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Regulating discrete-time homogeneous systems under arbitrary switching

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Abstract— **We consider discrete-time homogeneous control systems that undergo arbitrary switching. We propose an optimization-based, constructive method (which can be numerically realized) to generate a homogeneous control Lyapunov function and a homogeneous feedback law that stabilizes the origin for all possible switching scenarios. The established stability is robust with respect to small perturbations. We show that for linear systems the resulting Lyapunov function turns out to be convex. We also present a converse Lyapunov result where we state the equivalence of controllability to the origin and existence of a control Lyapunov function.**

I. INTRODUCTION

A dynamical system whose righthand side (of the differential or difference equation representing the system) can switch between the elements of a set of parametrized family of functions is called a switched system. The function (of time) with respect to which the switching occurs is called the switching signal. Switched systems have been actively investigated for various reasons, see [3], [6], [10]. The main reason, however, probably is that they can successfully represent many real-life and engineering systems which cannot be accurately modelled with more classical methods.

As it is usually the case in other subfields of control, the research on switched systems can be classified as stability analysis and control synthesis. The former focuses on problems like: find the conditions that, when satisfied by the system, would guarantee stability under any switching signal, see e.g. [1]; or classify switching signals for which the stability is guaranteed, see e.g. [13]. The work on control synthesis deals with finding a switching signal and/or a control input stabilizing the system, see e.g. [2]. In this work we are mainly interested in control synthesis. To be specific, we propose a recursive method to construct a feedback law (function of the state only) that robustly regulates the system *despite* the switching signal.

We generate our results for discrete-time homogeneous systems. Although the class of homogeneous systems is a small subclass of nonlinear systems, it includes systems with practical importance such as chained systems [12], systems in power form [7], and more importantly, linear systems, which are still not fully explored under switching. Switched homogeneous systems have not yet been able to attract many researchers. One of the few works on the subject is [4].

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The algorithm we propose to generate a feedback law also yields a continuous control Lyapunov function as a byproduct. Using that function and the results in [5] we show that the origin of the closed loop obtained via the generated feedback law is robustly asymptotically stable under any switching signal. The method is a natural extension of the recursive algorithm proposed in [11] and numerically realizable (for systems of low order at least). We show that the algorithm results in a convex control Lyapunov function for switched linear systems which then leads us to state an existence result. For the sake of completeness, we provide a converse Lyapunov result for switched homogeneous systems which states the equivalence of asymptotic and exponential controllability and existences of homogeneous and not necessarily homogeneous control Lyapunov functions.

II. NOTATION AND ASSUMPTIONS

We will consider the discrete-time *(control) system*

$$x^+ = \Gamma_q(x,\, u) \tag{1}$$

where $x \in \mathbb{R}^n$ is the *state*, $u \in \mathbb{R}^m$ is the *(control) input*, $q \in \{1,\, 2,\, \ldots,\, \bar{q}\} =: \mathcal{Q}$, $\bar{q}$ being a positive integer, is the *index* that switches the righthand side by selecting different transition maps from a parametrized family $\{\Gamma_q : q \in \mathcal{Q}\}$, and x^+ is the state at the next time instant. We will call a locally bounded map $\kappa : \mathbb{R}^n \to \mathbb{R}^m$ *feedback*. Given a feedback κ, the system $x^+ = \Gamma_q(x,\, \kappa(x))$ will be called the *closed loop*. The *solution* of a closed loop at time $k \in \mathbb{N}$, starting at the initial condition x, evolved under the influence of an index sequence $\mathbf{q} := \{q_0,\, q_1,\, \ldots\}$, with $q_i \in \mathcal{Q}$, is denoted by $\psi(k,\, x,\, \mathbf{q})$. (Notation $\mathbb{N}$ denotes the set of nonnegative integers.) Note that $\psi(0,\, x,\, \mathbf{q}) = x$ regardless of $\mathbf{q}$. System (1) is said to be *linear* if for each $q \in \mathcal{Q}$, $\Gamma_q(x,\, u) = A_q x + B_q u$, where $A_q \in \mathbb{R}^{n\times n}$ and $B_q \in \mathbb{R}^{n\times m}$. Notation $\overline{\mathbb{N}}$ stands for the set $\mathbb{N} \cup \{\infty\}$ and $\mathbb{E}$ denotes $[-\infty,\, \infty]$, so called the extended real line.

A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of *class* $\mathcal{K}_\infty$ if it is strictly increasing, continuous, zero at zero, and unbounded. (Notation $\mathbb{S}_{\geq \bar{s}}$ represents the set $\{s \in \mathbb{S} : s \geq \bar{s}\}$.) A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of *class* $\mathcal{KL}$ if for each fixed t, $\beta(\cdot,\, t)$ is nondecreasing and $\lim_{s\searrow 0} \beta(s,\, t) = 0$; and for each fixed s, $\beta(s,\, \cdot)$ is nonincreasing and $\lim_{t\to\infty} \beta(s,\, t) = 0$. When we write $\beta \in \mathcal{KL}$, we will mean that β is a class-$\mathcal{KL}$ function. Likewise for $\alpha \in \mathcal{K}_\infty$.

A *(state) dilation* Δ is such that for each $\lambda > 0$ (and it is undefined for λ nonpositive), $\Delta_\lambda = \mathrm{diag}(\lambda^{r_1},\, \lambda^{r_2},\, \ldots,\, \lambda^{r_n})$

with fixed $r_i > 0$. Likewise, δ denotes an input dilation, i.e. $\delta_\lambda u = \mathrm{diag}(\lambda^{p_1}, \lambda^{p_2}, \ldots, \lambda^{p_m})$.

Definition 1 *A transition map $\Gamma : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is said to be* homogeneous with respect to the dilation pair (Δ, δ) *if $\Gamma(\Delta_\lambda x, \delta_\lambda u) = \Delta_\lambda \Gamma(x, u)$ for all x, u, and λ.*

Definition 2 *A function $\sigma : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is said to be* homogeneous with respect to the dilation Δ with degree $d > 0$ *if $\sigma(\Delta_\lambda x) = \lambda^d \sigma(x)$ for all x and λ.*

Henceforth we will proceed under the following standing assumptions on system (1). **A1** There exists a dilation pair (Δ, δ) such that for each $q \in \mathcal{Q}$ the transition map Γ_q is homogeneous with respect to (Δ, δ). **A2** For each $q \in \mathcal{Q}$ the transition map Γ_q is continuous on $\mathbb{R}^n \times \mathbb{R}^m$. **A3** For each pair $R > r \geq 0$ there exists $U > 0$ such that for all $|x| \leq r$, $q \in \mathcal{Q}$, and $|u| \geq U$ we have $|\Gamma_q(x, u)| \geq R$.

Remark 1 *Assumption A3 is not crucial for most of the results to follow but rather put to guarantee the locally boundedness property of the stabilizing feedback which will come out as a minimizer to an optimization problem in which we do not penalize the input u. If system (1) is linear then A3 is equivalent to that B_q is full column rank for all q.*

Remark 2 *Whenever the system under consideration is linear, Assumptions A1-A2 comes for free since linear systems are homogeneous with respect to the standard dilations ($\Delta_\lambda = \lambda I_n$, $\delta_\lambda = \lambda I_m$) and continuous. ($I_i \in \mathbb{R}^{i \times i}$ is the identity matrix.)*

III. Constructing a feedback

Definition 3 *System (1) is said to be* strongly asymptotically controllable to the origin *if there exists a feedback κ and $\beta \in \mathcal{KL}$ such that for all x and $\mathbf{q}$, the solution of the closed loop $x^+ = \Gamma_q(x, \kappa(x))$ satisfies*

$$|\psi(k, x, \mathbf{q})| \leq \beta(|x|, k) \qquad \forall k \in \mathbb{N} \ . \tag{2}$$

Remark 3 *Note that under homogeneity, strong asymptotic controllability to the origin, which is a global definition, is equivalent to its local version in which (2) would hold for x that are in some neighborhood of the origin.*

Definition 4 *A continuous function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is said to be a* strong control Lyapunov function for $x^+ = \Gamma_q(x, u)$ *if there exist $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$ such that for all x we have*

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|) \tag{3}$$

$$\min_u \max_q V(\Gamma_q(x, u)) - V(x) \leq -\alpha_3(|x|) \ . \tag{4}$$

The next three definitions can be found in [8].

Definition 5 *A function $g : \mathbb{R}^p \to \mathbb{E}$ is* proper *if $g(\eta) < \infty$ for at least one $\eta \in \mathbb{R}^p$ and $g(\eta) > -\infty$ for all $\eta \in \mathbb{R}^p$.*

Definition 6 *A function $g : \mathbb{R}^p \to \mathbb{E}$ is* lower semicontinuous *at $\bar{\eta}$ if*

$$\lim_{\nu \searrow 0} \left\{ \inf_{|\eta - \bar{\eta}| \leq \nu} g(\eta) \right\} = g(\bar{\eta}) \ .$$

It is lower semicontinuous on $\mathbb{R}^p$ if this holds for all $\bar{\eta} \in \mathbb{R}^p$.

Definition 7 *A function $J : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{E}$ is* level-bounded in u locally uniformly in x *if for each $x \in \mathbb{R}^n$ and $c \in \mathbb{R}$ there exists a neighborhood $\mathcal{X}$ of x such that the set $\{(x, u) : x \in \mathcal{X},\ J(x, u) \leq c\}$ is bounded on $\mathbb{R}^n \times \mathbb{R}^m$.*

The following result resides in [8, Thm. 1.17].

Lemma 1 *For $J : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{E}$ that is proper, lower semicontinuous, and level-bounded in u locally uniformly in x let*

$$W(x) := \inf_u J(x, u) \ .$$

Then W is proper and lower semicontinuous on $\mathbb{R}^n$ and for each $x \in \mathbb{R}^n$ if $W(x) < \infty$ then there exists $u \in \mathbb{R}^m$ satisfying $J(x, u) = W(x)$. Moreover, W is continuous at x if $J(\cdot, u)$ is continuous at x for some u satisfying $J(x, u) = W(x) < \infty$.

Let us first pick some continuous positive definite function $\sigma : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ that is homogeneous with respect to Δ (of A1) with degree d. Then let $V_N : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ for $N \in \mathbb{N}$ be obtained through the recursive relation

$$V_{N+1}(x) := \sigma(x) + \inf_u \max_q V_N(\Gamma_q(x, u)) \tag{5}$$

with $V_0(x) := \sigma(x)$. Also, whenever it is finite-valued, we will let $V_\infty(x) := \lim_{N \to \infty} V_N(x)$.

Theorem 1 *Let system (1) be strongly asymptotically controllable to the origin. Then there exists $L \geq 1$ such that for all $N \in \overline{\mathbb{N}}$, V_N is continuous and satisfies for all x and λ*

$$\sigma(x) \leq V_N(x) \leq L\sigma(x) \ , \tag{6}$$

$$V_N(\Delta_\lambda x) = \lambda^d V_N(x) \ , \tag{7}$$

$$V_{N+1}(x) = \sigma(x) + \min_u \max_q V_N(\Gamma_q(x, u)) \ . \tag{8}$$

Proof. Let us begin with proving (7). Without loss of generality we take $d = 1$ in this proof. Suppose for some $N \in \mathbb{N}$ we have $V_N(\Delta_\lambda x) = \lambda V_N(x)$ for all x and λ. Then we can write by homogeneity of Γ_q and σ

$$\begin{aligned} V_{N+1}(\Delta_\lambda x) &= \sigma(\Delta_\lambda x) + \inf_u \max_q V_N(\Gamma_q(\Delta_\lambda x, u)) \\ &= \lambda\sigma(x) + \inf_u \max_q V_N(\Delta_\lambda \Gamma_q(x, \delta_{\lambda^{-1}} u)) \\ &= \lambda\sigma(x) + \inf_u \max_q \lambda V_N(\Gamma_q(x, u)) \\ &= \lambda \left\{ \sigma(x) + \inf_u \max_q V_N(\Gamma_q(x, u)) \right\} \\ &= \lambda V_{N+1}(x) \ . \end{aligned}$$

Note that $V_0(x) = \sigma(x)$ which satisfies (7) since σ is homogeneous. Hence we have (7) for all $N \in \mathbb{N}$ by induction. To show the $N = \infty$ case observe that

$$\begin{aligned} V_\infty(\Delta_\lambda x) &= \lim_{N \to \infty} V_N(\Delta_\lambda x) \\ &= \lim_{N \to \infty} \lambda V_N(x) \\ &= \lambda \left(\lim_{N \to \infty} V_N(x) \right) = \lambda V_\infty(x) \ . \end{aligned}$$

Now we prove (6). That $\sigma(x) \leq V_N(x)$ comes by definition. Since system is strongly asymptotically controllable to the origin there exist a feedback κ and $\tilde{\beta} \in \mathcal{KL}$ such that the closed loop solution satisfies for all x and $\mathbf{q}$

$$|\psi(k, x, \mathbf{q})| \leq \tilde{\beta}(|x|, k) \qquad \forall k \in \mathbb{N} .$$

Since σ is continuous, positive definite, and homogeneous, there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that $\alpha_1(|x|) \leq \sigma(x) \leq \alpha_2(|x|)$ for all x. Hence we can write

$$\sigma(\psi(k, x, \mathbf{q})) \leq \beta(\sigma(x), k) \qquad \forall k \in \mathbb{N} \tag{9}$$

where $\beta(s, t) := \alpha_2(\tilde{\beta}(\alpha_1^{-1}(s), t))$ is of class-$\mathcal{KL}$. Note that by (5) we have $V_{N+1}(x) \leq \sigma(x) + \max_q V_N(\Gamma_q(x, \kappa(x)))$. Now let $\bar{k} \in \mathbb{N}_{\geq 1}$ be such that $\beta(1, \bar{k}) \leq 2^{-1}$. Then we observe by (9) that $\sigma(\psi(\bar{k}, x, \mathbf{q})) \leq 2^{-1}$ for all $x \in \{z : \sigma(z) \leq 1\} =: \mathcal{B}_\sigma$ and $\mathbf{q}$. Let us define $L := 2\sum_{k=0}^{\bar{k}} \beta(1, k)$. Hence we have $V_{\bar{k}}(x) \leq 2^{-1}L$ for all $x \in \mathcal{B}_\sigma$. Let $\mathbf{q}_{\bar{k}}$ denote an index sequence $\{q_0, q_1, \ldots, q_{\bar{k}-1}\}$. Let us be given $x \in \mathcal{B}_\sigma$. Then for $i \in \mathbb{N}_{\geq 1}$ we can write from optimality arguments and homogeneity of V_N that

$$\begin{aligned} V_{(i+1)\bar{k}}(x) \;&\leq\; \max_{\mathbf{q}_{\bar{k}}} \Bigg\{ \sum_{k=0}^{\bar{k}-1} \sigma(\psi(k, x, \mathbf{q}_{\bar{k}})) + \\ &\qquad\qquad V_{i\bar{k}}(\psi(\bar{k}, x, \mathbf{q}_{\bar{k}})) \Bigg\} \\ &\leq\; 2^{-1}L + \max_{\mathbf{q}_{\bar{k}}} V_{i\bar{k}}(\psi(\bar{k}, x, \mathbf{q}_{\bar{k}})) \\ &=\; 2^{-1}L + \max_{\mathbf{q}_{\bar{k}}} 2^{-1} V_{i\bar{k}}(\Delta_2\psi(\bar{k}, x, \mathbf{q}_{\bar{k}})) \\ &=\; 2^{-1}L + 2^{-1} \max_{\mathbf{q}_{\bar{k}}} V_{i\bar{k}}(\Delta_2\psi(\bar{k}, x, \mathbf{q}_{\bar{k}})) \ . \end{aligned}$$

Note that $\Delta_2\psi(\bar{k}, x, \mathbf{q}_{\bar{k}}) \in \mathcal{B}_\sigma$ regardless of $\mathbf{q}_{\bar{k}}$. Therefore, recalling that $V_{\bar{k}}(x) \leq 2^{-1}L$, we can induce

$$V_{i\bar{k}}(x) \leq L \sum_{k=1}^{i} 2^{-k} \leq L \sum_{k=1}^{\infty} 2^{-k} = L$$

for all $i \in \overline{\mathbb{N}}_{\geq 1}$ and $x \in \mathcal{B}_\sigma$. Since $V_{N+1}(x) \geq V_N(x)$ we can write for all $N \in \overline{\mathbb{N}}$ that $V_N(x) \leq L$ provided that $x \in \mathcal{B}_\sigma$. Let $\eta \in \mathbb{R}^n$ be given. If $\eta = 0$ then $V_N(\eta) = 0$ and (6) holds. Suppose $\sigma(\eta) > 0$. Then observe that $\Delta_{\sigma(\eta)^{-1}}\eta \in \mathcal{B}_\sigma$. Therefore we can write

$$\begin{aligned} V_N(\eta) \;&=\; \sigma(\eta) V_N(\Delta_{\sigma(\eta)^{-1}}\eta) \\ &\leq\; L\sigma(\eta) \ . \end{aligned}$$

Thus we have (6).

We now prove continuity and (8). Suppose V_{N-1} is continuous for some $N \in \mathbb{N}_{\geq 1}$. We define $J_N : \mathbb{R}^n \times \mathbb{R}^m$ as

$$J_N(x, u) := \sigma(x) + \max_q V_{N-1}(\Gamma_q(x, u)) \ .$$

Note that J_N is continuous (hence lower semicontinuous) due to the continuity of σ, V_{N-1}, and $\Gamma_q(x, u)$ and the fact that maximum of continuous functions is continuous. It is proper by definition and level-bounded in u locally uniformly in x due to Assumption A3. Hence we can invoke Lemma 1 and obtain the continuity of V_N since $V_N(x) = \inf_u J_N(x, u)$. Also, again by Lemma 1, a minimizer exists and we have $V_N(x) = \min_u J_N(x, u)$. Note that V_0 is continuous since it is σ. As a result, by induction, V_N is continuous and (8) holds for all $N \in \mathbb{N}$.

At this point we pause shortly to claim that V_N converges to V_∞ uniformly on $\mathcal{B}_\sigma$. Suppose not. Then there would exist $\varepsilon > 0$ such that for each $N \in \mathbb{N}$ there would exist $x \in \mathcal{B}_\sigma$ such that $V_\infty(x) - V_N(x) > \varepsilon$. Let us pick $p = \lceil 2L^2\varepsilon^{-1} \rceil$ and let $x \in \mathcal{B}_\sigma$ then be such that $V_\infty(x) - V_p(x) > \varepsilon$. Then, by definition, for all index sequences $\mathbf{q}_p := \{q_0, q_1, \ldots, q_{p-1}\}$ we can write

$$V_p(x) \geq \sigma(\psi_0) + \sigma(\psi_1) + \ldots + \sigma(\psi_p) \tag{10}$$

where $\psi_0 = x$ and $\psi_{k+1} = \Gamma_{q_k}(\psi_k, \kappa_{p-k}(\psi_k))$ for $k \in \{0, 1, \ldots, p-1\}$ and where

$$\kappa_i(\eta) := \operatorname*{argmin}_u \max_q V_i(\Gamma_q(\eta, u)) \tag{11}$$

for $i \in \{1, 2, \ldots, p\}$. Let $\mathbf{q}$ be an infinite index sequence such that

$$V_\infty(x) = \sum_{k=0}^{\infty} \sigma(\psi(k, x, \mathbf{q}))$$

where $\psi(\cdot, x, \mathbf{q})$ is the solution to closed loop formed by κ_∞. Feedback κ_∞ is defined by (11) for $i = \infty$. Let $\mathbf{q}_p$ be the first p elements of $\mathbf{q}$. Recall that (10) holds for any index sequence hence for the $\mathbf{q}_p$ we pick. The terms in the sum (10) are all nonnegative. Hence there must exist some $\tilde{k} \in \{1, 2, \ldots, p\}$ such that $\sigma(\psi_{\tilde{k}}) \leq Lp^{-1} \leq \varepsilon(2L)^{-1}$ from (6), $x \in \mathcal{B}_\sigma$, and $p = \lceil 2L^2\varepsilon^{-1} \rceil$. Then from (10), (6), and how we picked $\mathbf{q}_p$ we can write

$$\begin{aligned} V_p(x) \;&\geq\; \sigma(\psi_0) + \ldots + \sigma(\psi_{\tilde{k}-1}) + V_\infty(\psi_{\tilde{k}}) - V_\infty(\psi_{\tilde{k}}) \\ &=\; V_\infty(x) - V_\infty(\psi_{\tilde{k}}) \\ &\geq\; V_\infty(x) - L\sigma(\psi_{\tilde{k}}) \\ &\geq\; V_\infty(x) - 2^{-1}\varepsilon \end{aligned}$$

which is a contradiction. Hence our claim holds.

As a result of this uniform convergence, V_∞ is continuous on $\mathcal{B}_\sigma$ since uniform limit of continuous functions is continuous (see, for instance, [9, Thm. 24.3]). The continuity of V_∞ on $\mathbb{R}^n$ then comes by homogeneity of V_∞.

Finally we prove (8) for $N = \infty$. Let us define $J_\infty(x, u) := \sigma(x) + \max_q V_\infty(\Gamma_q(x, u))$ which is continuous, proper, and level-bounded in u locally uniformly in x by the same arguments we had on J_N. Lemma 1 tells us that for each x a minimizer u exists such that $V_\infty(x) = J_\infty(x, u)$. Hence the result by (5). ■

The following corollary is of practical importance. It says that if N is large enough, then V_N generated by the recursive relation (5) can be used as a strong control Lyapunov function for system (1).

Corollary 1 *Suppose system (1) is strongly asymptotically controllable to the origin. Then for each $\mu < 1$ there exists $N_\mu \in \mathbb{N}$ such that for all $N \geq N_\mu$ and x we have*

$$\min_u \max_q V_N(\Gamma_q(x,\, u)) - V_N(x) \leq -\mu\sigma(x)\ . \qquad (12)$$

Proof. Recall that in the proof of Theorem 1 we have shown that V_N converges uniformly to V_∞ on $\mathcal{B}_\sigma$. Also recall that $V_{N+1}(x) \geq V_N(x)$ for all $N \in \mathbb{N}$ and x. Let us be given $\mu < 1$. Then there exists $N_\mu \in \mathbb{N}$ such that for all $x \in \mathcal{B}_\sigma$ and $N \geq N_\mu$ we have $V_\infty(x) - V_N(x) \leq 1 - \mu$ and hence

$$V_{N+1}(x) - V_N(x) \leq V_\infty(x) - V_N(x) \leq 1 - \mu\ .$$

Let us define $\mathcal{C}_\sigma := \{z \in \mathbb{R}^n : \sigma(z) = 1\}$. Note that $\mathcal{C}_\sigma \subset \mathcal{B}_\sigma$. Then from (8), for all $N \geq N_\mu$ and $x \in \mathcal{C}_\sigma$

$$\begin{aligned}\min_u \max_q\, & V_N(\Gamma_q(x,\, u)) - V_N(x)\\ &\leq \min_u \max_q V_N(\Gamma_q(x,\, u)) - V_{N+1}(x) + 1 - \mu\\ &\leq -1 + 1 - \mu = -\mu\ .\end{aligned}$$

Homogeneity (7) then brings us the result. ■

For $N \in \overline{\mathbb{N}}$, let us define feedback κ_N as

$$\kappa_N(x) := \operatorname*{argmin}_u \max_q V_N(\Gamma_q(x,\, u))\ . \qquad (13)$$

Note that κ_N satisfies for all x and λ

$$\max_q V_N(\Gamma_q(x,\, \kappa_N(x))) = \max_q V_N(\Gamma_q(x,\, \delta_\lambda \kappa_N(\Delta_{\lambda^{-1}} x)))$$

which implies $\kappa_N(\Delta_\lambda x) = \delta_\lambda \kappa_N(x)$ provided that the minimizer in (13) is unique for all x. Note however that, without loss of generality, we can assume $\kappa_N(\Delta_\lambda x) = \delta_\lambda \kappa_N(x)$ even if the minimizer is not unique for we can always choose a homogeneous κ_N from the set of minimizers.

A. On robustness

For robustness analysis let us consider difference inclusions. Let F be a set-valued map from $\mathbb{R}^n$ to the subsets of $\mathbb{R}^n$ and let $\psi(\cdot,\, x)$ denote a solution of the difference inclusion $x^+ \in F(x)$ starting from an initial condition x. Let $\mathcal{S}(x)$ denote the set of solutions starting from x. Let $\mathcal{B}$ be the unit closed ball in $\mathbb{R}^n$. The addition of two sets in $\mathbb{R}^n$, $\mathcal{W}$ and $\mathcal{Y}$, is defined as

$$\mathcal{W} + \mathcal{Y} := \{w + y \in \mathbb{R}^n : w \in \mathcal{W},\, y \in \mathcal{Y}\}\ .$$

For a continuous function $\varrho : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ a perturbed inclusion is defined as

$$x^+ \in F_\varrho(x) := F(x + \varrho(x)\mathcal{B}) + \varrho(x)\mathcal{B}$$

and a solution starting from x is denoted by $\psi_\varrho(\cdot,\, x)$ which is an element of the set $\mathcal{S}_\varrho(x)$.

Definition 8 *The origin is* robustly strongly asymptotically stable for $x^+ \in F(x)$ *if there exists $\varrho : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ continuous and positive definite and $\beta_\varrho \in \mathcal{KL}$ such that for all $x \in \mathbb{R}^n$, all solutions $\psi_\varrho \in \mathcal{S}_\varrho(x)$ satisfy*

$$|\psi_\varrho(k,\, x)| \leq \beta_\varrho(|x|,\, k) \qquad \forall k \in \mathbb{N}\ .$$

Definition 9 *A continuous function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is said to be a* strong Lyapunov function for $x^+ \in F(x)$ *if there exist $\alpha_1,\, \alpha_2,\, \alpha_3 \in \mathcal{K}_\infty$ such that for all x we have*

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|)$$

$$\max_{f \in F(x)} V(f) - V(x) \leq -\alpha_3(|x|)\ .$$

The following result is borrowed from [5].

Lemma 2 *Let F be a set-valued map from $\mathbb{R}^n$ to subsets of $\mathbb{R}^n$ and for each $x \in \mathbb{R}^n$ let $F(x)$ be nonempty. Then for the difference inclusion $x^+ \in F(x)$, if there exists a strong Lyapunov function then the origin is robustly strongly asymptotically stable.*

Definition 10 *A feedback κ is said to be* robustly stabilizing *for system (1) if the origin is robustly strongly asymptotically stable for $x^+ \in F(x)$, where*

$$F(x) := \bigcup_{q \in \mathcal{Q}} \Gamma_q(x,\, \kappa(x))\ .$$

Theorem 2 *Suppose system (1) is strongly asymptotically controllable to the origin. Then there exists $p \in \mathbb{N}$ such that for all $N \in \overline{\mathbb{N}}_{\geq p}$, κ_N is robustly stabilizing for system (1).*

Proof. Let $\mu = 2^{-1}$ and N_μ then be given by Corollary 1. Take $p = N_\mu$. Let $N \in \overline{\mathbb{N}}_{\geq p}$. Since σ is continuous, positive definite, and homogeneous there exist $\alpha_1,\, \alpha_2 \in \mathcal{K}_\infty$ such that $\alpha_1(|x|) \leq \sigma(x) \leq \alpha_2(|x|)$ for all x. By (6) therefore we can write for all x

$$\alpha_1(|x|) \leq V_N(x) \leq \alpha_3(|x|) \qquad (14)$$

where $\alpha_3(s) := L\alpha_2(s)$ is a class-$\mathcal{K}_\infty$ function. Let us define

$$F_N(x) := \bigcup_{q \in \mathcal{Q}} \Gamma_q(x,\, \kappa_N(x))\ .$$

From (12) and (13) we then have

$$\begin{aligned}\max_{f \in F_N(x)} V_N(f) - V_N(x) &\\ &= \max_q V_N(\Gamma_q(x,\, \kappa_N(x))) - V_N(x)\\ &\leq -2^{-1}\sigma(x)\\ &\leq -\alpha_4(|x|) \qquad (15)\end{aligned}$$

where $\alpha_4(s) := 2^{-1}\alpha_1(s)$ is a class-$\mathcal{K}_\infty$ function. We know by Theorem 1 that V_N is continuous. By (14) and (15) therefore V_N is a strong Lyapunov function for $x^+ \in F_N(x)$. Result follows from Definition 10 and Lemma 2. ■

B. Linear systems and convexity

Next result says that if σ is chosen quadratic and the system is linear, V_N of (5) turns out to be convex for all N.

Corollary 2 *Suppose system (1) is linear and strongly asymptotically controllable to the origin. Let $\sigma(x) = x^T Q x$ for some positive definite $Q \in \mathbb{R}^{n \times n}$. Then V_N is convex and satisfies $V_N(\lambda x) = \lambda^2 V_N(x)$ for all λ, x, and $N \in \overline{\mathbb{N}}$.*

Proof. Observe that $\sigma(\lambda x) = \lambda^2 \sigma(x)$. Then that $V_N(\lambda x) = \lambda^2 V_N(x)$ directly comes from Theorem 1 and the linearity of the system. Let us prove convexity. Suppose for some $N \in \mathbb{N}$, V_N is convex. Let $x, y \in \mathbb{R}^n$, $\mu \in [0, 1]$, and $\bar{\mu} = 1 - \mu$. Now we claim that

$$\min_u \max_q V_N(A_q(\mu x + \bar{\mu} y) + B_q u) \leq$$
$$\mu \min_u \max_q V_N(A_q x + B_q u) + \bar{\mu} \min_u \max_q V_N(A_q y + B_q u).$$

Suppose not. Then there would exist $v, w \in \mathbb{R}^m$ such that

$$\begin{aligned}
\min_u \max_q & V_N(A_q(\mu x + \bar{\mu} y) + B_q u) \\
&> \mu \max_q V_N(A_q x + B_q v) + \bar{\mu} \max_q V_N(A_q y + B_q w) \\
&\geq \max_q \{\mu V_N(A_q x + B_q v) + \bar{\mu} V_N(A_q y + B_q w)\} \\
&\geq \max_q V_N(\mu(A_q x + B_q v) + \bar{\mu}(A_q y + B_q w)) \\
&= \max_q V_N(A_q(\mu x + \bar{\mu} y) + B_q(\mu v + \bar{\mu} w)) \\
&\geq \min_u \max_q V_N(A_q(\mu x + \bar{\mu} y) + B_q u)
\end{aligned}$$

which poses a contradiction. Hence our claim holds. Note that σ is convex since $Q > 0$. Therefore, from (5), the definition of convexity, and the fact that the sum of convex functions is convex, we can at once obtain the convexity of V_{N+1}. Since V_0 is σ, which is convex, the result follows for all $N \in \mathbb{N}$ by induction. Note that system is strongly asymptotically controllable to the origin, which implies, by Theorem 1, that V_∞ exists. Recall that the limit of convex functions is convex. Thus V_∞ is also convex. ■

Next result is a trivial consequence of Corollaries 1-2.

Theorem 3 *Suppose system (1) is linear and strongly asymptotically controllable to the origin. Then there exists a convex strong control Lyapunov function that is homogeneous of degree two with respect to the standard dilation.*

Remark 4 *Note that if V is a convex strong control Lyapunov function for a switched linear system $x^+ = A_q x + B_q u$, then it is also a strong control Lyapunov function for the switched linear system $x^+ = \widehat{A}_q x + \widehat{B}_q u$ provided that each member of the family $\{(\widehat{A}_q, \widehat{B}_q)\}$ lies within the convex hull generated by the members of the family $\{(A_q, B_q)\}$.*

IV. A CONVERSE LYAPUNOV RESULT

Definition 11 *System (1) is said to be* strongly exponentially controllable to the origin with respect to $\sigma : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ *if there exists a feedback κ, $M \geq 1$, and $\rho > 0$ such that for all x and $\mathbf{q}$ the solution of the closed loop $x^+ = \Gamma_q(x, \kappa(x))$ satisfies*

$$\sigma(\psi(k, x, \mathbf{q})) \leq M \exp(-\rho k)\sigma(x) \qquad \forall k \in \mathbb{N}\ .$$

Theorem 4 *The following are equivalent.*

(i) *System (1) is strongly asymptotically controllable to the origin.*
(ii) *System (1) is strongly exponentially controllable to the origin with respect to σ.*
(iii) *There exists a strong control Lyapunov function for $x^+ = \Gamma_q(x, u)$.*
(iv) *For each $r > 0$ there exists a strong control Lyapunov function for $x^+ = \Gamma_q(x, u)$ that is homogeneous with degree r.*

Proof. We have (i) ⇒ (iv) by Corollary 1. Obvious are (ii) ⇒ (i) and (iv) ⇒ (iii). Hence it suffices to show (iii) ⇒ (i) and (iv) ⇒ (ii). Let us begin with the former. Let V be a strong control Lyapunov function for $x^+ = \Gamma_q(x, u)$ and $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$ come from Definition 4. Then from (3) and (4) we can write

$$\min_u \max_q V(\Gamma_q(x, u)) \leq V(x) - \alpha_3(\alpha_2^{-1}(V(x)))\ . \quad (16)$$

Let us define $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ as $\gamma(s) := s - \alpha_3(\alpha_2^{-1}(s))$. Note that $\gamma(s) < s$ for all $s > 0$. Having defined γ let us define $\beta_V \in \mathcal{KL}$ as such

$$\beta_V(s, t) := \gamma^k(s) \qquad \forall t \in [k, k+1) \quad (17)$$

where $k \in \mathbb{N}$ and $\gamma^{k+1}(s) = \gamma(\gamma^k(s))$ with $\gamma^0(s) = s$. Let us also define feedback κ as

$$\kappa(x) := \operatorname*{argmin}_u \max_q V(\Gamma_q(x, u))\ . \quad (18)$$

Then for all x and $\mathbf{q}$ we have by (16), (17), and (18)

$$V(\psi(k, x, \mathbf{q})) \leq \beta_V(V(x), k) \qquad \forall k \in \mathbb{N}$$

where ψ is the closed-loop solution to $x^+ = \Gamma_q(x, \kappa(x))$. Therefore we can write by (3)

$$|\psi(k, x, \mathbf{q})| \leq \beta(|x|, k) \qquad \forall k \in \mathbb{N}$$

where $\beta(s, t) := \alpha_1^{-1}(\beta_V(\alpha_2(s), t))$ is a class-$\mathcal{KL}$ function. Hence (iii) ⇒ (i) is shown.

Now suppose we have (iv). Then there exists a homogeneous strong control Lyapunov function V^{h} that has the same degree of homogeneity with σ, that is $r = d$. Without loss of generality let the degree d be unity. Function V^{h} satisfies (3) and (4) for some $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$ and we can obtain (16). Let function γ be defined as above and let $\mu \in (\gamma(1), 1)$. Then by (16) for all $x \in \{z \in \mathbb{R}^n : V^{\mathrm{h}}(z) = 1\} =: \mathcal{C}_V$ we have

$$\min_u \max_q V^{\mathrm{h}}(\Gamma_q(x, u)) \leq \mu\ .$$

Let us be given some x with $V^{\mathrm{h}}(x) > 0$. Let $\lambda := V^{\mathrm{h}}(x)$. Note that $\Delta_{\lambda^{-1}} x \in \mathcal{C}_V$. Then we can write

$$\begin{aligned}
\min_u \max_q & V^{\mathrm{h}}(\Gamma_q(x, u)) \\
&= \min_u \max_q V^{\mathrm{h}}(\Gamma_q(\Delta_\lambda \Delta_{\lambda^{-1}} x, \delta_\lambda \delta_{\lambda^{-1}} u)) \\
&= \min_u \max_q V^{\mathrm{h}}(\Delta_\lambda \Gamma_q(\Delta_{\lambda^{-1}} x, \delta_{\lambda^{-1}} u)) \\
&= \min_u \max_q \lambda V^{\mathrm{h}}(\Gamma_q(\Delta_{\lambda^{-1}} x, u)) \\
&\leq \mu\lambda = \mu V^{\mathrm{h}}(x)\ .
\end{aligned}$$

As a consequence the solution of the closed-loop $x^+ = \Gamma_q(x, \kappa^{\mathrm{h}}(x))$ satisfies, where

$$\kappa^{\mathrm{h}}(x) := \operatorname*{argmin}_u \max_q V^{\mathrm{h}}(\Gamma_q(x, u))\ ,$$

for all x and $\mathbf{q}$

$$V^{\mathrm{h}}(\psi(k, x, \mathbf{q})) \le \mu^k V^{\mathrm{h}}(x) \qquad \forall k \in \mathbb{N} \ . \tag{19}$$

Since both σ and V^{h} are continuous, positive definite, and homogeneous with same degree of homogeneity, there exist positive constants ℓ_1, ℓ_2 such that for all x

$$\ell_1 \sigma(x) \le V^{\mathrm{h}}(x) \le \ell_2 \sigma(x) \ . \tag{20}$$

Combining (19) and (20) yields

$$\begin{aligned}\sigma(\psi(k, x, \mathbf{q})) &\le \ell_1^{-1}\ell_2 \mu^k \sigma(x) \\ &= M \exp(-\rho k)\sigma(x) \qquad \forall k \in \mathbb{N}\end{aligned}$$

where $M := \ell_1^{-1}\ell_2$ and $\rho := -\ln(\mu)$. Hence we have (ii) and the result follows. ■

V. A NUMERICAL EXAMPLE

For a numerical demonstration, we picked a second order system $x^+ = \Gamma_q(x, u)$ with $q \in \{1, 2\}$ where

$$\{\Gamma_q(x, u)\} = \left\{ \begin{bmatrix} x_1 + u \\ x_2 + |x_1|u \end{bmatrix} , \begin{bmatrix} x_1 + 1.25u \\ x_2 + 1.5625|x_1|u \end{bmatrix} \right\} .$$

Note that Γ_q is homogeneous with respect to (Δ, δ) where $\Delta_\lambda = \mathrm{diag}(\lambda, \lambda^2)$ and $\delta_\lambda = \lambda$. We recursively computed V_N for $N = \{0, 1, \ldots, 10\}$ via (5) where we took $\sigma(x) = (x_1^4 + x_2^2)^{1/2}$. Fig. 1 shows the sublevel sets $\{z : V_N(z) = 1\}$ where $N = 0$ for the outermost curve and the the curves shrink toward the center subsequently with increasing N. In Fig. 2 we show the simulation results of the closed loop $x^+ = \Gamma_q(x, \kappa_{10}(x))$ (where κ_{10} is computed via (13)) for three different initial conditions under arbitrary switching sequences.

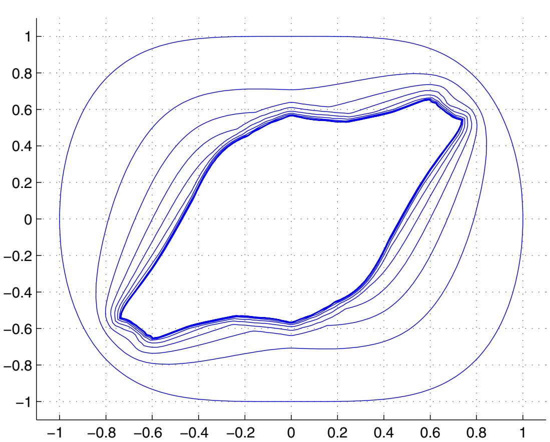

Fig. 1. Sublevel sets $\{z : V_N(z) = 1\}$ for $N = 0, 1, \ldots, 10$.

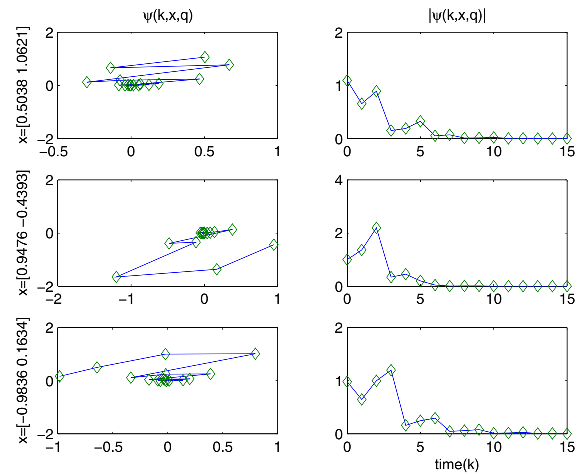

Fig. 2. Simulation results for various initial conditions under arbitrary switching sequences. On left are the phase plots $\psi(k, x, \mathbf{q})$. On right are the norms $|\psi(k, x, \mathbf{q})|$ versus time.

VI. CONCLUSION

For discrete-time homogeneous control systems that undergo arbitrary switching, we presented a constructive method to generate a control Lyapunov function and a robustly stabilizing feedback law. We showed that the generated Lyapunov function is convex for switched linear systems. We also presented a converse Lyapunov result where we state the equivalence of controllability to the origin and existence of a control Lyapunov function. We demonstrated our results on a second order switched nonlinear system via simulations.

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