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Linear Algebraic Techniques for Quantum Dynamical Decoupling

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Abstract— **Suppressing decoherence is one of the most challenging problems in the control of quantum dynamical systems. Dynamical Decoupling is an open loop decoherence control technique based on high-frequency and high-amplitude periodic controls. Here, we reformulate the effects of the basic strategy in terms of linear, symmetric matrix equations. Such a reformulation proves to be useful both in the analysis and in the synthesis of the needed unitary control actions. A general framework is provided, and simple, but significant, particular cases are studied in detail.**

I. INTRODUCTION

Quantum Control is emerging as a challenging discipline, with applications ranging from quantum computation [7], to metrology and spectroscopy. Its typical tasks, state steering and quantum operation realization, have been effectively reformulated in terms of classical control problems. Thus, not only physicists but an increasing number of control engineers start to work in the field, applying their well established methods to these newborn problems (see e.g. [1], [3], [11]). Nevertheless, the critical issue in quantum control is yet far from a satisfactory solution: Decoherence and dissipation plague every realistic model of a quantum system. A qualitative formulation of decoherence control in terms of classical control theory is easy: We require insensitiveness of the control strategy with respect to the uncontrolled interactions of the system with the environment. This analogy with a classical robustness problem is not only formal: In quantum optics, for instance, by weakly monitoring the system one can implement classical feedback control design for the resulting stochastic model and introduce useful robustness features [4], [12]. For open-loop, general decoupling techniques, like *Dynamical Decoupling* (DD) proposed by L. Viola et al. [15], [13], [14], however, the contribution from control engineering is still missing. In this paper we reformulate some aspects of the DD technique in control theoretic terms. We employ a linear algebra approach to get some insight, and to generalize the Bang-Bang decoupling scheme proposed in [15]. This also permits to analyze the effectiveness of a given control sequence. In the proposed framework, we provide a synthesis method in the case of a known interaction Hamiltonian. Some of the proofs of the results in this paper are missing or only sketched. The full proofs will appear in a forthcoming, more complete version of the paper [10].

II. THE DYNAMICAL DECOUPLING METHOD

In its basic formulation [15], Dynamical Decoupling is essentially a periodic control strategy that relies on a high-frequency, unbounded control approximation[1].

We consider an n-dimensional quantum system $\mathcal{S}$ coupled to an environment (heat-bath) $\mathcal{B}$. We employ the standard quantum mechanical description associating to a quantum system a complex Hilbert space [9]. Let $\mathcal{H}_\mathcal{S}$ and $\mathcal{H}_\mathcal{B}$ be the complex Hilbert spaces associated, respectively, to the system and to the environment, so that $\dim(\mathcal{H}_\mathcal{S}) = n$. As usual [9], the coupled time evolution of $\mathcal{S}$ and $\mathcal{B}$ is described in the Kronecker product space $\mathcal{H} := \mathcal{H}_\mathcal{S} \otimes \mathcal{H}_\mathcal{B}$ and is driven by the joint Hamiltonian

$$H_0 \quad = \quad H_\mathcal{S} \otimes I_\mathcal{B} + I_\mathcal{S} \otimes H_\mathcal{B} + H_{\mathcal{SB}} \tag{1}$$

where $H_\mathcal{S}$ is the system drift Hamiltonian, and $H_\mathcal{B}$ is the bath drift Hamiltonian. The mixed term $H_{\mathcal{SB}}$ accounts for the *interactions* we want to suppress. As it is done in [15], [13], [14], we assume that $H_{\mathcal{SB}}$ is the result of a *finite* number of interaction terms of the form $S \otimes B$ with S being an operator acting on $\mathcal{H}_\mathcal{S}$ and B being an operator acting on $\mathcal{H}_\mathcal{B}$. Since the first two terms of the sum in the right-hand side of (1) are of this form, as well, we can write H_0 as

$$H_0 = \sum_{i=1}^{p} S_i \otimes B_i. \tag{2}$$

To control the evolution, we rely on a time-dependent control Hamiltonian $H_1(t)$ that acts on the system space alone. In DD, $H_1(t)$ is cyclic: the associated propagator $U_1(t)$ is periodic:

$$U_1(t) = U_1(t + T_c), \tag{3}$$

for some $T_c > 0$. This implies that $U_1(kT_c) = I_\mathcal{S}, \quad k \in \mathbb{N}$.

It is now convenient to consider the time-dependent change of basis in $\mathcal{H}_\mathcal{S}$ induced by $U_1(t)$ (known in the physical literature as *interaction picture*, see e.g.[9]). Then, the total Hamiltonian reads:

$$\tilde{H}(t) := \sum_{i=1}^{p} U_1^\dagger(t) S_i U_1(t) \otimes B_i, \tag{4}$$

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[1]Some assumptions can be relaxed or modified, see e.g. [14].

where $U^\dagger$ indicates the transposed conjugate of U (the Hermitian adjoint in general). The associated propagator $\tilde{U}(t)$ can be conveniently expanded in Magnus series [6]. For $t = T_c$ we have:

$$\tilde{U}(T_c) := e^{-i(\bar{H}^{(0)}+\bar{H}^{(1)}+\bar{H}^{(2)}+\ldots)T_c}, \tag{5}$$

where we choose suitable measurement units to have $\hbar = 1$ (as usual) and

$$\bar{H}^{(0)} := \frac{1}{T_c}\int_0^{T_c} \tilde{H}(s)ds.$$

We argue below that the higher order terms can be neglected. Given a finite time horizon for decoupling T, we apply N control cycles in $[0, T]$, $T = NT_c$. We consider the (ideal) situation $N \to \infty$. Under these assumptions, the higher order terms in (5) are negligible [15], and we are left with:

$$\tilde{U}(T_c) \approx e^{-i\bar{H}^{(0)}T_c}.$$

Suppose we choose a piecewise constant control propagator:

$$U_1(t) \equiv G_j, \quad j\Delta t \le t \le (j+1)\Delta t, \tag{6}$$

with $\mathcal{G} = \{G_j\}_{j=1}^{n_g}$ a finite set of unitary operators and $\Delta t := T_c/n_g$. We have:

$$\begin{aligned} \bar{H}^{(0)} &= \frac{1}{T_c}\int_0^{T_c}\left(\sum_{i=1}^{p} U_1^\dagger(t) S_i U_1(t) \otimes B_i\right) ds \\ &= \sum_i \left(\frac{1}{n_g}\sum_{j=1}^{n_g} G_j^\dagger S_i G_j\right) \otimes B_i. \end{aligned} \tag{7}$$

In DD [15], $\mathcal{G}$ is a (finite) group, and the control algebra $\mathcal{C}_s$ is the Lie algebra generated by $\mathcal{G}$. Hence, the expression between brackets in Equation (7) defines the projection of every S onto the *commutant* of $\mathcal{C}_s$, i.e. the set of operators commuting with every operator in $\mathcal{C}_s$. If $\mathcal{C}_s$ is the whole space of Hermitian operators defined on the system's Hilbert space, the commutant consists only of the scalar matrices, $\lambda I_{\mathcal{S}}$, $\lambda \in \mathbb{R}$. This means that the effective, mean Hamiltonian $\bar{H}^{(0)}$ on a control cycle is reduced to a scalar matrix. Moreover, at every integer multiple of T_c the interaction frame coincides with the initial frame, since $U_1(kT_C) = e^{-i\lambda} I$.
This leaves the initial state of the system unchanged (up to an irrelevant global phase factor), "stabilizing" the state.

Thus, with this decoherence control strategy we can "average" the interactions with the environment in a way such that the mean effects on the system of interest are negligible.

III. Linear Matrix Equation Approach

The first contribution of this letter is to propose an approach that generalizes the above setting to the case when $\mathcal{G}$ is not a group. Let $\mathcal{B}(\mathcal{S})$ the set of $n\times n$ complex matrices, where n is the dimension of the system's Hilbert space. As long as we face unknown interaction with the environment, the aim of an effective decoupling strategy is the following:

Problem 1: Find a set of unitary operators $\mathcal{G} = \{G_j\}_{j=1}^{n_g}$ such that, given any Hermitian matrix $X = X^\dagger \in \mathcal{B}(\mathcal{S})$, there exists $\lambda \in \mathbb{R}$ such that:

$$\sum_{j=1}^{n_g} G_j^\dagger X G_j = \lambda I_{\mathcal{S}}. \tag{8}$$

We can consider the following *disturbance rejection* analysis problem instead:

Problem 2: Given a set of unitary operators $\mathcal{G} = \{G_j\}_{j=1}^{n_g}$, find the subspace of $\mathcal{B}(\mathcal{S})$ of Hermitian matrices $X = X^\dagger$ such that:

$$\sum_{j=1}^{n_g} G_j^\dagger X G_j = \lambda I_{\mathcal{S}}, \tag{9}$$

for some $\lambda \in \mathbb{R}$.
These Hermitian matrices correspond to the rejected interactions, once fixed the control sequence $\mathcal{G}$. Such an analysis problem can be studied independently from the assumptions on the structure of $\mathcal{G}$.

If $n_g = 1$, the unique solution of (9) is trivially $X = \lambda I_{\mathcal{S}}$. For any $n_g > 1$ we have the solution $X = \frac{\lambda}{n_g} I_{\mathcal{S}}$. Hence, defining

$$\hat{G}[X] := \sum_{j=1}^{n_g} G_j^\dagger X G_j,$$

the whole space of the solution is of form $\lambda I_{\mathcal{S}} \oplus \ker \hat{G}$.

Since (almost) all the physically relevant operators are self adjoint, the solutions of (9) that are meaningful for the decoupling problem are Hermitian matrices $X = X^\dagger$. Hence, we are interested in the *real* linear space of the Hermitian matrices contained in $\ker(\hat{G})$. The following result addresses this issue.

Theorem 1: If $\dim\ker(\hat{G}) = k$, then there is a k-dimensional *real* subspace of Hermitian operators $\mathcal{V}$ such that $\mathcal{V} \subset \ker(\hat{G})$.
The proof is essentially based on an iterative procedure we sketch below. Consider the complex subspace $\mathcal{K} := \ker(\hat{G})$ of $\mathbb{C}^{n\times n}$. Let $K_1, \ldots, K_p$ be a basis of $\mathcal{K}$. Notice that $\hat{G}(X) = 0$ implies $\hat{G}(X^\dagger) = 0$. Let

$$V_1 := \begin{cases} K_1 + K_1^\dagger & \text{if} \quad K_1 \neq -K_1^\dagger \\ iK_1 & \text{if} \quad K_1 = -K_1^\dagger. \end{cases}$$

Then V_1 is Hermitian and there exists a new basis $V_1, L_2, \ldots, L_k$ of $\mathcal{K}$. Such procedure can be iterated so as to obtain k linearly independent matrices in $\ker(\hat{G})$.

An general way to study these symmetric matrix equations is to introduce the vec$(\cdot)$ function [5], [8]. If X is an $n \times n$ matrix,

$$\text{vec}(X) := \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{pmatrix},$$

where X_i are the columns of X. It can be proven that

$$\mathrm{vec}(AXB) = A \otimes B^T \mathrm{vec}(X),$$

where $\otimes$ is the standard Kronecker product. Thus, Equation (9) can be always translated to a standard vector linear system:

$$L\,\mathrm{vec}(X) = \lambda\,\mathrm{vec}(I), \tag{10}$$

with:

$$L := \sum_{i=1}^{n_g} G_i^\dagger \otimes G_i^T. \tag{11}$$

The analysis of the rejected interaction space is now reduced to determine the kernel of L, since we know that the scalar matrices are always particular solutions for the linear system.

A. The case of two control actions

For instance, the $n_g = 2$ case can always be reduced, via the change of basis associated to G_1, to the study of the symmetric, homogeneous Stein's equation ($G := G_2 G_1^\dagger$):

$$X + G^\dagger X G = 0. \tag{12}$$

In this case, $L = I_{\mathcal{S}} + G^\dagger \otimes G^T$. To characterize the dimension of the kernel of L we have to count the -1 eigenvalues of $G^\dagger \otimes G^T$. A useful theorem is the following [5]:

Theorem 2: If $\lambda_1, ..., \lambda_n$ are the eigenvalues of $A \in \mathbb{C}_{n\times n}$ and $\mu_1, ..., \mu_m$ are the eigenvalues of $B \in \mathbb{C}_{m\times m}$, then the eigenvalues of

$$\phi(A,B) := \sum_{i,j=0}^{p} c_{ij} A^i \otimes B^j, \tag{13}$$

are the $n \times m$ numbers:

$$\{\phi(\lambda_r, \lambda_s)\}_{r=1,...,n;s=1,...,m},$$

with the Kronecker product substituted by the standard product.
Hence, we have the following:

Proposition 1 (Mixing Condition): Equation (12) admits a 2-dimensional (complex) matrix subspace of solutions for every couple of opposite eigenvalues of G.
We can also give a bound on the effectiveness of the $n_g = 2$ decoupling.

Proposition 2: Let G be a unitary, $n\times n$ complex matrix. Then $G^\dagger \otimes\ G^T$ admits up to $n^2/2$ eigenvectors relative to eigenvalue -1 in the case of n even, $(n^2-1)/2$ if n is odd. The proof may be found in [10].

IV. A Synthesis Problem

The analysis presented in the previous section suggests also a synthesis method to decouple the system from the environment in a simple case. Consider a coupling Hamiltonian of the form:

$$H = \sum_{i=1}^{p} S_i \otimes B_i, \tag{14}$$

with the additional condition:

$$[S_i, S_j] = 0, \quad \forall i,j. \tag{15}$$

Proposition 3: Consider an n-dimensional system and a Hamiltonian (2) that satisfies the constraints (14)-(15). Then there exists a control Hamiltonian of the form $u(t)H_d$ that is sufficient to ensure decoupling, with $u(t)$ a periodic, impulsive control function.
In [10], we give a constructive proof we sketch here. Each S_i can be always decomposed as $S = \frac{\mathrm{Tr}(S_i)}{n} I_{\mathcal{S}} + Z_i, \quad \mathrm{Tr}(Z_i) = 0$. For the structure of the problem we can restrict our attention to Z_i, and move to a basis in which each Z_i is simultaneously diagonal. Choosing without loss of generality $G_0 = I_{\mathcal{S}}$, a n-steps decoupling strategy for Z_i must guarantee:

$$Z_i + G_1^\dagger Z_i G_1 + G_2^\dagger Z_i G_2 + \ldots + G_{n-1}^\dagger Z_i G_{n-1} = 0. \tag{16}$$

We can consider the $n-1$ unitary operations G_j corresponding to cyclic permutations of the eigenvalues, a set of *circulant matrices*. Matrices G_j are *circulant* of the following form:

$$G_1 = \begin{pmatrix} 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 0 & 0 \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix},$$
$$G_2 = (G_1)^2, \ldots, G_{n-1} = (G_1)^{n-1}.$$

With these unitary control actions and considering a general Z, Equation (16) becomes:

$$\begin{aligned} & Z + G_1^\dagger Z G_1 + \ldots + G_{n-1}^\dagger Z G_{n-1} \\ & = \mathrm{diag}\left\{\alpha_1, \ldots, \alpha_{n-1}, -\sum_i \alpha_i\right\} + \\ & + \mathrm{diag}\left\{-\sum_i \alpha_i, \alpha_1, \ldots, \alpha_{n-1}\right\} + \\ & + \ldots + \mathrm{diag}\left\{\alpha_2, \ldots, \alpha_{n-1}, -\sum_i \alpha_i, \alpha_1\right\} = 0. \end{aligned}$$

This choice is effective for all the commuting Z_i simultaneously, as it is apparent from of Equation (7). Since the G_i are invertible, they admit a matrix logarithm (see e.g. [2] and references therein). With H_d defined as above, it is straightforward to see that the desired unitary control actions G_i can be realized choosing H_d as control Hamiltonian, and a impulsive control function function. Some remarks on the result:

- Being all the generators of the desired unitary control actions proportional, the physical implementation of the controller has been remarkably simplified. Fixed H_d, one has to tune only the relative strength of the pulses;

- The strategy is intrinsically robust with respect to error in the estimate of the spectrum of the interaction terms. That is, every matrix that commutes with the nominal one is averaged out. If the system Hilbert space dimension is n, we are averaging out a whole $n-1$ dimensional real subspace of zero-trace, Hermitian matrices.
- Obviously, this strategy can be applied not only to remove system-bath interactions, but even to remove undesired components of the free system dynamics. It suffices to consider the system Hamiltonian terms $H_{\mathcal{S}} \otimes I_{\mathcal{B}}$ as the disturbance to be rejected.

V. Applications

For the simplest model one can think of, a two-dimensional quantum system (a *qbit*) coupled to an environment, the solution emerging from the proposes strategy recollects directly in the well-known *spin-echo* technique, that itself inspirited the development of Dynamical Decoupling (see e.g. [15] and references therein, to which we remaind for the discussion of the single qbit case). Let us consider a couple of slightly more sophisticated models, of both didactic and actual interest.

A. Two qubits coupled via Ising interaction

Consider two 2-dimensional quantum systems, labelled by superscripts $^{(1)}, ^{(2)}$. The Hilbert space for the joint description is $\mathcal{H}^{(1)} \otimes \mathcal{H}^{(2)} \sim \mathbb{C}^2 \otimes \mathbb{C}^2$. The Hamiltonian driving the (whole) system is given of the following form (*Ising interaction*):

$$
\begin{aligned}
H &= \frac{\omega_1}{2}\sigma_z^{(1)} \otimes I^{(2)} + \frac{\omega_2}{2} I^{(1)} \otimes \sigma_z^{(2)} + \\
&\quad + J_z \sigma_z^{(1)} \otimes \sigma_z^{(2)},
\end{aligned} \tag{17}
$$

with $\omega^{(i)}, J_z$ real parameters, and $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Suppose the aim of the control strategy be "freezing" the dynamics, i.e. average out the total Hamiltonian H.

The Hamiltonian in the matrix representation induced by the choice of the canonical basis in $\mathcal{H}^{(1)} \otimes \mathcal{H}^{(2)}$, reads:

$$
\begin{aligned}
H = \frac{1}{2}\mathrm{diag}(&\omega_1 + \omega_2 + J_z, \omega_1 - \omega_2 - J_z, \\
&-\omega_1 - \omega_2 - J_z, -\omega_1 - \omega_2 + J_z).
\end{aligned}
$$

Since the Hamiltonian is already diagonal, our synthesis recipe prescripts to implement the 4 circulant matrices generated by the powers of:

$$
G_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.
$$

In fact, it follows easily that for $G_i = (G_1)^i$:

$$
\sum_{i=1}^{4} G_i^\dagger H G_i = 0.
$$

B. Two interacting qbits coupled with the environment

Now consider the following setting, in which we introduce an unspecified quantum environment $\mathcal{B}$, that moves the representation to $\mathcal{H}^{(1)} \otimes \mathcal{H}^{(2)} \otimes \mathcal{H}^{\mathcal{B}}$:

- Each qbit is coupled with the environment via a $\sigma_z^{(i)} \otimes B_i$ term that doesn't affect the other one;
- The qbits are still interacting via the Ising coupling, $J_z \sigma_z^{(1)} \otimes \sigma_z^{(2)}$.

The total Hamiltonian becames:

$$
\begin{aligned}
H &= \frac{\omega_1}{2}\sigma_z^{(1)} \otimes I^{(2)} \otimes B_1 + \frac{\omega_2}{2} I^{(1)} \otimes \sigma_z^{(2)} \otimes B_2 + \\
&\quad + J_z \sigma_z^{(1)} \otimes \sigma_z^{(2)} \otimes I^{\mathcal{B}}.
\end{aligned} \tag{18}
$$

Since all the terms concerning the two qbits are still diagonal in the natural basis of their joint subspace, the same strategy as before can be successfully applied. This is due to the remarkable symmetry of the Hamiltonian, in essence to the fact that the interactions for every qbit act via the same matrix, $\sigma_z^{(i)}$. However, the choice of the z-axis is conventional for each qbit. The strategy, in both examples, inherit intrinsical robustness with respect to imperfect knowledge of the Hamiltonian parameters $\omega^{(i)}, J_z$.

VI. Conclusions

In this paper we present an alternative point of view for the dynamical decoupling method, that is one general decoherence control technique for quantum models of open systems. The problem has been decomposed, in control engineering terms, in an analysis and a synthesis part. The first is essentially a disturb-rejection problem. It has been studied in terms of the kernel of a linear operator. We have proposed a general algebraic framework, which leads to results in the general case and to quite complete characterization of the two-controls case.

The second part aims to find a set of unitary operators, the control actions, that suppress the undesired interactions. The case of given, commuting "disturb" interactions has been discussed and efficiently solved, as well as exemplified.

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