by

$$\{\pm(f_l - f_m) : 1 \leq l < m \leq n\}.$$

Here

$$f_m(iD) = d_{mm}, \qquad 1 \leq m \leq n$$

where we use d_{ij} denote the entries of D. The reflections in $\mathfrak{h}$ about the plane

$$\{X \in \mathfrak{h} | f_l(X) - f_m(X) = 0\}$$

is just exchanging d_{ll} and d_{mm}, so the Weyl group is just permutations of the diagonal entries, and the Weyl chambers are given by $\mathfrak{h}_\sigma = \{iD | d_{\sigma(1)\sigma(1)} > d_{\sigma(2)\sigma(2)} > ... > d_{\sigma(n)\sigma(n)}\}$ for all permutations σ.

Definition 3 (Weyl Orbit) Given the Cartan decomposition $\mathfrak{g} = \mathfrak{p} + \mathfrak{k}$, let $\mathfrak{h} \subset \mathfrak{p}$ represent a maximal abelian subalgebra. We use the notation $\Delta_{X_d} = \mathfrak{h} \bigcap Ad_K(X_d)$ to denote the maximal commuting set contained in the adjoint orbit of X_d. The set Δ_{X_d} is called the Weyl orbit of X_d. We use $\mathfrak{c}(X_d) = \{\sum_{i=1}^n \alpha_i X_i | \alpha_i \geq 0, \sum \alpha_i = 1, X_i \in \Delta_{X_d}\}$, to denote the convex hull of the Weyl orbit of X_d, with vertices given by the elements of the Weyl orbit of X_d.

The Weyl orbit intersects the closure of each Weyl chamber precisely at one point [2].

Remark 4 In Example 1, the Weyl orbit of $X_d \in \mathfrak{p}$ is all the diagonal matrices whose diagonal elements are permutations of the eigenvalues of X_d.

Theorem 1 (Kostant's Convexity Theorem[4]): Let $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ be a Cartan Decomposition and $\mathfrak{h} \subset \mathfrak{p}$ represent a maximal abelian subalgebra, $X_d \in \mathfrak{p}$. Let $\Gamma : \mathfrak{p} \to \mathfrak{h}$, be the orthogonal projection of $\mathfrak{p}$ onto $\mathfrak{h}$, with respect to the killing form. Then $\Gamma(Ad_K(X_d)) = \mathfrak{c}(X_d)$, where $\mathfrak{c}(X_d)$ is the convex hull of the Weyl orbit of X_d as defined above.

Let $\mathfrak{h}^*$ be the dual of $\mathfrak{h}$, and $\{e_1, e_2, ..., e_l\}$ be a basis of $\mathfrak{h}^*$, a root $\alpha = \sum_{i=1}^l r_i e_i$ is said to be positive if there exists i such that $r_1 = 0, r_2 = 0, ..., r_{i-1} = 0, r_i > 0$. A root is said to be simple positive if it's positive and can't be written as sum of two positive roots.

Let $\Sigma = \{\beta_1, ..., \beta_l\} \subseteq \Delta_p$ be the set of simple positive roots. Now let

$$\mathfrak{h}_+ = \{x \in \mathfrak{h} | \beta_i(x) > 0 \text{ for } i = 1, 2, ..., l\}$$

then $\mathfrak{h}_+$ is a Weyl chamber in $\mathfrak{h}$.

Any $x \in \mathfrak{h}$ is W-conjugate to a unique element in $\overline{\mathfrak{h}}_+$ [2]P.293.

Remark 5 In Example 1, one set of simple positive roots Σ is given by

$$\Sigma = \{f_m - f_{m+1} : 1 \leq m < n\}$$

so $\mathfrak{h}_+ = \{iD | d_{11} > d_{22} > ... > d_{nn}\}$ is a Weyl chamber.

The Killing form induces an isomorphism $\mathfrak{h} \to \mathfrak{h}^*$. Let x_i be the element in $\mathfrak{h}$ corresponding to β_i under this isomorphism, we define a convex cone $\mathfrak{h}_p$ as:

$$\mathfrak{h}_p = \{x \in \mathfrak{h} | x = \sum_{i=1}^{l} r_i x_i, r_i \geq 0\}$$

Theorem 2 ([4])*:*Let $x \in \overline{\mathfrak{h}}_+$. Then for any $y \in \mathfrak{h}$:
(1) $y \in \mathfrak{c}(x)$ iff $x - w(y) \in \mathfrak{h}_p$ for all $w \in W(G, K)$.
(2) If $y \in \overline{\mathfrak{h}}_+$ then $y \in \mathfrak{c}(x)$ iff $x - y \in \mathfrak{h}_p$.

Remark 6 In Example 1, the element x_m corresponding to $f_m - f_{m+1}$ is iD^m, where $(D^m)_{mm} = 1$, $(D^m)_{(m+1)(m+1)} = -1$ and all other entries are 0. The convex cone then is given by $\mathfrak{h}_p =$

$$\{iD | d_{11} \geq 0, d_{11} + d_{22} \geq 0, ..., \sum_{j=1}^{n-1} d_{jj} \geq 0, \sum_{j=1}^{n} d_{jj} = 0\}$$

so in this example, $y \in \mathfrak{c}(x)$ iff the diagonal entry of $-iy$ majorized by the diagonal entry of $-ix$.

III. Main Result

Now consider again the equation

$$\dot{U}(t) = [X_d(t) + \sum_{j=1}^{m} v_j(t) X_j] U(t), \qquad U \in G, \quad (4)$$

and a priori, we assume no bound on v_j. If, for some time $t \geq 0$, $U(t) = U'$, i.e. the control $v(t) = (v_1(t), v_2(t), ..., v_m(t))$ steers this system (4) to U' in t units of time, then we say U' is attainable or reachable at time t.

Definition 4 (Reachable Set): The set of all $U' \in G$ attainable from U_0 within time T will be denoted by $R(U_0, T)$.

Remark 7 From the right invariance of control system(4), it follows that $R(U_0, T) = R(e, T)U_0$, we use $\overline{R}(U_0, T)$ to denote its closure.

Definition 5 (Infimizing Time): Given $U_F \in G$, we define the *infimizing time* for U_F as

$$t^*(U_F) = \inf \{t \geq 0 | \; U_F \in \overline{R}(e, t)\}$$

Now we state the main result of this paper.
Let G be a connected semi-simple Lie group over $\mathbb{R}$ and K a compact subgroup of G. Let $\mathfrak{g}$ and $\mathfrak{k}$ be their Lie algebras respectively, such that $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$, $\mathfrak{p} = \mathfrak{k}^{\perp}$ is a Cartan decomposition, let $\mathfrak{h}$ be a maximal abelian subalgebra in $\mathfrak{p}$, and $\mathfrak{h}_+$ be a Weyl chamber.

Theorem 3 Consider the right invariant control system (4) on G, assume $\{X_i\}_{LA} = \mathfrak{k}$, $X_d(t) \in \mathfrak{p}$. At each time point t, the Weyl orbit $\Delta_{X_d(t)}$ intersects $\overline{\mathfrak{h}}_+$ at one point, denote it as $\mathrm{Ad}_{k_+}(X_d(t))$, let

$$X_+(T) = \int_0^T \mathrm{Ad}_{k_+}(X_d(\tau)) \mathrm{d}\tau$$

The reachable set for the system within time T is

$$R(e, T) = \{K_1 \exp(Y) K_2 | K_1, K_2 \in K, Y \in \mathfrak{c}(X_+(T))\}$$

Consequently for $U_F = K_1 \exp(Y) K_2$ where $K_1, K_2 \in K$, and $Y \in \mathfrak{h}$,

$$t^*(U_F) = min\{t \geq 0 | \exp(Y) \in \exp(\mathfrak{c}(X_+(t)))\}$$

Proof: Let $H = \exp(\mathfrak{h})$, we know that $G = KHK$. Since it does not consume any time generating the K part, it suffices to prove that the reachable set for the H part is $\exp[\mathfrak{c}(X_+(T))]$.

Let $U(t) = K_1(t) A(t) K_2(t)$ be a trajectory of the control system, where $K_1(t)$, $K_2(t) \in K$, $A(t) \in H$. Then $A(t) = K_1'(t) U(t) K_2'(t)$, where $K_1'(t) = K_1^{-1}(t)$, $K_2'(t) = K_2^{-1}(t)$. So $\frac{dA(t)}{dt} =$

$$\begin{aligned}
&\dot{K_1'}(t) U(t) K_2'(t) + K_1'(t) \dot{U}(t) K_2'(t) + K_1'(t) U(t) \dot{K_2'}(t) \\
=&\mathfrak{k}_1(t) K_1'(t) U(t) K_2'(t) + K_1'(t) [X_d(t) \\
&+ \sum_{i=1}^{m} v_i(t) X_i] U(t) K_2'(t) + K_1'(t) U(t) K_2'(t) \mathfrak{k}_2(t) \\
=&\mathfrak{k}_1(t) A(t) + [\mathrm{Ad}_{K_1'(t)}(X_d(t)) + \mathfrak{k}_3(t)] A(t) \\
&+ \mathrm{Ad}_{A(t)}(\mathfrak{k}_2(t)) A(t) \\
=&[\mathrm{Ad}_{K_1'(t)}(X_d(t)) + \mathfrak{k}_1(t) + \mathfrak{k}_3(t) + \mathrm{Ad}_{A(t)}(\mathfrak{k}_2(t))] A(t) \\
=&a(t) A(t)
\end{aligned}$$

where $\mathfrak{k}_1(t), \mathfrak{k}_2(t), \mathfrak{k}_3(t) \in \mathfrak{k}$,

$$\mathfrak{k}_3(t) = \mathrm{Ad}_{K_1'(t)}(\sum_{i=1}^{m} v_i(t) X_i),$$

$$a(t) = [\mathrm{Ad}_{K_1'(t)}(X_d(t)) + \mathfrak{k}_1(t) + \mathfrak{k}_3(t) + \mathrm{Ad}_{A(t)}(\mathfrak{k}_2(t))].$$

Since the tangent space of H at $A(t)$ is $\mathfrak{h}A(t)$, we only need to look at the part of $a(t)$ belonging to $\mathfrak{h}$. Notice that

$$\begin{aligned}
B(\mathrm{Ad}_{A(t)}(\mathfrak{k}_2(t)), \mathfrak{h}) &= B(\mathfrak{k}_2(t), \mathrm{Ad}_{A^{-1}(t)}(\mathfrak{h})) \\
&= B(\mathfrak{k}_2(t), \mathfrak{h}) = 0
\end{aligned} \quad (5)$$

so $\mathrm{Ad}_{A(t)}(\mathfrak{k}_2(t))$ is orthogonal to $\mathfrak{h}$. $\mathfrak{k}_1(t) + \mathfrak{k}_3(t) \in \mathfrak{k}$ is also orthogonal to $\mathfrak{h}$, then the $\mathfrak{h}$ part of $a(t)$ is just $\Gamma(Ad_{K_1'(t)}(X_d(t)))$ (recall $\Gamma : \mathfrak{p} \to \mathfrak{h}$ is the orthogonal projection onto $\mathfrak{h}$), since we can generate the K part instantly, $K_1'(t)$ can take any value in K. Using Kostant's convexity theorem, we can write the evolution of $A(t)$ as

$$\dot{A}(t) = \Omega(t) A(t), \qquad A(0) = e \quad (6)$$

where $\Omega(t) \in \mathfrak{c}(X_d(t))$.

The reachable set for (6) within time T is $\exp(\mathbf{S}(e,T))$, where

$$\mathbf{S}(e,T) = \{\int_0^\tau \Omega(t)\mathrm{d}t) | 0 \le \tau \le T, \Omega(t) \in \mathfrak{c}(X_d(t))\},$$

note that $0 \in \mathfrak{c}(X_d(t))$ for all t, if $\tau < T$, we can extend $\Omega(t)$ by appending 0, so WLOG we assume the integration is always over the whole interval [0,T], i.e.

$$\mathbf{S}(e,T) = \{\int_0^T \Omega(t)\mathrm{d}t | \Omega(t) \in \mathfrak{c}(X_d(t))\}.$$

It's now equivalent to show $\mathbf{S}(e,T) = \mathfrak{c}(X_+(T))$.

We first show $\mathbf{S}(e,T)$ is a convex set containing $\mathfrak{c}(X_+(T))$.
$\forall a_1, a_2 \in \mathbf{S}(e,T)$, assume

$$a_1 = \int_0^T \Omega_1(t)\mathrm{d}t$$

$$a_2 = \int_0^T \Omega_2(t)\mathrm{d}t$$

where $\Omega_1(t), \Omega_2(t) \in \mathfrak{c}(X_d(t))$. $\forall \gamma \in [0,1]$,

$$\gamma a_1 + (1-\gamma)a_2 = \int_0^T \gamma\Omega_1(t) + (1-\gamma)\Omega_2(t)\mathrm{d}t$$

since $\mathfrak{c}(X_d(t))$ is a convex set, $\gamma\Omega_1(t) + (1-\gamma)\Omega_2(t) \in \mathfrak{c}(X_d(t))$, so $\gamma a_1 + (1-\gamma)a_2 \in \mathbf{S}(e,T)$ by definition, which means $\mathbf{S}(e,T)$ is convex.

If we take $\Omega(t) = w(Ad_{k_+}(X_d(t))) \in \mathfrak{c}(X_d(t))$, $w \in W(G,K)$, then $w(X_+(T)) =$

$$\int_0^T w(Ad_{k_+}(X_d(t)))\mathrm{d}t = \int_0^T \Omega(t)\mathrm{d}t \in \mathbf{S}(e,T),$$

i.e. all the vertices of $\mathfrak{c}(X_+(T))$ are in **S**(e,T), which gives $\mathbf{S}(e,T) \supset \mathfrak{c}(X_+(T))$.

Second we show $\mathbf{S}(e,T) \subset \mathfrak{c}(X_+(T))$:
$\forall a \in \mathbf{S}(e,T)$, assume

$$a = \int_0^T \Omega(t)\mathrm{d}t, \qquad \Omega(t) \in \mathfrak{c}(X_d(t))$$

$\forall w \in W(G,K)$,

$$X_+(T) - w(a) = \int_0^T \mathrm{Ad}_{k_+}(X_d(t)) - w(\Omega(t))\mathrm{d}t$$

Theorem 2 tells that $\mathrm{Ad}_{k_+}(X_d(t)) - w(\Omega(t)) \in \mathfrak{h}_p$ for all t, since $\mathfrak{h}_p$ is a convex cone,

$$\int_0^T \mathrm{Ad}_{k_+}(X_d(t)) - w(\Omega(t))\mathrm{d}t \in \mathfrak{h}_p$$

i.e.

$$X_+(T) - w(a) \in \mathfrak{h}_p, \qquad \forall w \in W(G,K)$$

which means $a \in \mathfrak{c}(X_+(T))$, so $\mathbf{S}(e,T) \subset \mathfrak{c}(X_+(T))$.

Q.E.D

Remark 8 Observe the assumption $X_d(t) \in \mathfrak{p}$ is without loss of generality. The part of $X_d(t)$ contained in $\mathfrak{k}$ can be removed by unbounded controls.

IV. EXAMPLES

We illustrate the theorem by some examples.

A. $G/K = SU(n)/SO(n)$

For the control system

$$\dot{U}(t) = [X_d(t) + \sum_{j=1}^m v_j(t)X_j]U(t), \qquad U(0) = I, \quad (7)$$

on $SU(n)$, assume $\{X_j\}_{LA} = so(n)$, $X_d(t) = iS(t)$, $S(t)$ is real traceless symmetric matrix. Let $\lambda^\downarrow(t)$ be a vector whose entries are the eigenvalues of $S(t)$ in decreasing order, and $\mu(T) = \int_0^T \lambda^\downarrow(t)\mathrm{d}t$, then the reachable set of the system within time T is all the matrices with the form

$$K_1 \exp(iY) K_2$$

where K_1 and K_2 belong to $SO(n)$, Y is a real diagonal matrix and the diagonal entries $(Y_{11}, Y_{22}, ..., Y_{nn})$ majorized by $\mu(T)$.

We now show that how the result given in the introduction also follows from this example. Note that there exists an isomorphism between $\mathfrak{su}(n)$ and $\mathfrak{sl}(n,\mathbb{R})$, given by $\mathfrak{k} \to \mathfrak{k}$, $\mathfrak{p} \to i\mathfrak{p}$. For the control system (1), if $A(t)$ is traceless symmetric, let $\lambda^\downarrow(t)$ be eigenvalues of $A(t)$ arranged in decreasing order and let

$$\mu(T) = \int_0^T \lambda^\downarrow(t)\mathrm{d}t$$

where the integration is performed for each entry of the vector. Then the closure of all the points that can be reached at time T ($T > 0$) is given by the set $K_1 \ \exp(D) \ K_2$ where K_1,$K_2 \in SO(n)$, D is a diagonal matrix whose diagonal entries $(d_{11}, d_{22}, ..., d_{nn})$ lies in the convex hull of $\mu(T)$ and all its permutations, i.e. the diagonal of D is majorized by $\mu(T)$ ($D \prec \mu(T)$). For general $A(t)$, we first make a change of a variable $Z(t) = \exp(\int_0^T -Tr(A(\tau))\mathrm{d}\tau)Y(t)$ to reduce the system to $SL(n,\mathbb{R})$, and get rid of the skew symmetric part of $A(t)$ as described in Remark 8. The results in the introduction then follows from above example.

B. $G/K = SU(p+q)/S(U(p) \times U(q)), 0 < p \le q$

Let $G = SU(p+q)$, $\mathfrak{g} = su(p+q)$, assume

$$\mathfrak{k} = \{X_j\}_{LA} = \{\begin{pmatrix} A & \mathbf{0} \\ \mathbf{0} & B \end{pmatrix} | \begin{array}{c} A \in u(p), B \in u(q) \\ Tr(A+B) = 0 \end{array} \}$$

$$\mathfrak{p} = \mathfrak{k}^\perp = \{\begin{pmatrix} \mathbf{0} & Z \\ -Z^\dagger & \mathbf{0} \end{pmatrix} | Z \text{ is } p \times q \text{ complex matrix}\}$$

$X_d(t) \in \mathfrak{p}$ and $\mathfrak{g} = \mathfrak{p} + \mathfrak{k}$ is a Cartan decomposition.

$$\mathfrak{h} = \{\begin{pmatrix} \mathbf{0} & iD & \mathbf{0} \\ iD & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} | D \text{ is p×p real diagonal matrices}\}$$

is a maximal abelian subalgebra in $\mathfrak{p}$. The restricted roots Δ_p are given by ($\pm$ signs read independently)

$$\Delta_p = \{\pm f_l, \pm 2 f_l, \pm f_l \pm f_m) : 1 \le l < m \le p\}.$$

with multiplicities $2(q-p)$, 1, and 2, respectively, where

$$f_m(\mathfrak{h}) = \frac{1}{2} d_{mm}, \qquad 1 \le m \le p$$

Consider all the reflections in $\mathfrak{h}$ about the planes $\alpha(H) = 0$ ($\alpha \in \Delta_p$), and use Proposition 2, it is found the Weyl group is either permutations, or sign flips, or permutations with sign flips, of the diagonal entries of D.

The set Σ of simple positive roots are given by

$$\Sigma = \{f_m - f_{m+1}, c f_p : 1 \le m < p\}$$

here $c = 1$ if $p < q$, $c = 2$ if $p = q$. Then

$$\mathfrak{h}_+ = \{\begin{pmatrix} \mathbf{0} & iD & \mathbf{0} \\ iD & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} | d_{11} > d_{22} > ... > d_{pp} > 0\}$$

is a Weyl chamber. So at each time point, assume

$$X_d(t) = \begin{pmatrix} \mathbf{0} & Z(t) \\ -Z^\dagger(t) & \mathbf{0} \end{pmatrix},$$

then $\exists U_1 \in U(p), U_2 \in U(q)$, such that

$$\begin{pmatrix} U_1 & 0 \\ 0 & U_2 \end{pmatrix} \in SU(p+q)$$

and $U_1 Z(t) U_2^\dagger = iD(t)$, where $D(t)$ is a real diagonal matrix with $d_{11}(t) \ge d_{22}(t) \ge ... \ge d_{pp}(t) \ge 0$. If let $\lambda^\downarrow(t)$ denote the vector $[d_{11}(t), d_{22}(t), ..., d_{pp}(t)]$ and let

$$\mu(T) = \int_0^T \lambda^\downarrow(t) \mathrm{d}t$$

where the integration is performed for each entry of the vector. Then the reachable set of system (4) in time T is all the matrices with the form $K_1 \exp(iY) K_2$ where K_1 and K_2 belong to

$$\{\begin{pmatrix} U_1 & 0 \\ 0 & U_2 \end{pmatrix} \in SU(p+q) | U_1 \in U(p), U_2 \in U(q)\},$$

$Y = \begin{pmatrix} \mathbf{0} & D & \mathbf{0} \\ D & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}$ with D a diagonal matrix whose diagonal entries lies in the convex hull of $\mu(T)$ and all its permutations with arbitrary sign flips, i.e. the absolute value of diagonal of D is majorized by $\mu(T)$.

In brief, for the control system

$$\dot{U}(t) = [X_d(t) + \sum_{j=1}^m v_j(t) X_j] U(t), \qquad U(0) = I, \quad (8)$$

on $SU(p+q)$, $0 < p \le q$, assume

$$\{X_j\}_{LA} = \{\begin{pmatrix} A & \mathbf{0} \\ \mathbf{0} & B \end{pmatrix} | \begin{matrix} A \in u(p), B \in u(q) \\ Tr(A+B) = 0 \end{matrix} \},$$

$$X_d(t) = \begin{pmatrix} \mathbf{0} & Z(t) \\ -Z(t)^\dagger & \mathbf{0} \end{pmatrix},$$

where $Z(t)$ is $p \times q$ complex matrix. Let $\lambda^\downarrow(t)$ be a vector whose entries are the positive square root of the eigenvalues of $Z(t)Z(t)^\dagger$ in decreasing order, and $\mu(T) = \int_0^T \lambda^\downarrow(t) \mathrm{d}t$, then the reachable set of the system within time T is all the matrices with the form $K_1 \exp(iY) K_2$ where K_1 and K_2 belong to

$$\{\begin{pmatrix} U_1 & 0 \\ 0 & U_2 \end{pmatrix} \in SU(p+q) | U_1 \in U(p), U_2 \in U(q)\},$$

$Y = \begin{pmatrix} \mathbf{0} & D & \mathbf{0} \\ D & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}$ with D a real $p \times p$ diagonal matrix whose diagonal entries lies in the convex hull of $\mu(T)$ and all its permutations with arbitrary sign flips, i.e. the absolute value of diagonal of D is majorized by $\mu(T)$.

C. $G/K = SU(4)/SU(2) \otimes SU(2)$

This case models the problem of control of coupled spin-1/2. The unitary evolution of two coupled spins is an element in $SU(4)$. External independent excitation of the spins by radio frequency electro magnetic field generates unitary transformations in the subgroups $SU(2) \otimes SU(2)$, which is the control subgroup for this problem.

The Lie algebra $\mathfrak{g} = \mathfrak{su}(4)$ has a Cartan decomposition $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$, where

$$\begin{aligned} \mathfrak{k} &= \text{span } \frac{i}{2}\{\sigma_x^1,\ \sigma_y^1,\ \sigma_z^1,\ \sigma_x^2,\ \sigma_y^2,\ \sigma_z^2\}, \\ \mathfrak{p} &= \text{span } \frac{i}{2}\{\sigma_x^1\sigma_x^2,\ \sigma_x^1\sigma_y^2,\ \sigma_x^1\sigma_z^2,\ \sigma_y^1\sigma_x^2, \\ &\qquad \sigma_y^1\sigma_y^2,\ \sigma_y^1\sigma_z^2,\ \sigma_z^1\sigma_x^2,\ \sigma_z^1\sigma_y^2,\ \sigma_z^1\sigma_z^2\}. \end{aligned} \quad (9)$$

Here σ_x, σ_y, and σ_z are the Pauli matrices,

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and $\sigma_\alpha^1 \sigma_\beta^2 = \sigma_\alpha^1 \otimes \sigma_\beta^2$.

$$\mathfrak{h} = \text{span } \frac{i}{2}\{\sigma_x^1\sigma_x^2,\ \sigma_y^1\sigma_y^2, \sigma_z^1\sigma_z^2\} \quad (10)$$

is a maximal abelian subalgebra in $\mathfrak{p}$.

Now we compute the Weyl group $W(G, K)$. Let $X = \frac{i}{2}(c_1\sigma_x^1\sigma_x^2 + c_2\sigma_y^1\sigma_y^2 + c_3\sigma_z^1\sigma_z^2) \in \mathfrak{h}$. Identify $\mathfrak{h}$ with $\mathbb{R}^3$, then $X = [c_1, c_2, c_3]$. The roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$ are:

$$\Delta_\mathfrak{p} = i\{\pm f_l \pm f_m) : 1 \le l < m \le 3\}. \quad (11)$$

where $f_i(X) = c_i$.

For $\alpha = i(f_1 - f_3) \in \Delta_\mathfrak{p}$, the plane $\alpha(X) = 0$ in $\mathfrak{h}$ is the set $\{X \in \mathbb{R}^3 | X u^T = 0\}$, where $u = [1, 0, -1]$.

The reflection of $X = [c_1, c_2, c_3]$ with respect to the plane $\alpha(X) = 0$ is

$$s_\alpha(X) = X - \frac{2uu^T}{\|u\|^2} X = [c_3, c_2, c_1]. \quad (12)$$

Similarly, we can compute all the reflections s_α as follows:

$$s_{i(f_3-f_2)}(X) = [c_1, c_3, c_2], \quad s_{i(f_2+f_3)}(X) = [c_1, -c_3, -c_2],$$

$$s_{i(f_2-f_1)}(X) = [c_2, c_1, c_3], \quad s_{i(f_1+f_2)}(X) = [-c_2, -c_1, c_3],$$

$$s_{i(f_1-f_3)}(X) = [c_3, c_2, c_1], \quad s_{i(f_1+f_3)}(X) = [-c_3, c_2, -c_1].$$

From Proposition 2, the Weyl group $W(G,K)$ is generated by s_α. Therefore, it's equivalent to either permutations of the elements of $[c_1, c_2, c_3]$, or permutations with sign flips of two elements.

$\Sigma = i\{f_1 - f_2, f_2 - f_3, f_2 + f_3\}$ is a set of simple positive roots. Then, we obtain a Weyl chamber

$$\mathfrak{h}_+ = \{[c_1, c_2, c_3] | c_1 > c_2 > |c_3|\}$$

Let $\{x_1, x_2, x_3\}$ be the dual of the simple positive roots,

$$x_1 = [1, -1, 0], x_2 = [0, 1, -1], x_3 = [0, 1, 1]$$

then $\mathfrak{h}_p = \{\sum_{i=1}^3 r_i x_i, r_i \geq 0\}$, let

$$x = c_1\sigma_x^1\sigma_x^2 + c_2\sigma_y^1\sigma_y^2 + c_3\sigma_z^1\sigma_z^2$$

$$y = c_1'\sigma_x^1\sigma_x^2 + c_2'\sigma_y^1\sigma_y^2 + c_3'\sigma_z^1\sigma_z^2$$

$[c_1, c_2, c_3], [c_1', c_2', c_3'] \in \overline{\mathfrak{h}}_+$, then according to theorem 2, $y \in \mathfrak{c}(x)$ if and only if

$$([c_1, c_2, c_3] - [c_1', c_2', c_3']) \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix}^{-1} \geq 0$$

i.e.

$$c_1' \leq c_1$$

$$c_1' + c_2' - c_3' \leq c_1 + c_2 - c_3$$

$$c_1' + c_2' + c_3' \leq c_1 + c_2 + c_3$$

this is exactly the partial order obtained in [5].

In relation to the control system in Theorem 3, the drift $X_d(t)$ takes the general form

$$X_d(t) = \frac{i}{2}\sum_{i,j} J_{ij}(t)\sigma_i^1\sigma_j^2 \qquad i,j \in \{x,y,z\}.$$

We assume $\{X_j\}_{LA} = \mathfrak{k}$, there exists $U(t), V(t) \in SU(2) \otimes SU(2)$, such that

$$U(t)X_d(t)V(t) = \frac{i}{2}\{\alpha_1(t)\sigma_x^1\sigma_x^2 + \alpha_2(t)\sigma_y^1\sigma_y^2 + \alpha_3(t)\sigma_z^1\sigma_z^2\}$$

where $\alpha_1(t) \geq \alpha_2(t) \geq |\alpha_3(t)|$.

Let $X_+(T) = [\int_0^T \alpha_1(\tau)\mathrm{d}\tau, \int_0^T \alpha_2(\tau)\mathrm{d}\tau, \int_0^T \alpha_3(\tau)\mathrm{d}\tau]$, then the reachable set within time T is all the matrices with the form

$$K_1 \exp(iY) K_2$$

where K_1 and K_2 belong to $SU(2) \otimes SU(2)$ and

$$Y = c_1\sigma_x^1\sigma_x^2 + c_2\sigma_y^1\sigma_y^2 + c_3\sigma_z^1\sigma_z^2$$

s.t $c_1 \geq c_2 \geq |c_3|$ and

$$c_1 \leq \int_0^T \alpha_1(\tau)\mathrm{d}\tau$$

$$c_1 + c_2 - c_3 \leq \int_0^T \alpha_1(\tau)\mathrm{d}\tau + \int_0^T \alpha_2(\tau)\mathrm{d}\tau - \int_0^T \alpha_3(\tau)\mathrm{d}\tau$$

$$c_1 + c_2 + c_3 \leq \int_0^T \alpha_1(\tau)\mathrm{d}\tau + \int_0^T \alpha_2(\tau)\mathrm{d}\tau + \int_0^T \alpha_3(\tau)\mathrm{d}\tau$$

REFERENCES

[1] N.Khaneja, R.Brockett and S.J.Glaser, "Time Optimal Control in spin systems", *Phys. Rev. A* **63**, 032308(2001).

[2] S. Helgason, *Differential Geometry, Lie Groups, and Symmetric Spaces* (Academic Press) (1978).

[3] J A. Wolf, *Spaces of Constant Curvature* (Publish or Perish, Inc., 1984).

[4] B. Kostant, "On Convexity, the Weyl Group and the Iwasawa decomposition" *Ann. Sci. Ecole Norm. Sup.* **6**, 413-455 (1973).

[5] C.H.Bennett, J.I.Cirac, M.S.Leifer, D.W.Leung, N.Linden, S.Popescu and G.Vudal, "Optimal simulation of two-qubit Hamiltonians using general local operations", *Phys. Rev. A* **66**, 012305(2002).

[6] R.W.Brockett, "System theory on group manifolds and coset spaces" *SIAM J. Control*, Vol. pp. 85-108, 1963.