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Gain scheduled state feedback control of discrete-time systems with time-varying uncertainties: an LMI approach

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***Abstract*— This paper addresses the problems of stabilization and $\mathcal{H}_\infty$ control by means of state feedback parameter-dependent gains applied to discrete-time linear systems whose matrices are affected by arbitrarily time-varying parameters belonging to a polytope. The solution of the proposed design conditions, written as a finite set of linear matrix inequalities at the polytope vertices, allows to obtain a parameter-dependent gain (i.e. a gain scheduled controller) as an analytical function of the parameters. The proposed strategy is different from similar approaches in the literature, that are based on discretizations of the space of parameters to determine interpolated control gains, or that assume special structures for the time-varying parameters or even suppose that some of the system matrices are fixed and time-invariant in order to have a convex design problem. Numerical examples illustrate the efficiency of the conditions given in the paper.**

I. INTRODUCTION

The design of gain scheduled controllers has been an important issue in systems theory and control applications for decades (see the survey papers [1], [2] and references therein). Basically, this technique focus on determining the control gain as a function of the system time-varying parameters, supposed to be available in real time. A classical way to compute a gain scheduled controller for a given linear parameter-varying (LPV) model of a plant, that usually comes from the linearization of the nonlinear model of the plant around operating points, follows the steps: i) determine a grid in the space of parameters to choose a family of plants and design one local controller for each plant, ii) based on the values of the parameters (measured or estimated on-line), schedule the control gains using some interpolation method, iii) assess the closed-loop system stability and performance. Although the system performance can be improved by means of increasing the precision of the discretization of the space of parameters (at the price of increasing the computational burden) this approach may be unreliable, since the global stability and performance are only assessed through simulation. Another problem is that the rates of variation of the time-varying parameter are not taken into account in the design, which may lead to instability or poor performance in the case of fast time-varying parameters [3], [4].

More recently, several design approaches based on Lyapunov functions attempt to provide gain scheduled controllers to cope with time-varying parameters with bounded or unbounded rates of variation, some of them using linear matrix inequalities (LMIs – see [5]), that are numerically attractive due to their solvability via polynomial time algorithms [6]. For instance, in the case of parameters with arbitrary rates of variation, when the plant and the controller admit a representation given by linear fractional transformation (LFT), a stabilizing controller can be determined solving a convex problem with a finite number of LMIs [7], [8], [9], [10], [11]. Also in the context of arbitrary parametric variations, the quadratic stability was used to provide convex LMI design of LPV controllers with $\mathcal{H}_\infty$ guaranteed performance for linear time-varying systems in polytopic domains for which the control matrices are supposed to be fixed and time-invariant [12], [13], [14]. It is interesting to note that the previous mentioned works avoid gridding procedures by means of restrictive assumptions on the structure of the time-varying parameters or by considering some matrices of the system as fixed and time-invariant. In the case of parameters with bounded rates of variation, some works address the design of parameter-dependent (i.e. gain scheduled) control gains for linear time-varying systems in polytopic domains using a discretization of the parametric space in a finite number of points that can assure the design specifications [15], [16], [17]. One problem with these approaches is that to improve the performance, it is necessary to increase the precision of the grid, thus increasing the computational effort rapidly. A convex condition written as a finite set of LMIs using only the vertices of the polytope and the bounds on the rates of variation of the system parameters has been given in [18], allowing to obtain stabilizing state feedback gain scheduled controllers.

The aim of this paper is to provide LMI convex conditions to design state feedback gain scheduling controllers to cope with stabilization and $\mathcal{H}_\infty$ performance for discrete-time linear systems with time-varying parameters belonging to a polytope with arbitrary rates of variation. It is worthy of mention that previous results in the literature that deal with the same problem using the Lyapunov approach, as [19], [20], assume that the control matrices are fixed and time-invariant. Here, all the system matrices are supposed to be affected by the time-varying parameters which can vary arbitrarily, being useful for instance to cope with problems of actuator failures that can be modeled as variations in the columns of the control matrices. Moreover, there is no use of grids in the parametric space nor restrictive assumptions for the uncertainty structure. Only the vector of time-varying parameters and the vector of the state variables are supposed to be available at each sampling instant to synthesize the

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control signal. Numerical examples show an improvement in the stabilizabilization and in the $\mathcal{H}_\infty$ performance provided by the conditions given in the paper when compared to other similar techniques from the literature.

II. PROBLEM FORMULATION

Consider the discrete-time linear system

$$x(k+1) = A(\alpha(k))x(k) + B_1(\alpha(k))w(k) + B_2(\alpha(k))u(k) \quad (1)$$

$$z(k) = C(\alpha(k))x(k)+D_1(\alpha(k))w(k)+D_2(\alpha(k))u(k) \quad (2)$$

where $x(k) \in \mathbb{R}^n$ is the state, $w(k) \in \mathbb{R}^r$ is an exogenous input, $u(k) \in \mathbb{R}^m$ is the control input and $z(k) \in \mathbb{R}^p$ is the system output. All the system matrices, $A(\alpha(k)) \in \mathbb{R}^{n\times n}$, $B_1(\alpha(k)) \in \mathbb{R}^{n\times r}$, $B_2(\alpha(k)) \in \mathbb{R}^{n\times m}$, $C(\alpha(k)) \in \mathbb{R}^{p\times n}$, $D_1(\alpha(k)) \in \mathbb{R}^{p\times r}$, $D_2(\alpha(k)) \in \mathbb{R}^{p\times m}$ depend on time-varying parameters, belonging to the polytope

$$\mathcal{D} = \Big\{(A, B_1, B_2, C, D_1, D_2)(\alpha(k)) \ : \ (A, B_1, B_2, C, D_1, D_2)(\alpha(k)) = \sum_{j=1}^{N} \alpha_j(k)(A, B_1, B_2, C, D_1, D_2)_j, \ \sum_{j=1}^{N} \alpha_j(k) = 1, \ \alpha_j(k) \geq 0, \ j = 1, \ldots, N\Big\} \quad (3)$$

The vector of parameters $\alpha(k) = [\alpha_1(k) \cdots \alpha_N(k)]'$, $\sum_{j=1}^{N} \alpha_j(k) = 1, \ \alpha_j(k) \geq 0, \ j = 1, \ldots, N$ is supposed to be available in real time (measured or estimated).

Assume that system (1)-(2) is subject to the state feedback control law with a parameter-dependent gain given by

$$u(k) = K(\alpha(k))x(k) \quad , \quad K(\alpha(k)) \in \mathbb{R}^{m\times n} \quad (4)$$

which allows to represent the closed-loop system as

$$x(k + 1) = A_{cl}(\alpha(k))x(k) + B_1(\alpha(k))w(k) \quad (5)$$

$$z(k) = C_{cl}(\alpha(k))x(k) + D_1(\alpha(k))w(k) \quad (6)$$

with

$$A_{cl}(\alpha(k)) = A(\alpha(k)) + B_2(\alpha(k))K(\alpha(k)) \quad (7)$$

$$C_{cl}(\alpha(k)) = C(\alpha(k)) + D_2(\alpha(k))K(\alpha(k)) \quad (8)$$

This paper focuses on the following two problems.

Problem 1: Suppose $w(k) = \mathbf{0}$. Find a parameter-dependent gain $K(\alpha(k))$ such that the closed-loop system

$$x(k + 1) = A_{cl}(\alpha(k))x(k) \quad (9)$$

with the state feedback control law (4) and $A_{cl}(\alpha(k))$ given by (7) is stable for any arbitrary time variation of the parameters $\alpha(k)$ in the polytope $\mathcal{D}$ given by (3).

Problem 2: Suppose $x(0) = \mathbf{0}$. Find a parameter-dependent gain $K(\alpha(k))$ such that the closed-loop system (5)-(8) is stable for any arbitrary time variation of the parameters $\alpha(k)$ in the polytope $\mathcal{D}$ given by (3), and also assuring that, for any input $w(k) \in \ell_2$, the system output $z(k) \in \ell_2$ such that

$$\|z(k)\|_2 < \gamma\|w(k)\|_2 \quad (10)$$

for a finite $\gamma > 0$, called an $\mathcal{H}_\infty$ guaranteed cost for the system.

III. STABILIZABILITY

A convex LMI condition which is sufficient to solve Problem 1 is given by the next theorem.

Theorem 1: If there exist symmetric positive definite matrices $S_j \in \mathbb{R}^{n\times n}$ and matrices $G_j \in \mathbb{R}^{n\times n}$ and $F_j \in \mathbb{R}^{m\times n}$, $j = 1, \ldots, N$ such that the LMIs

$$M_{ij} \triangleq \begin{bmatrix} G_j + G_j' - S_j & G_j'A_j' + F_j'B_{2j}' \\ \star & S_i \end{bmatrix} > 0, \quad i = 1, \ldots, N \ , \ j = 1, \ldots, N \quad (11)$$

$$M_{ijk} \triangleq \begin{bmatrix} G_j + G_j' + G_k + G_k' - S_j - S_k & T_{12} \\ \star & 2S_i \end{bmatrix} > 0, \quad i = 1, \ldots, N \ , \ j = 1, \ldots, N - 1 \ , \ k = j + 1, \ldots, N \quad (12)$$

$$T_{12} \triangleq G_j'A_k' + G_k'A_j' + F_j'B_{2k}' + F_k'B_{2j}' \quad (13)$$

have a solution, then the stability of the closed-loop system (9) is assured by the state feedback control law (4) with the parameter-dependent gain

$$K(\alpha(k)) = F(\alpha(k))G(\alpha(k))^{-1} \quad (14)$$

with

$$(F, G, S)(\alpha(k)) = \sum_{j=1}^{N} \alpha_j(k)(F, G, S)_j \ , \quad \sum_{j=1}^{N} \alpha_j(k) = 1 \ , \ \alpha_j(k) \geq 0 \ , \ j = 1, \ldots, N \quad (15)$$

Proof: Consider the parameter-dependent Lyapunov function

$$v(x(k)) = x(k)'P(\alpha(k))x(k) \quad (16)$$

with

$$P(\alpha(k)) = \sum_{j=1}^{N} \alpha_j(k)P_j \ , \ P_j = P_j' > 0 \ , \ \sum_{j=1}^{N} \alpha_j(k) = 1 \ , \quad \alpha_j(k) \geq 0 \ , \ j = 1, \ldots, N \quad (17)$$

Notice that

$$v(x(k + 1)) = x(k + 1)'P(\alpha(k + 1))x(k + 1) \quad (18)$$

where $P(\alpha(k + 1))$ can be rewritten as

$$P(\beta(k)) = \sum_{i=1}^{N} \beta_i(k)P_i \ , \ P_i = P_i' > 0 \ , \ \sum_{i=1}^{N} \beta_i(k) = 1 \ , \quad \beta_i(k) \geq 0 \ , \ i = 1, \ldots, N \quad (19)$$

Using (9) and $P(\alpha(k+1)) = P(\beta(k))$, one can rewrite (18) as

$$v(x(k+1)) = x(k)' A_{cl}(\alpha(k))' P(\beta(k)) A_{cl}(\alpha(k)) x(k) \tag{20}$$

Taking into account (16) and (20), the difference function $\Delta v(x(k)) \triangleq v(x(k+1)) - v(x(k))$ is given by

$$x(k)'\Big(A_{cl}(\alpha(k))'P(\beta(k))A_{cl}(\alpha(k)) - P(\alpha(k))\Big)x(k) \tag{21}$$

Recall that system (9) is stable if

$$A_{cl}(\alpha(k))'P(\beta(k))A_{cl}(\alpha(k)) - P(\alpha(k)) < 0 \tag{22}$$

or, using Schur complement, if

$$\begin{bmatrix} P(\alpha(k)) & A_{cl}(\alpha(k))'P(\beta(k)) \\ \star & P(\beta(k))) \end{bmatrix} > 0 \tag{23}$$

Multiplying (23) at left and at right by

$$\begin{bmatrix} P(\alpha(k))^{-1} & \mathbf{0} \\ \mathbf{0} & P(\beta(k))^{-1} \end{bmatrix}$$

and taking the change of variables $P(\alpha(k))^{-1} = S(\alpha(k))$ $P(\beta(k))^{-1} = S(\beta(k))$ into account one has

$$\begin{bmatrix} S(\alpha(k)) & S(\alpha(k))A_{cl}(\alpha(k))' \\ \star & S(\beta(k)) \end{bmatrix} > 0 \tag{24}$$

which is equivalent to (23). In addition, expression (24) is equivalent to

$$\begin{bmatrix} G(\alpha(k)) + G(\alpha(k))' - S(\alpha(k)) & J_{12} \\ \star & S(\beta(k)) \end{bmatrix} > 0 \tag{25}$$

with $J_{12} = G(\alpha(k))'A_{cl}(\alpha(k))'$, since if (24) has as solution $S(\alpha(k))$ and $S(\beta(k))$, then (25) is feasible with $G(\alpha(k)) = G(\alpha(k))' = S(\alpha(k))$. Conversely, multiplying (25) at left by $T = \begin{bmatrix} -A_{cl}(\alpha(k)) & \mathbf{I} \end{bmatrix}$ and at right by T', one has $S(\beta(k)) - A(\alpha(k))S(\alpha(k))A(\alpha(k))' > 0$, which is the Schur complement of (24).

Replacing $A_{cl}(\alpha(k))$ by (7) and making the change of variables $F(\alpha(k)) = K(\alpha(k))G(\alpha(k))$, it is possible to rewrite (25) as

$$\begin{bmatrix} G(\alpha(k)) + G(\alpha(k))' - S(\alpha(k)) & L_{12} \\ \star & S(\beta(k)) \end{bmatrix} > 0 \tag{26}$$

with $L_{12} = G(\alpha(k))'A(\alpha(k))' + F(\alpha(k))'B_2(\alpha(k))'$. Using (3), (15) and

$$S(\beta(k)) = \sum_{i=1}^{N} \beta_i(k) S_i \ , \ S_i = S_i' > 0 \ , \ \sum_{i=1}^{N} \beta_i(k) = 1 \ , \quad \beta_i(k) \geq 0 \ , \ i = 1, \ldots, N$$

and taking into account $\sum_{j=1}^{N} \alpha_j(k) = 1$, $\sum_{i=1}^{N} \beta_i(k) = 1$, (26) can be rewritten as

$$\sum_{j=1}^{N} \alpha_j(k)^2 \sum_{i=1}^{N} \beta_i(k) M_{ij} + \sum_{j=1}^{N-1} \sum_{k=j+1}^{N} \alpha_j(k)\alpha_k(k) \sum_{i=1}^{N} \beta_i(k) M_{ijk} > 0 \tag{27}$$

with M_{ij} defined in (11) and M_{ijk} defined in (12)-(13). Notice, finally, that if the conditions of Theorem 1 are feasible one has $M_{ij} > 0$, $M_{ijk} > 0$, which is sufficient to assure (27) for all $\alpha(k)$, $\sum_{j=1}^{N} \alpha_j(k) = 1$, $\alpha_j(k) \geq 0$, $j = 1, \ldots, N$, $\beta(k)$, $\sum_{i=1}^{N} \beta_i(k) = 1$, $\beta_i(k) \geq 0$, $i = 1, \ldots, N$, thus assuring the stability of the closed-loop system for any arbitrary time variation of the parameters in the polytope $\mathcal{D}$. ■

As a first remark, notice that Theorem 1 provides a convex condition with $N + N^2 + N^2(N-1)/2$ LMIs (including $S_j > 0$, $j = 1, \ldots, N$) whose solution allows to obtain a stabilizing gain scheduled controller with no need of discretization of the parametric space neither use of interpolation between controllers locally designed. The parameter-dependent gains is determined analytically by (14)-(15) as a nonlinear function of the parameter vector.

A second important remark is that the conditions of Theorem 1 are particularly useful to deal with systems where the matrices $(A, B_2)(\alpha(k))$ are supposed to be time-varying. Recall that similar conditions in the literature [19] deal with the problem of design of LPV gains using LMIs but only for the special case of $B_2(\alpha(k)) = B_2$, i.e., the case of control matrices fixed and time-invariant. The conditions in [19] cannot cope, for instance, with the problem of actuator failures, modeled as time-varying entries in the control matrix.

A third remark is that the conditions of Theorem 1 contain the stabilizability conditions given in [21] to design via LMIs a robust state feedback stabilizing gain for the system with time-varying matrices $(A, B_2)(\alpha(k))$. The condition from [21] states that if there exist symmetric positive definite matrices $S_j \in \mathbb{R}^{n \times n}$, $j = 1, \ldots, N$, and matrices $G \in \mathbb{R}^{n \times n}$ and $F \in \mathbb{R}^{m \times n}$ such that

$$\begin{bmatrix} G + G' - S_j & G'A_j' + F'B_{2j}' \\ \star & S_i \end{bmatrix} > 0, \quad i = 1, \ldots, N \ , \ j = 1, \ldots, N \tag{28}$$

then the gain $K = FG^{-1}$ stabilizes the closed-loop system. However, there are systems for which (28) fails to provide a feasible solution and the conditions of Theorem 1 provide a parameter-dependent stabilizing gain thanks to the extra variables G_j, F_j, $j = 1, \ldots, N$, as illustrated in the sequel by means of a numerical example.

Finally, as a fourth remark, considering a practical implementation of the gain scheduled controller designed through the conditions of Theorem 1, one has that given *a priori* the system vertices $(A, B)_j$, $j = 1, \ldots, N$, the feasibility of (11)-(12) allows to obtain a set of matrices $(F, G)_j$, $j = 1, \ldots, N$, that can be stored in a memory. Then, based on the values of the vector of parameters (on-line measured or estimated), one can determine $(F, G)(\alpha(k))$ using (15) to obtain the parameter-dependent gain $K(\alpha(k))$ through (14).

IV. $\mathcal{H}_\infty$ CONTROL

A sufficient convex solution to Problem 2 is given by the next theorem.

Theorem 2: If there exist symmetric positive definite matrices $S_j \in \mathbb{R}^{n\times n}$ and matrices $G_j \in \mathbb{R}^{n\times n}$ and $F_j \in \mathbb{R}^{m\times n}$, $j = 1, \ldots, N$ such that the optimization problem

$$\min \quad \mu$$

subject to

$$N_{ij} \triangleq \begin{bmatrix} G_j + G_j' - S_j & \mathbf{0} & Q_{13} & Q_{14} \\ \star & \mathbf{I} & B_{1j}' & D_{1j}' \\ \star & \star & S_i & \mathbf{0} \\ \star & \star & \star & \mu\mathbf{I} \end{bmatrix} > 0, \tag{29}$$

$$i = 1, \ldots, N \ , \ j = 1, \ldots, N$$

$$Q_{13} \triangleq G_j' A_j' + F_j' B_{2j}'$$

$$Q_{14} \triangleq G_j' C_j' + F_j' D_{2j}'$$

$$N_{ijk} \triangleq \begin{bmatrix} R_{11} & \mathbf{0} & R_{13} & R_{14} \\ \star & \mathbf{I} & B_{1j}' + B_{1k}' & D_{1j}' + D_{1k}' \\ \star & \star & 2S_i & \mathbf{0} \\ \star & \star & \star & 2\mu\mathbf{I} \end{bmatrix} > 0,$$

$$i = 1, \ldots, N \ , \ j = 1, \ldots, N-1 \ , \ k = j+1, \ldots, N \tag{30}$$

$$R_{11} \triangleq G_j + G_j' + G_k + G_k' - S_j - S_k$$

$$R_{13} \triangleq G_j' A_k' + G_k' A_j' + F_j' B_{2k}' + F_k' B_{2j}'$$

$$R_{14} \triangleq G_j' C_k' + G_k' C_j' + F_j' D_{2k}' + F_k' D_{2j}'$$

has a solution, then the stability of the closed-loop system (5)-(8) with an $\mathcal{H}_\infty$ guaranteed cost given by

$$\gamma = \sqrt{\mu^\star} \quad , \quad \mu^\star = \min \mu \tag{31}$$

is assured by the state feedback control law (4) with the parameter-dependent gain (14)-(15) for any arbitrary time variation of the parameters $\alpha(k)$ in the polytope $\mathcal{D}$.

Proof: The stability of the system with an $\mathcal{H}_\infty$ guaranteed cost is assured if

$$\Delta v(x(k)) + z(k)' z(k) - \mu \ w(k)' w(k) < 0 \tag{32}$$

Replacing $\Delta v(x(k))$ and $z(k)$ by expressions (21) and (6) respectively, inequality (32) can be rewritten as $\theta(k)' U \theta(k) < 0$ with $\theta(k)' = \begin{bmatrix} x(k)' & w(k)' \end{bmatrix}$ and

$$U = \begin{bmatrix} U_{11} & U_{12} \\ \star & U_{22} \end{bmatrix} \tag{33}$$

$$U_{11} = A_{cl}(\alpha(k))' P(\beta(k)) A_{cl}(\alpha(k)) - P(\alpha(k)) + C_{cl}(\alpha(k))' C_{cl}(\alpha(k))$$

$$U_{12} = A_{cl}(\alpha(k))' P(\beta(k)) B_1(\alpha(k)) + C_{cl}(\alpha(k))' D_1(\alpha(k))$$

$$U_{22} = B_1(\alpha(k))' P(\beta(k)) B_1(\alpha(k)) + D_1(\alpha(k))' D_1(\alpha(k)) - \mu\mathbf{I}$$

Notice that imposing $U < 0$ in (33), one guarantees that (32) holds for all $x(k) \neq \mathbf{0}$, $w(k) \neq \mathbf{0}$. Using Schur complement, (33) is equivalent to

$$\begin{bmatrix} P(\alpha(k)) & V_{12} & \mathbf{0} & C_{cl}(\alpha(k))' \\ \star & P(\beta(k)) & V_{23} & \mathbf{0} \\ \star & \star & \mathbf{I} & D_1(\alpha(k))' \\ \star & \star & \star & \mu\mathbf{I} \end{bmatrix} > 0 \tag{34}$$

$$V_{12} = A_{cl}(\alpha(k))' P(\beta(k))$$

$$V_{23} = P(\beta(k)) B_1(\alpha(k))$$

Following steps similar to those in the proof of Theorem 1, one can find that (34) is equivalent to

$$\begin{bmatrix} X_{11} & \mathbf{0} & G(\alpha(k))' A_{cl}(\alpha(k))' & X_{14} \\ \star & \mathbf{I} & B_1(\alpha(k))' & D_1(\alpha(k))' \\ \star & \star & S(\beta(k)) & \mathbf{0} \\ \star & \star & \star & \mu\mathbf{I} \end{bmatrix} > 0 \tag{35}$$

$$X_{11} = G(\alpha(k)) + G(\alpha(k))' - S(\alpha(k))$$

$$X_{14} = G(\alpha(k))' C_{cl}(\alpha(k))'$$

Under the definitions of the matrices given previously and using the problem constraints $\sum_{j=1}^N \alpha_j(k) = 1$, $\sum_{i=1}^N \beta_i(k) = 1$, (35) can be rewritten as

$$\sum_{j=1}^N \alpha_j(k)^2 \sum_{i=1}^N \beta_i(k) N_{ij} + \sum_{j=1}^{N-1} \sum_{k=j+1}^N \alpha_j(k)\alpha_k(k) \sum_{i=1}^N \beta_i(k) N_{ijk} > 0 \tag{36}$$

with N_{ij} defined in (29) and N_{ijk} defined in (30). Finally, if the conditions of Theorem 2 hold, one has that $N_{ij} > 0$, $N_{ijk} > 0$, which is sufficient to assure (35) for all $\alpha(k)$, $\sum_{j=1}^N \alpha_j(k) = 1$, $\alpha_j(k) \geq 0$, $j = 1, \ldots, N$, $\beta(k)$, $\sum_{i=1}^N \beta_i(k) = 1$, $\beta_i(k) \geq 0$, $i = 1, \ldots, N$, thus assuring the stability with an $\mathcal{H}_\infty$ guaranteed cost for the closed-loop system under any arbitrary time variation of the parameters in the polytope $\mathcal{D}$. ■

Some remarks are now in order. First, it is interesting to recall that the problem of design via LMIs of LPV controllers to the class of systems under investigation here was addressed with a similar methodology in [20], but also only for the case where matrices $(B_2, D_2)(\alpha(k))$ are considered as fixed and time-invariant. Theorem 2 deals with a more general case, where these matrices are supposed to be arbitrarily time-varying. Second remark, the extension of the robust stabilizability condition (28) to cope with $\mathcal{H}_\infty$ performance is straightforward: if there exist symmetric positive definite matrices $S_j \in \mathbb{R}^{n\times n}$, $j = 1, \ldots, N$, and matrices $G \in \mathbb{R}^{n\times n}$ and $F \in \mathbb{R}^{m\times n}$ such that $\min \mu$ subject to (29) with $G_j = G$, $F_j = F$ has a solution, then $K = FG^{-1}$ assures the closed-loop stability with γ given by (31). It is interesting to mention that there are systems for which such robust gain cannot be determined or, if one can obtain the fixed gain, it may lead to a poor performance (high values of γ) compared to those obtained with parameter-dependent gains given by Theorem 2, as illustrated by the

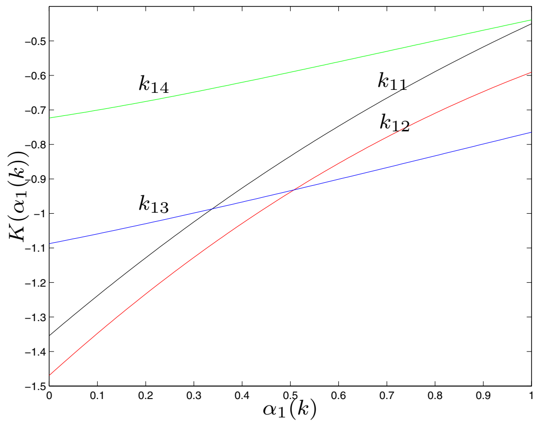

Fig. 1. Entries of the parameter-dependent stabilizing gain (39) obtained from Theorem 1 as a function of $\alpha_1(k)$.

second example in the sequel. Notice also that the first and the fourth remarks of Theorem 1 are valid for Theorem 2.

V. NUMERICAL EXAMPLES

Example 1: Consider a system with $N = 2$ vertices randomly generated given by

$$A_1 = \begin{bmatrix} 0.3158 & 0.2261 & 0.4781 & 0.4588 \\ 0.0473 & 0.6081 & 0.2509 & 0.2790 \\ 0.1581 & 0.4883 & 0.9031 & 0.6497 \\ 0.7402 & 0.4455 & 0.8582 & 0.1879 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0.7926 & 0.9792 & 0.4006 & 0.7986 \\ 0.4810 & 0.1183 & 0.1389 & 0.2469 \\ 0.7169 & 0.8413 & 0.6237 & 0.4428 \\ 0.7596 & 0.8886 & 0.8093 & 0.2588 \end{bmatrix} \quad (37)$$

$$B_{21} = \begin{bmatrix} 0.6594 \\ 0.9802 \\ 0.2648 \\ 0.9155 \end{bmatrix}, \quad B_{22} = \begin{bmatrix} 0.0816 \\ 0.1295 \\ 0.5339 \\ 0.7519 \end{bmatrix} \quad (38)$$

and the problem of stabilizability (Problem 1). The conditions from [19] to design LPV stabilizing gains cannot be applied here since $B_{21} \neq B_{22}$, implying in a time-varying control matrix $B_2(\alpha(k))$. Moreover, condition (28) fails in providing a stabilizing robust state feedback control gain to this system. On the other hand, Theorem 1 allows to determine a stabilizing parameter-dependent gain in the form

$$K(\alpha_1(k)) = \begin{bmatrix} k_{11}(\alpha_1(k)) \\ k_{12}(\alpha_1(k)) \\ k_{13}(\alpha_1(k)) \\ k_{14}(\alpha_1(k)) \end{bmatrix}' \quad (39)$$

whose entries are shown in Fig. 1 as a function of $\alpha_1(k)$, with $\alpha_2(k) = 1 - \alpha_1(k)$.

Observe the clear nonlinear behavior of entries $k_{11}(\alpha_1(k))$ and $k_{12}(\alpha_1(k))$. In the case of practical implementation of the gain scheduled controller, matrices F_j and G_j, $j = 1, 2$, calculated *a priori* using the conditions of Theorem 1, can be stored in a memory. From the knowledge of $\alpha_1(k)$ (supposed available in real time), one evaluates $F(\alpha_1(k))$ and $G(\alpha_1(k))$ using (15) and then the control gain $K(\alpha_1(k))$ is obtained from (14). Notice that there is no use of gridding in the parametric space to obtain the gain scheduled controller. The closed-loop stability is assured by the parameter-dependent Lyapunov matrix $P(\alpha_1(k)) = S(\alpha_1(k))^{-1}$, where

$$S(\alpha_1(k)) = \alpha_1(k)S_1 + (1 - \alpha_1(k))S_2 \quad (40)$$

with

$$S_1 = \begin{bmatrix} 0.3650 & -0.0192 & -0.0296 & -0.0265 \\ -0.0192 & 0.3564 & -0.0660 & -0.0612 \\ -0.0296 & -0.0660 & 0.3139 & -0.0529 \\ -0.0265 & -0.0612 & -0.0529 & 0.3139 \end{bmatrix} \quad (41)$$

$$S_2 = \begin{bmatrix} 0.3481 & -0.0531 & -0.0204 & -0.0782 \\ -0.0531 & 0.3399 & -0.0677 & -0.0901 \\ -0.0204 & -0.0677 & 0.3640 & -0.0414 \\ -0.0782 & -0.0901 & -0.0414 & 0.3031 \end{bmatrix} \quad (42)$$

obtained from Theorem 1.

Example 2: Consider a system with vertices

$$A_1 = \begin{bmatrix} 0.28 & -0.315 \\ 0.63 & -0.84 \end{bmatrix}, \quad B_{11} = B_{21} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad (43)$$

$$C_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad D_{11} = D_{21} = \begin{bmatrix} 0 \end{bmatrix} \quad (44)$$

$$A_2 = \begin{bmatrix} 0.52 & 0.77 \\ -0.7 & -0.07 \end{bmatrix}, \quad B_{12} = B_{11}, \quad B_{22} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad (45)$$

$$C_2 = C_1, \quad D_{12} = D_{22} = \begin{bmatrix} 0 \end{bmatrix} \quad (46)$$

and the problem of $\mathcal{H}_\infty$ control (Problem 2). This system was studied in [20] (second example), in the special case where $B_{21} = B_{22}$, implying that $B(\alpha(k))$ is fixed and time-invariant. The aim here is to investigate the more general case $B_{21} \neq B_{22}$. Using the extension of condition (28) to cope with $\mathcal{H}_\infty$ stabilizability, as discussed in the second remark given after the proof of Theorem 2, one can find this system is stabilizable through the fixed state feedback gain

$$K = \begin{bmatrix} 0.2784 & 0.6364 \end{bmatrix} \quad (47)$$

with an $\mathcal{H}_\infty$ guaranteed cost given by $\gamma_K = 6.4938$. It is possible to improve the system performance using the parameter-dependent control strategy, provided by Theorem 2, yielding the solution

$$F_1 = \begin{bmatrix} -2.3079 & 3.3073 \end{bmatrix}, \quad F_2 = \begin{bmatrix} 1.4653 & -1.8766 \end{bmatrix} \quad (48)$$

$$G_1 = \begin{bmatrix} 5.0467 & -0.6525 \\ -1.2692 & 7.4506 \end{bmatrix}, \quad G_2 = \begin{bmatrix} 7.6666 & -4.1665 \\ -0.0571 & 7.6578 \end{bmatrix} \quad (49)$$

$$S_1 = \begin{bmatrix} 6.9920 & 1.4329 \\ 1.4329 & 9.8275 \end{bmatrix}, \quad S_2 = \begin{bmatrix} 6.8224 & -4.2757 \\ -4.2757 & 10.6498 \end{bmatrix} \quad (50)$$

which allows to compute a gain scheduled controller that stabilizes the closed-loop system with an $\mathcal{H}_\infty$ guaranteed cost given by $\gamma_{T2} = 4.9187$, which represents a reduction of 24.3% in the value of γ_K, providing better rejection of disturbances.

VI. CONCLUSION

This paper has presented convex LMI conditions to design gain scheduled controllers suitable to stabilize and to assure $\mathcal{H}_\infty$ performance to discrete-time linear systems which depend on arbitrarily time-varying parameters in a polytope. Differently from previous approaches, there are no restrictive assumptions on the uncertainty structure and all the system matrices are supposed affected by time-varying parameters. The proposed design conditions are written as a finite set of LMIs at the vertices of the polytope, which avoids gridding the parametric space and the parameter-dependent gains here are obtained through an analytical expression, without interpolations. The closed-loop stability and performance are assured by a parameter-dependent Lyapunov function, obtained from the solution of the proposed LMIs. Numerical examples have shown how the conditions given in the paper allow to improve the stabilizability and the $\mathcal{H}_\infty$ performance compared to similar conditions from the literature.

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