

Pre-Filter Design for Tracking Error Specifications in MIMO-QFT

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Abstract— This paper presents a novel approach to solve the MIMO-QFT problem for tracking error specification through a method of obtaining exact bounds for the design of individual elements of pre-filter. The paper specifically deals with the appropriate transformation of the MIMO system to the equivalent SISO problems, which allows easy design to find the feedback compensator and pre-filter. A linearized model of quadruple-tank process is used to show the effectiveness of the proposed method.

I. INTRODUCTION

QUANTITATIVE feedback theory (QFT) is a two-degree-of-freedom (2DOF) feedback control design method for uncertain plants as shown in Fig. 1. The general problem in QFT is usually how to design the feedback compensator and pre-filter such that, the desired specifications are satisfied in the region of uncertainty. The feedback compensator, $G(s)$, reduces the effects of uncertainties and the pre-filter, $F(s)$, shifts the response to the desired region.

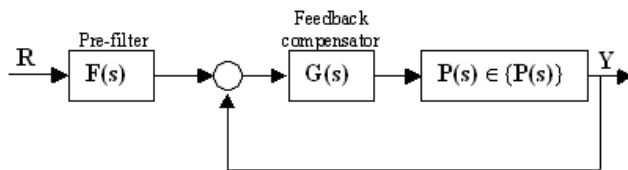


Fig. 1. Two-degree-of-freedom control structure.

In most QFT problems the desired specifications are assumed to be only on the magnitude of the closed loop transfer functions termed as *conventional* approaches in this paper. These specifications put the responses of the controlled system between two lower and upper bounds, [5]-[7]. The pre-filter is simply designed in the conventional QFT approach, particularly when the MATLAB[®] QFT-Toolbox, [6] is used.

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A fewer works employ tracking error specifications as desired specifications, [1]-[2], [4]-[5] and [8]. Constraining the magnitude of the difference between the closed loop transfer function and reference model to lie within a disk can make a better engineering sense, [4]. The idea of tracking error specifications has been proposed by Yaniv and Chait [8] for Multi-Input, Multi-Output (MIMO) uncertain systems. By selecting the reference model as pre-filter, they lost one degree of freedom. In [4], the QFT controllers were designed such that the closed loop transfer function is bounded within a disk around some nominal performance with 2DOF control system. In [4], in order to design the pre-filter, a constraint was introduced for each column of the pre-filter matrix, and then a non-linear search was used to find the individual elements of pre-filter matrix, i.e., $f_{ij}(s)$. The obtained pre-filter, may be complex and of high order. In [2], a novel methodology through the concepts of model-matching problem and unstructured uncertainty has been provided which leads to simple and low order controllers. It may be over-design as a result of a poor choice of the nominal plant in [2] and [4]. Since the over-bounding is used for pre-filter design in both of proposed methods, [2] and [4] lead to conservative solutions.

Hence, it should be mentioned *the optimum pre-filter design for tracking error specifications as simple as the conventional methods is a new subject in QFT.*

This paper presents a novel approach to solve MIMO-QFT problem for tracking error specification through a simple design procedure. The 2DOF structure of the QFT design is preserved here. Using an appropriate transformation of the MIMO system to the equivalent Single-Input, Single-Output (SISO) problems, a method of obtaining exact bounds for the design of individual elements of pre-filter is introduced. Then the individual elements of the pre-filter matrix are separately designed using appropriately generated bounds based on the proposed method in [5] for SISO-QFT problems. The generated bounds can be easily plotted in log polar complex plane by MATLAB[®] QFT-Toolbox. The choice of nominal plant is arbitrary in the proposed method such as conventional QFT design. Since the over-bounding is

not used for pre-filter design in the new method, conservatism will also be reduced in comparison with the existing methods.

The paper is organized as follows: In section II the problem is formulated. Section III presents the feedback compensator design. The pre-filter design is discussed in section IV and finally in section V, the linearized quadruple-tank process is used to show the effectiveness of the proposed method.

II. PROBLEM FORMULATION

Consider the 2DOF feedback control system with equal numbers in inputs and outputs, shown in Fig. 1. The real tracking error specification $\mathbf{E}(s)$ is defined as the magnitude of difference between the reference model, $\mathbf{M}(s)$ and the transfer matrix of the controlled system, $\mathbf{T}_m(s)$ as given by equation (1).

$$\begin{aligned} \mathbf{E}(s) &= \mathbf{M}(s) - \mathbf{T}_m(s) \\ \mathbf{E}(s) &= [e_{ij}(s)], \mathbf{M}(s) = [m_{ij}(s)], \mathbf{T}_m(s) = [t_{mij}(s)] \end{aligned} \quad (1)$$

The target is to design $\mathbf{G}(s)$ and $\mathbf{F}(s)$, to meet the desired tracking error specification, $\mathbf{E}_d(s)$, given by equation (2) for all plants in the region of uncertainty.

$$|\mathbf{M}(j\omega) - \mathbf{T}(j\omega)\mathbf{F}(j\omega)|_{ij} \leq |\mathbf{E}_d(j\omega)|_{ij} \quad (2)$$

$$\mathbf{E}_d(s) = [e_{dij}(s)]$$

$\mathbf{T}(s)$ represents the complementary matrix transfer function of the controlled system. Capital-Bold notations have been used for matrix transfer functions in this paper.

III. FEEDBACK COMPENSATOR DESIGN

In this section, first the MIMO problem is transformed to equivalent SISO problems. Then, an appropriate disturbance rejection model is provided to achieve desired feedback compensator for each equivalent SISO problem.

With $\mathbf{G}(s) = \text{diag}\{g_{ii}(s)\}$, $\mathbf{F}(s) = [f_{ij}(s)]$, it can be shown that the closed loop transfer matrix, $\mathbf{T}_m(s)$ can be decomposed as follows, [7]:

$$\mathbf{T}_m(s) = \boldsymbol{\alpha}^{-1}(s) \hat{\mathbf{T}}(s) \quad (3)$$

where

$$\boldsymbol{\alpha}(s) = \begin{bmatrix} 1 & -\alpha_{12}(s) & \cdots & -\alpha_{1N}(s) \\ -\alpha_{21}(s) & 1 & \cdots & -\alpha_{2N}(s) \\ \vdots & \vdots & \ddots & \vdots \\ -\alpha_{N1}(s) & -\alpha_{N2}(s) & \cdots & 1 \end{bmatrix}$$

$$\hat{\mathbf{T}}(s) = \begin{bmatrix} \frac{l_1(s)f_{11}(s)}{1+l_1(s)} & \frac{l_1(s)f_{12}(s)}{1+l_1(s)} & \cdots & \frac{l_1(s)f_{1N}(s)}{1+l_1(s)} \\ \frac{l_2(s)f_{21}(s)}{1+l_2(s)} & \frac{l_2(s)f_{22}(s)}{1+l_2(s)} & \cdots & \frac{l_2(s)f_{2N}(s)}{1+l_2(s)} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{l_N(s)f_{N1}(s)}{1+l_N(s)} & \frac{l_N(s)f_{N2}(s)}{1+l_N(s)} & \cdots & \frac{l_N(s)f_{NN}(s)}{1+l_N(s)} \end{bmatrix}$$

$$\alpha_{ij}(s) = -\frac{\left(\frac{q_{ii}(s)}{q_{ij}(s)}\right)}{1+q_{ii}(s)g_{ii}(s)}, l_i(s) = q_{ii}(s)g_{ii}(s), \mathbf{P}^{-1}(s) = \left[\frac{1}{q_{ij}(s)}\right]$$

N is the number of equivalent SISO subsystems.

By substituting equation (3) into equation (1), it can be shown that:

$$e_{ij}(s) = \sum_{k=1, \neq i}^N \alpha_{ik}(s)e_{kj}(s) - \sum_{k=1, \neq i}^N \alpha_{ik}(s)m_{kj}(s) + m_{ij}(s) - t_{ij}(s) \quad (4)$$

Using the Schwartz inequality on the right-hand side of equation (4) we have:

$$\begin{aligned} |e_{ij}(j\omega)| &= \left| \sum_{k=1, \neq i}^N \alpha_{ik}(s)(e_{kj}(s) - m_{kj}(s)) + m_{ij}(s) - t_{ij}(s) \right|_{s=j\omega} \\ &\leq \left| \sum_{k=1, \neq i}^N \alpha_{ik}(s)(e_{kj}(s) - m_{kj}(s)) \right|_{s=j\omega} + \left| m_{ij}(s) - t_{ij}(s) \right|_{s=j\omega} \end{aligned} \quad (5)$$

Defining $S_i(s) = 1/(1+l_i(s))$ as the sensitivity function of i th equivalent SISO subsystem and substituting it into $\alpha_{ij}(s)$, we have $\alpha_{ij}(s) = -S_i(s) \frac{q_{ii}(s)}{q_{ij}(s)}$. Hence

equation (5) can be rewritten as:

$$\begin{aligned} |e_{ij}(j\omega)| &\leq |S_i(j\omega)| \sum_{k=1, \neq i}^N \left| \frac{q_{ii}(j\omega)}{q_{ik}(j\omega)} \right| (|e_{kj}(j\omega)| + |m_{kj}(j\omega)|) \\ &\quad + |m_{ij}(j\omega) - T_{ii}(j\omega)f_{ij}(j\omega)| \end{aligned} \quad (6)$$

where $T_{ii}(s)$ is the complementary sensitivity function of each equivalent SISO subsystem given by:

$$T_{ii}(s) = l_i(s)/(1+l_i(s)).$$

As in standard QFT, the design specifications are used to over-bound any unknown functions on the right-hand side of equation (6).

Using the notations of $M_{ij}(\omega) := |m_{ij}(j\omega)|$ and

$E_{dij}(\omega) := |e_{dij}(j\omega)|$, we have:

$$|m_{ij}(j\omega) - T_{ii}(j\omega)f_{ij}(j\omega)| \leq |\gamma_{ij}(j\omega)| \quad (7)$$

where

$$\begin{aligned} |\gamma_{ij}(j\omega)| &= E_{d_{ij}}(\omega) \\ &- |S_i(j\omega)| \sum_{k=1, \neq i}^N \left| \frac{q_{ii}(j\omega)}{q_{ik}(j\omega)} \right|_{\max} (E_{d_{kj}}(\omega) + M_{kj}(\omega)) \end{aligned}$$

Equation (7) describes the MIMO problems in the form of SISO problem, which has been used in [1]. It specially deals with a method of obtaining the exact bounds for the design of individual elements of pre-filter, *the main objective of this paper*.

The right hand side of equation (7) should be positive; therefore a constraint on sensitivity function will be obtained as the cost paid for transformation of the MIMO problem to the equivalent SISO problem given by:

$$|S_i(j\omega)| \leq \min_{j=1, \dots, N} \left\{ \frac{E_{d_{ij}}(\omega)}{\sum_{k=1, \neq i}^N \left| \frac{q_{ii}}{q_{ik}} \right|_{\max} (E_{d_{kj}}(\omega) + M_{kj}(\omega))} \right\} \quad (8)$$

In equation (8), in order to attenuate the interactions between the subsystems, the worst case of

$$\left\{ \frac{E_{d_{ij}}(\omega)}{\sum_{k=1, \neq i}^N \left| \frac{q_{ii}}{q_{ik}} \right|_{\max} (E_{d_{kj}}(\omega) + M_{kj}(\omega))} \right\} \text{ is considered.}$$

The design approach for the output disturbance rejection problem is based upon the performance specification that the disturbance has no effect on the steady state output. There are some methods to minimise, the effect of a disturbance input on the output of a control system, such that:

$$|S_i| = \left| \frac{Y_i(j\omega)}{D_i(j\omega)} \right| < |T_{D_i}(j\omega)| \quad (9)$$

where $T_{D_i}(s)$ is the output disturbance rejection model.

Considering equations (8) and (9), it is sufficient to select the output disturbance rejection model for each equivalent subsystem as given by:

$$|T_{D_i}| \leq \min_{j=1, \dots, N} \left\{ \frac{E_{d_{ij}}(\omega)}{\sum_{k=1, \neq i}^N \left| \frac{q_{ii}}{q_{ik}} \right|_{\max} (E_{d_{kj}}(\omega) + M_{kj}(\omega))} \right\} \quad (10)$$

where $D_i(s)$ is the output disturbance entering to the i th subsystem as shown in Fig. 2.

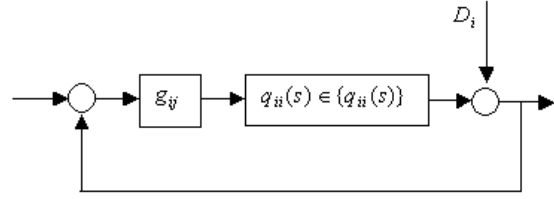


Fig. 2. Control Structure of i th equivalent SISO problem.

Using MATLAB[®] QFT-Toolbox and based on equation (10), the permitted bounds on the loop function $l_i(s) = q_{ii}(s)g_{ii}(s)$ can be generated to design the desired feedback compensator for each equivalent SISO problem. Equation (10) guarantees the robust performance. In order to have robust stability, it is sufficient to design the feedback compensator for each equivalent SISO problem such that the loop function does not intersect the critical point $(-180^\circ, 0dB)$ and does not enter into stability U-Contours, [6].

IV. PRE-FILTER DESIGN

In the pervious section it has been shown that how the MIMO problem is transformed to the equivalent SISO problems given by equation (7). Appropriate bounds have also been proposed for feedback compensator design. In this section, pre-filter is designed according to the method proposed in [5].

In [5], the pre-filter $f(s)$, has been designed for SISO-QFT problems such that:

$$|M(j\omega) - T(j\omega)f(j\omega)| \leq |E_d(j\omega)|, \quad (11)$$

where $M(s)$, $T(s)$ and $E_d(s)$ are reference model, complementary sensitivity function and desired error respectively and are known.

By comparing equations (7) and (11), it is obvious that using the novel transformation proposed in this paper, the pre-filter design approach [5] is directly applicable to MIMO-QFT problems.

Using $m_{ij} = ye^{j\beta}$, $T_{ii} = xe^{j\alpha}$ and $f_{ij} = re^{j\phi}$, equation (7) is transformed to the polar coordinates as

$$\begin{aligned} ((y \cos(\beta) - xr \cos(\phi + \alpha))^2 + \\ ((y \sin(\beta) - xr \sin(\phi + \alpha))^2 \leq \gamma_{ij}^2 \end{aligned} \quad (12)$$

In equation (12) m_{ij} , T_{ii} and γ_{ij} are known and f_{ij} is unknown. The solution of (12) for r , for a given plant case and design frequency, and over $\phi \in [-360, 0]$ will divide the complex plane of $f_{ij}(s)$ into acceptable and

unacceptable regions. The intersection of the regions provides an exact bound for the design of $f_{ij}(s)$.

Equation (7), which directly yields design bounds for pre-filter design, can be compared to the proposed approach [4]. In [4] the pre-filter is designed by the following equation.

$$\mathbf{F}(s) = \mathbf{F}_o(s) + \delta\mathbf{F}(s) \quad (13)$$

where $\mathbf{F}_o(s) = \mathbf{T}_o^{-1}(s)\mathbf{M}(s)$. $\mathbf{F}_o(s)$ and $\mathbf{T}_o(s)$ are pre-filter and complementary sensitivity function for nominal plant respectively. $\mathbf{M}(s)$ represents the reference model. By computing the magnitude of error and using over-bound specifications, a lower bound on magnitude of the elements of the j th columns of $\delta\mathbf{F}(s)$, $|\delta f_{kj}| \leq |\delta f_j^*|$ has been obtained where:

$$|\delta f_j^*(j\omega)| \leq \min_i \left\{ (E_{dij} - |\mathbf{M} - \mathbf{TF}_o|_{ij}) \min_{P \in \{P\}} \left\{ \frac{1}{\sum_{k=1}^N |t_{mik}|} \right\} \right\} \quad (14)$$

A nonlinear search (starting from $\mathbf{F}_o(s)$), is used to find the individual elements of $\mathbf{F}(s)$, which causes difficulties without necessarily improving the designed pre-filter. It may also result in a pre-filter unnecessarily high order. As opposed to conventional QFT methods, there may also be over-design as a result of a poor choice of nominal plant in [4]. While the advantages of the new approach are as follows:

- 1- The individual elements of the pre-filter matrix are separately designed using the individual generated bounds given by equation (12). The generated bounds can be easily plotted in log polar complex plane by MATLAB[®] QFT-Toolbox.
- 2- The choice of nominal plant is also arbitrary such as conventional QFT design, which leads to minimum over-design in comparison with the proposed method [4].
- 3- Because of avoiding the nonlinear search, the new methodology is easier. The designed pre-filter will also be lower order.
- 4- Since the over-bounding is not used for pre-filter design in the new method, conservatism will be reduced in comparison with the existing method [4].

V. ILLUSTRATIVE EXAMPLE (QUADRUPLE-TANK PROCESS)

Quadruple-Tank process is used to illustrate many concepts in multivariable control; particularly performance limitations due to right half plane zeros. Here the linearized model is used to show the effectiveness of the novel approach. The process is

consisted of four interconnected water tanks and two pumps shown in Fig. 3.

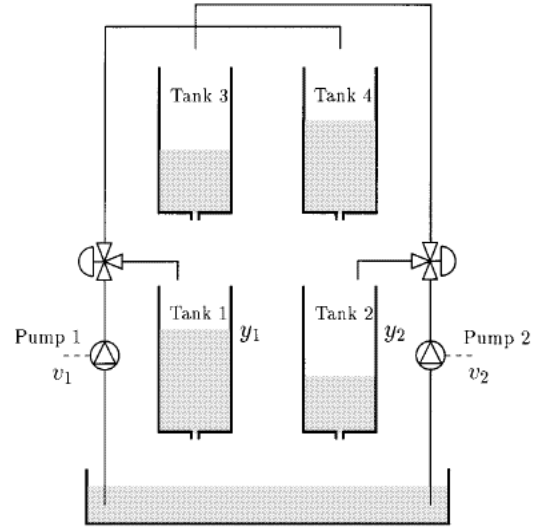


Fig. 3. Quadruple-tank process.

Its inputs are the voltages to the two pumps and the outputs are the water level in the lower two tanks. The plant includes a zero that can be placed in both sides of the imaginary axis by changing a valve. A linearized transfer function matrix has been modeled as follows in [3]:

$$\mathbf{P}(s) = \begin{bmatrix} \frac{\gamma_1 \alpha_1}{1 + sT_1} & \frac{(1 - \gamma_2) \alpha_1}{(1 + sT_1)(1 + sT_3)} \\ \frac{(1 - \gamma_1) \alpha_2}{(1 + sT_2)(1 + sT_4)} & \frac{\gamma_2 \alpha_2}{1 + sT_2} \end{bmatrix}$$

The parameters $\gamma_1, \gamma_2 \in (0,1)$ are determined from how the valves are set prior to experiment. The flow to tank 1 is proportional to γ_1 and the flow to tank 4 is proportional to $1 - \gamma_1$ and similarly for γ_2 with respect to tank 2 and tank 3. It is shown that the system is non-minimum phase for $0 < \gamma_1 + \gamma_2 < 1$ and minimum phase for $1 < \gamma_1 + \gamma_2 < 2$.

We restrict the discussion to $T_1 = \dots = T_4 = 1$, $\alpha_1 = \alpha_2 = 1$ and $\gamma_1 = \gamma_2 = \gamma$ for $0 < \gamma_1 + \gamma_2 < 1$. The nominal plant has been chosen at $\gamma = 1/3$ similar to the one used in [3].

The reference model and desired error specifications are also selected as:

$$\mathbf{M}(s) = \begin{bmatrix} \frac{1}{(s/0.2+1)} & 0 \\ 0 & \frac{1}{(s/0.2+1)} \end{bmatrix}, \quad E_{dij} = \frac{0.1(s/0.01+1)}{(s/0.1+1)}, \quad i, j = 1, 2$$

Since plant is highly coupled, a static pre-compensator is employed to achieve diagonal dominance at low frequencies. By $\mathbf{P}^{-1}(s)|_{s=0}$ and $\gamma = 1/3$, the static pre-compensator is obtained as $\begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix}$. Then the compensated plant used for QFT design is given by:

$$\mathbf{P}^{\wedge}(s) = \begin{bmatrix} \frac{-(\gamma s + 3\gamma - 2)}{(s+1)^2} & \frac{2\gamma s + 3\gamma - 1}{(s+1)^2} \\ \frac{2\gamma s + 3\gamma - 1}{(s+1)^2} & \frac{-(\gamma s + 3\gamma - 2)}{(s+1)^2} \end{bmatrix}$$

Using equation (10), the output disturbance rejection model within the performance bandwidth $\omega \in [0, 0.2]$ for each isolated subsystem is selected by:

$$T_{Di}(s) = \frac{19s(s/0.1+1)}{(s/0.04+1)(s/0.13+1)} \quad i=1,2$$

Fig. 4 shows the selected output disturbance rejection model satisfies equation (10). In Fig. 4, to avoid conservatism in control design, the output disturbance rejection model rises a little upper than the sensitivity constraint over lower frequencies.

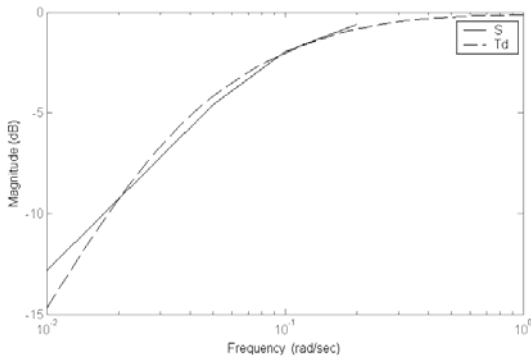


Fig. 4. Output disturbance rejection model.

Using the MATLAB[®] QFT-Toolbox, the related design bounds are generated for each equivalent subsystem. At higher frequencies robust stability bound has been considered using the constraint on complementary sensitivity matrix as $|T_{ii}(s)| \leq 1.5$. The composite bounds (robust performance and robust stability bounds) have been illustrated in Fig. 5. Feedback controller should be designed so that loop function, $l_i(s)$, lies above the design bounds and does not enter the U-contours. Fig. 5 shows that the following controller satisfies the design bounds:

$$g_{ii}(s) = \frac{55(s/0.5+1)}{(s/0.01+1)(s/20+1)} \quad i=1,2$$

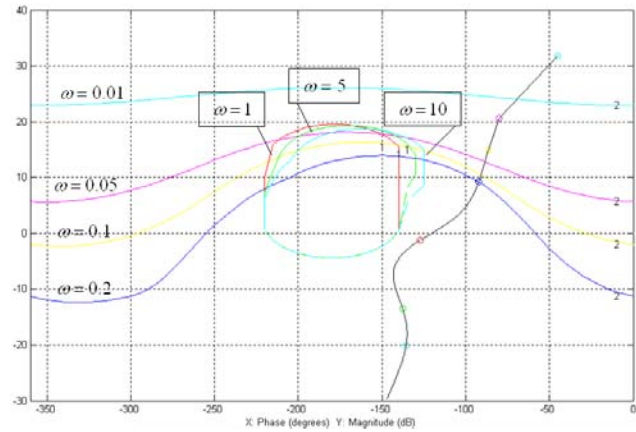


Fig. 5. Feedback compensator design bounds, $\omega = \{0.01, 0.05, 0.1, 0.2\}$ are related to robust performance bounds, $\omega = \{1, 5, 10\}$ are related to robust stability bounds.

It is obvious that by selecting $f_{ij}(s) = 0$, equation (7) is satisfied for off-diagonal elements (i.e., $i \neq j$). Equation (12) is used to generate design bounds of $f_{ii}(s)$. The solution of (12) divides the complex plane into acceptable and unacceptable regions at each frequency as shown in Fig. 6. $f_{ii}(s)$ should be designed to lie into the accepted bounds at each frequency. Fig. 6 shows that by $f_{ii}(s) = 1/(s/0.2+1)$, the design bounds are satisfied by a simple transfer function.

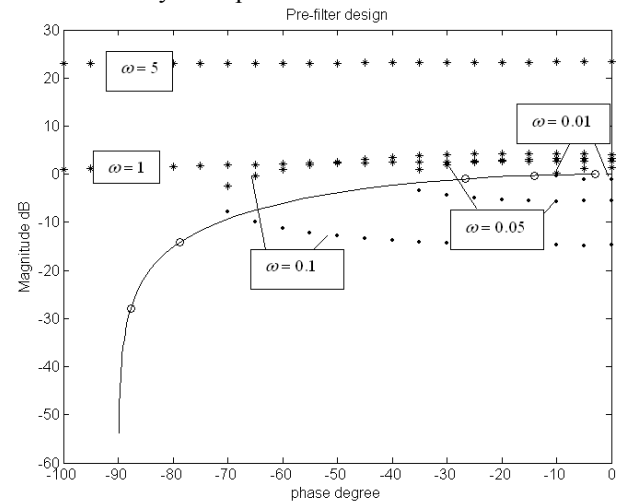


Fig. 6. Bounds for pre-filter design in complex plane, Upper bounds (star) and Lower bounds (dot).

Fig. 7 shows that the desired error specifications are satisfied for several plant cases in the uncertain region. Corresponding unit step response and control effort are also depicted in Fig.'s 8 and 9, respectively. They show appropriate signals to achieve desired performance.

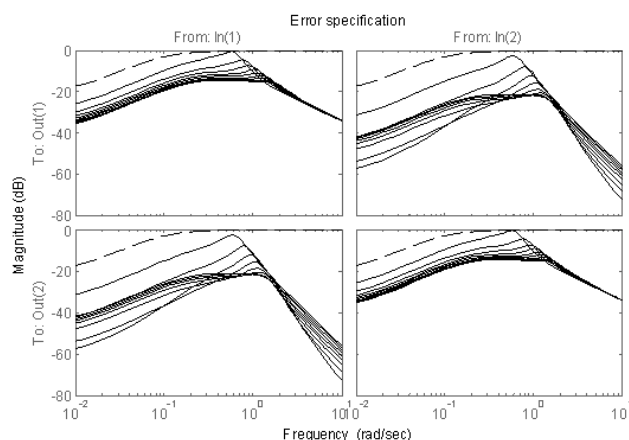


Fig. 7. Tracking error specifications for several plant cases.

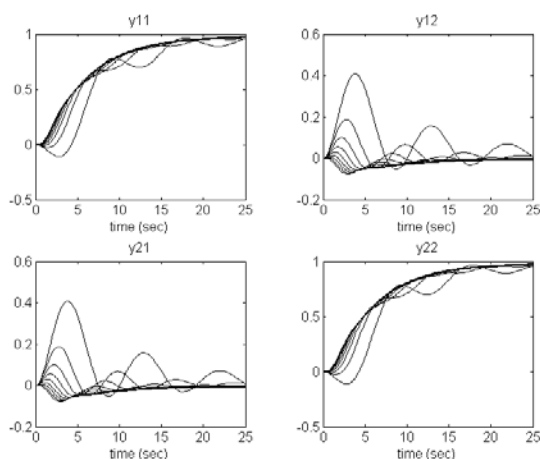


Fig. 8. Unit step responses for several plant cases.

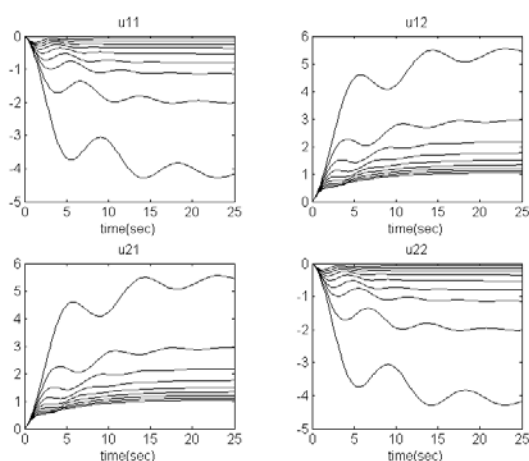


Fig. 9. Control signals.

VI. CONCLUSION

The contribution in this paper is a novel design approach in MIMO-QFT problems for tracking error specification through a simple pre-filter design in comparison with the existing methods. The steps of the proposed method summarize in transformation of the MIMO problem to the equivalent SISO problems and QFT controllers design. The pre-filter is designed using the bound generation in complex plane. The proposed method has been compared with the existing methods.

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