Proceedings of the
44th IEEE Conference on Decision and Control, and
the European Control Conference 2005
Seville, Spain, December 12-15, 2005



# Regularity of the Adjoint Variable in Optimal Control under State Constraints

Hélène Frankowska

*Abstract*— **It is well known that the adjoint state of the Pontryagin maximum principle may be discontinuous whenever the optimal trajectory lies partially on the boundary of constraints. Still we prove that if the associated Hamiltonian $H(t,x,\cdot)$ is differentiable and the constraints are sleek, then every optimal trajectory is continuously differentiable. Moreover if for all $x$ on the boundary of constraints, $H'_p(t,x,\cdot)$ is strictly monotone in directions normal at $x$ to the set of constraints, then the adjoint state is also continuous on interior of its interval of definition. Finally, we identify a class of constraints for which the adjoint state is absolutely continuous or even Lipschitz on this open interval. This allows us to derive necessary conditions for optimality in the form of variational differential inequalities, maximum principle and modified transversality conditions.**

## I. INTRODUCTION

Consider a control system

$$x'(t) = f(t, x(t), u(t)), \quad u(t) \in U(t) \quad \text{a.e. in } [0,1] \qquad (1)$$

under state constraints

$$x(t) \in K \quad \text{for all} \quad t \in [0,1], \qquad (2)$$

where $U$ is a measurable set-valued map from $[0,1]$ into nonempty closed subsets of a complete separable metric space $\mathcal{Z}$, $f : [0,1]\times\mathbf{R}^n\times\mathcal{Z} \to \mathbf{R}^n$ and $K$ is a closed subset of $\mathbf{R}^n$. Denote by $S^K_{[0,1]}$ the set of all absolutely continuous solutions to (1) satisfying state constraints (2).

In this paper we are interested by regularity of minimizers of the Bolza optimal control problem under state constraints

$$\begin{array}{l}\min\Big\{\varphi(x(0),x(1)) + \int_0^1 L(t,x(t),u(t))dt \mid \\ x \in S^K_{[0,1]},\ (x(0),x(1)) \in K_1\},\end{array} \qquad (3)$$

where $\varphi : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$ is locally Lipschitz , $L : [0,1] \times \mathbf{R}^n\times\mathcal{Z} \to \mathbf{R}$ and $K_1 \subset \mathbf{R}^n\times\mathbf{R}^n$ is closed. We also assume that $f,\ L$ are measurable with respect to the first variable and continuous with respect to the second and third variables.

Consider an optimal trajectory/control pair $(z,\bar{u})$. Under some regularity assumptions on data it satisfies the following necessary condition for optimality: there exist $\lambda \in \{0,1\}$, an absolutely continuous $p : [0,1] \to \mathbf{R}^n$ and a mapping $\psi : [0,1] \to \mathbf{R}^n$, $\psi \in NBV([0,1])$ (space of normalized functions with bounded variation on $[0,1]$) not vanishing simultaneously such that

$i)$ for a positive Radon measure $\mu$ on $[0,1]$ and a Borel measurable $\nu(\cdot) : [0,1] \to \mathbf{R}^n$ satisfying $\mu$-almost everywhere $\nu(s) \in N_K(z(s)) \cap B$ (where $N_K(z(s))$ denotes the normal cone to $K$ at $z(s)$ and $B$ the closed unit ball) we have $\psi(t) = \int_{[0,t]} \nu(s)d\mu(s)$ for all $t \in (0,1]$,

$ii)$ $p(\cdot)$ is a solution to the adjoint system

$$-p' = f'_x(s,z(s),\bar{u}(s))^*(p+\psi) - \lambda L'_x(s,z(s),\bar{u}(s)) \quad (4)$$

satisfying the maximum principle

$$\begin{array}{l}\langle p(s)+\psi(s), z'(s)\rangle - \lambda L(s,z(s),z'(s)) = \\ \max_{u\in U(s)}(\langle p(s)+\psi(s), f(s,z(s),u)\rangle - \lambda L(s,z(s),u))\end{array}$$

and the transversality condition

$$(p(0), -p(1)-\psi(1)) \in \lambda\nabla\varphi(z(0),z(1)) + N_{K_1}(z(0),z(1)).$$

The above necessary conditions are called normal if $\lambda = 1$.

The Hamiltonian $H : [0,1] \times \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$ associated to the above Bolza problem is defined by

$$H(t,x,p) = \sup_{u\in U(t)} (\langle p, f(t,x,u)\rangle - L(t,x,u)). \qquad (5)$$

For all $(t,x) \in [0,1] \times \mathbf{R}^n$, $H(t,x,\cdot)$ is convex and

$$\begin{array}{l}\partial_p H(t,x,p) = \{f(t,x,u) \mid u \in U(t), \\ H(t,x,p) = \langle p, f(t,x,u)\rangle - L(t,x,u)\},\end{array} \qquad (6)$$

where $\partial_p H(t,x,p)$ denotes the subdifferential of convex analysis of $H(t,x,\cdot)$ at $p$. Thus in the normal case,

$$z'(t) \in \partial_p H(t,z(t),p(t)+\psi(t)) \text{ a.e. in } [0,1].$$

When there is no state and end point constraints, then $\lambda = 1$, $\psi = 0$, and the above inclusion allows to deduce regularity of the derivative $z'$ from regularity of $H'_p$. Indeed if $H'_p$ is continuous (respectively locally Lipschitz), then $z'$ is continuous (respectively absolutely continuous). This fails to be true in general because in the constrained case

$$z'(t) = H'_p(t,z(t),p(t)+\psi(t)) \text{ a.e. in } [0,1] \qquad (7)$$

and $\psi$ may be discontinuous.

In this paper we focus our attention on regularity of optimal solutions and of mapping $\psi$ for sleek $K$ in the normal case. (The state constraints $K$ are sleek, if for every $x \in K$ the contingent cone to $K$ at $x$ coincides with Clarke's tangent cone to $K$ at $x$). We show that if $H$ is continuous, then the function $(0,1) \ni t \mapsto H(t,z(t),p(t)+\psi(t))$ is continuous, even though $\psi$ may be discontinuous. If in addition $H(t,z(t),\cdot)$ is differentiable, then $z \in C^1$ and the mapping

$$(0,1) \ni t \mapsto H'_p(t,z(t),p(t)+\psi(t))$$

Work supported in part by the European Community's Human Potential Programme under contract HPRN-CT-2002-00281, Evolution Equations.

H. Frankowska is with CNRS and École Polytechnique, Paris, France franko@shs.polytechnique.fr

is continuous (we prove this result without assuming continuity of $H'_p$). Moreover for all $t \in (0,1)$,

$$\langle \psi(t) - \psi(t-), z'(t) \rangle = 0,$$

$$H'_p(t, z(t), p(t) + \psi(t)) = H'_p(t, z(t), p(t) + \psi(t-)).$$

Thus jumps of $\psi$ occur only in the directions orthogonal to derivatives of $z$ (see Theorem 2.5). Furthermore, $\psi$ is continuous on $(0,1)$ provided that $H'_p(t, z(t), \cdot)$ is strictly monotone in the directions normal to constraints at $z(t)$.

We also propose sufficient conditions for $\psi$ to be absolutely continuous or Lipschitz on $(0,1)$, and for $z'$ to be Lipschitz on $[0,1]$ (see Theorem 3.1). To obtain this result we use some ideas of proofs from [13], [15], but we impose a monotonicity assumption on the Hamiltonian with respect to normals to constraints (instead of supposing the strict convexity of the Lagrangian) and consider control systems that are not affine with respect to controls. In Example 2 we discuss relations of our assumptions to those of [15].

The above regularity results imply : if for a trajectory/control pair $(z, \bar{u})$ the normal constrained maximum principle $i) - ii)$ holds true with absolutely continuous $\psi$, then there exists an *absolutely continuous* mapping $q : [0,1] \to \mathbf{R}^n$ satisfying the differential variational inequalities

$$-q' \in f'_x(s, z(s), \bar{u}(s))^* q - L'_x(s, z(s), \bar{u}(s)) - N_K(z(s)), \tag{8}$$

the maximum principle

$$\langle q(s), z'(s) \rangle - L(s, z(s), \bar{u}(s)) = H(s, z(s), q(s)) \text{ a.e.} \tag{9}$$

and the transversality condition

$$\begin{aligned} &(q(0), -q(1)) \in \nabla\varphi(z(0), z(1)) + \\ &+ N_{K_1}(z(0), z(1)) + N_K(z(0)) \times N_K(z(1)), \end{aligned} \tag{10}$$

where $N_K(z(s))$ (resp. $N_{K_1}(z(0), z(1))$) denotes Clarke's normal cone to $K$ at $z(s)$ (resp. to $K_1$ at $(z(0), z(1))$).

The Maximum Principle with regular costate was proved by Gamkrelidze in [12] for smooth optimal trajectories. Then a number of papers were written on this subject without restrictions imposed on $z$, but using measures in the definition of costate (see for instance [9]). We refer to [16] for extended discussions on the constrained maximum principle and further references and to [1] for the Russian bibliography on the subject. In this paper we go another way around. We impose some assumptions on the Hamiltonian and constraints to deduce absolute continuity of $\psi$.

Regularity of $\psi$ can be used for further investigation of smoothness of optimal control $\bar{u}$. Indeed, if for all $q \in \mathbf{R}^n$ there exists exactly one $u(t, q)$ such that

$$H(t, z(t), q) = \langle q, f(t, z(t), u(t, q)) \rangle - L(t, z(t), u(t, q))$$

then, by the maximum principle, $\bar{u}(t) = u(t, p(t) + \psi(t))$ for almost all $t$. Thus regularity of $\bar{u}$ depends upon regularity of the mapping $u(\cdot, \cdot)$ and $p + \psi$. For instance Lipschitz continuity of $\psi$ on $(0,1)$ implied Lipschitz continuity of minimizing controls in [13, Hager] for linear control systems, convex Lagrangian and convex state constraints, in [8, Dontchev & Hager] for the LQR problem under affine state constraints and in [14, Malanowski] for both control system and Lagrangian nonlinear with respect to the state. Very recently Shvartsman and Vinter [15] considered the case of fully nonlinear constraints

$$K = \{x \mid h_j(x) \leq 0, \ j = 1, ..., m\}$$

with $h_j \in C^{1,1}_{loc}$ (actually in their paper $h_j$ are also time dependent). In their work the system is supposed to be affine with respect to controls and the Lagrangian $L(t, x, \cdot)$ is smooth and strictly convex. Under various sets of conditions they show that the above mapping $u(\cdot, \cdot)$ is locally Lipschitz.

Since the adjoint system is never used in this paper, results of Sections 3, 4 and 5 can be applied with various maximum principles, including their non smooth versions (see for instance [1], [16]).

Let $X$ be a real Banach space, $B$ denote the closed unit ball in $X$. A set $C \subset X$ is called a cone if it is nonempty and for all $\lambda \geq 0$ and $v \in C$ we have $\lambda v \in C$. The negative polar cone of $C$ is denoted by $C^-$. Let $K \subset \mathbf{R}^n$ be closed and $x \in K$. The contingent cone to $K$ at $x$ is defined by

$$T_K(x) = \{v \in \mathbf{R}^n \mid \quad \liminf_{h \to 0+} \frac{dist(x + hv, K)}{h} = 0\}.$$

$K$ is called sleek if the set-valued map $K \ni x \rightsquigarrow T_K(x)$ is lower semicontinuous Every convex set is sleek. For other examples of sleek sets see [2]. The negative polar $N_K(x) := T_K(x)^-$ is called the normal cone to $K$ at $x \in K$. If $K$ is sleek, then $N_K(x)$ is equal to Clarke's normal cone to $K$ at $x$ and $K \ni x \rightsquigarrow N_K(x)$ has closed graph.

*Proposition 1.1 ([6]):* Let $K$ be closed and $z : [0,1] \to K$ be so that $t \rightsquigarrow N_K(z(t))$ has closed graph. Let $\psi \in NBV([0,1])$ be such that for some scalar positive Radon measure $\mu$ on $[0,1]$ and a selection $\nu(s) \in N_K(z(s)) \cap B$ $\mu - a.e.$ we have $\psi(t) = \int_{[0,t]} \nu(s) d\mu(s), \forall\, t \in (0,1]$. Then

$$\psi(0+) \in N_K(z(0)) \;\&\; \psi(t) - \psi(t-) \in N_K(z(t)) \;\forall\, t \in (0,1].$$

Let $z : [0,1] \to \mathbf{R}^n$ be a Lipschitz function. For every $t \in [0,1]$ set $\partial^* z(t) = \text{Limsup}_{s \to t}\{z'(s)\}$, where Limsup denotes the upper set-valued limit. (See for instance [2] for the corresponding definition). If $\partial^* z(t)$ is a singleton, then $z$ is differentiable at $t$ and $\{z'(t)\} = \partial^* z(t)$ (see [4]).

Recall that any function $f : [0,1] \to \mathbf{R}^n$ of bounded variation on $[0,1]$ has right and left limits $f(0+)$ and $f(1-)$.

The space $NBV([0,1])$ (Normalized Bounded Variations) is the space of functions $f$ of bounded variation on $[0,1]$, which are continuous from the right on $(0,1)$ and such that $f(0) = 0$. The norm of $f \in NBV([0,1])$ is the total variation of $f$ on $[0,1]$.

In this paper when we say measurable or almost everywhere without refereeing to a precise measure, we always mean the Lebesgue measure.

## II. $C^1$-MINIMIZERS AND CONTINUITY OF THE ADJOINT STATE

Denote by $\partial\varphi(z(0), z(1))$ the generalized gradient of $\varphi$ at $(z(0), z(1))$.

*Definition 2.1:* A trajectory/control pair $(z,\bar{u})$ of (1), (2) with $(z(0),z(1)) \in K_1$ satisfies the constrained maximum principle if there exist $\lambda \in \{0,1\}$, $\psi \in NBV([0,1])$ and an absolutely continuous $p(\cdot) : [0,1] \to \mathbf{R}^n$ not vanishing simultaneously such that

$$(p(0), -p(1)-\psi(1)) \in \lambda\partial\varphi(z(0),z(1)) + N_{K_1}(z(0),z(1)) \tag{11}$$

$$\begin{aligned}&\langle p(s)+\psi(s), z'(s)\rangle - \lambda L(s,z(s),\bar{u}(s)) = \\ &\sup_{u\in U(s)}\langle p(s)+\psi(s), f(s,z(s),u)\rangle - \lambda L(s,z(s),u)\end{aligned} \tag{12}$$

a.e. in $[0,1]$ and

$$\begin{aligned}&\psi(0+) \in N_K(z(0)),\ \psi(t)-\psi(t-) \in N_K(z(t)) \\ &\psi(t) = \textstyle\int_{[0,t]} \nu(s)d\mu(s) \quad \forall\, t \in (0,1]\end{aligned} \tag{13}$$

for a positive (scalar) Radon measure $\mu$ on $[0,1]$ and a Borel measurable $\nu(\cdot) : [0,1] \to \mathbf{R}^n$ satisfying

$$\nu(s) \in N_K(z(s)) \cap B \quad \mu - a.e. \tag{14}$$

The constrained maximum principle is normal if $\lambda = 1$.

*Remark 2.2:* Notice that we did not invoke the adjoint system in the above definition. In fact it will not be needed in this paper. On the other hand many maximum principles that exist in the literature differ just in the adjoint system and transversality conditions. We also never use the particular form of the transversality condition of the above definition. In this way results of this paper may be applied with any maximum principle under state constraints, including non smooth versions (see for instance [1], [16]), provided $\psi$ is right continuous and the jump conditions (13) hold true. This implies that the maximum principles of [16] have to be written in a slightly different way (with right continuous instead of left continuous multipliers).

*Lemma 2.3:* Let $K \subset \mathbf{R}^n$ be a closed set, $z : [0,1] \to K$ be a Lipschitz function and $t \in [0,1]$ be so that $z(t) \in \partial K$. Then for every $n \in N_K(z(t))$ we have

i) if $z$ is differentiable at $t \in (0,1)$, then $\langle n, z'(t)\rangle = 0$.

ii) for every set $A \subset [0,1]$ of zero measure there exist $s_i \to t,\ s_i \notin A,\ s_i \le t$ and $t_i \to t,\ t_i \notin A,\ t_i \ge t$ such that

$$t > 0 \implies \lim_{i\to\infty}\langle n, z'(s_i)\rangle \ge 0$$

$$t < 1 \implies \lim_{i\to\infty}\langle n, z'(t_i)\rangle \le 0.$$

Let $(z,\bar{u})$ be a trajectory control pair of (1) and let the set-valued map $F : [0,1]\times\mathbf{R}^n \rightsquigarrow \mathbf{R}^{n+1}$ be defined by

$$F(t,x) = \{(L(t,x,u)+v, f(t,x,u)) \,|\, u \in U(t),\ v \ge 0\}. \tag{15}$$

*Theorem 2.4:* Assume that $(z,\bar{u})$ satisfies the normal constrained maximum principle with some $p,\ \psi$, that $z$ is Lipschitz and $K$ is sleek. If $graph(F)$ is closed and $H$ is continuous on $graph(z)\times\mathbf{R}^n$, then the function

$$[0,1] \ni t \mapsto H(t,z(t),p(t)+\psi(t))$$

is continuous on $(0,1)$ and upper semicontinuous at 0, 1.

*Proof —* We only prove the first statement. Set $\phi(t) = H(t,z(t),p(t)+\psi(t))$. Then $\phi$ is right continuous on $(0,1)$. Fix $0 \le t \le 1$. Define $n := \psi(t)-\psi(t-) \in N_K(z(t))$ if $t > 0$ and $n = \psi(0+)$ otherwise. By (12) and Lemma 2.3 if $t < 1$, then there exist $t_i \to t,\ t_i \ge t$ such that

$$\begin{gathered}H(t_i,z(t_i),p(t_i)+\psi(t_i)) = \\ \langle p(t_i)+\psi(t_i), z'(t_i)\rangle - L(t_i,z(t_i),u(t_i))\end{gathered}$$

and $\lim_{i\to\infty}\langle n, z'(t_i)\rangle \le 0$. Taking a subsequence and keeping the same notations, we may assume that for some $u \in U(t),\ v_0 \ge 0$, we have $\lim_{i\to\infty} z'(t_i) = f(t,z(t),u)$,

$$\lim_{i\to\infty} L(t_i,z(t_i),u(t_i)) = L(t,z(t),u) + v_0.$$

Thus

$$\begin{aligned}&\phi(t+) = \langle n, f(t,z(t),u)\rangle + \\ &+\langle p(t)+\psi(t+)-n, f(t,z(t),u)\rangle - L(t,z(t),u) - v_0.\end{aligned}$$

It follows that $\phi(0+) \le \phi(0)$ and if $0 < t < 1$, then $\phi(t) \le \phi(t-)$. Thus $\phi$ is upper semicontinuous at zero. Similarly, if $t > 0$, then for some $v \in U(t),\ v_1 \ge 0$

$$\phi(t-) \le \langle p(t)+\psi(t), f(t,z(t),v)\rangle - L(t,z(t),v) - v_1 \le \phi(t).$$

Thus, $\phi$ is continuous on $(0,1)$ and u.s.c. at 1. □

*Theorem 2.5:* Assume that $(z,\bar{u})$ satisfies the normal constrained maximum principle with some $p,\ \psi$, that $z$ is Lipschitz, $K$ is sleek, $graph(F)$ is closed, $H$ is continuous on $graph(z)\times\mathbf{R}^n$ and $H(t,z(t),\cdot)$ is differentiable for all $t \in [0,1]$. Then $z \in C^1([0,1])$, the mapping

$$(0,1) \ni t \mapsto H'_p(t,z(t),p(t)+\psi(t))$$

is continuous and $z'(t) = H'_p(t,z(t),p(t)+\psi(t))$ for every $t \in (0,1)$. Furthermore

$$\langle \psi(t)-\psi(t-), z'(t)\rangle = 0 \ \ \forall\, t \in (0,1),$$

$$\langle \psi(0+), z'(0)\rangle \le 0, \ \ \langle \psi(1)-\psi(1-), z'(1)\rangle \ge 0$$

and for a measurable function $u(t) \in U(t)$ such that $u = \bar{u}$ almost everywhere

$$z'(t) = f(t,z(t),u(t)) \ \ \forall\ \ t \in [0,1],$$

$$\begin{gathered}\forall\ \ t \in [0,1),\ H(t,z(t),p(t)+\psi(t+)) = \\ \langle p(t)+\psi(t+), f(t,z(t),u(t))\rangle - L(t,z(t),u(t)),\end{gathered}$$

$$\begin{gathered}H(1,z(1),p(1)+\psi(1-)) = \\ \langle p(1)+\psi(1-), f(1,z(1),u(1))\rangle - L(1,z(1),u(1)).\end{gathered}$$

Moreover,

$$\begin{aligned}&H'_p(t,z(t),p(t)+\psi(t)) = \\ &H'_p(t,z(t),p(t)+\psi(t-)) \ \ \forall\ \ t \in (0,1).\end{aligned} \tag{16}$$

*Proof —* Define the subset $\mathcal{D} \subset [0,1]$ of full measure by

$$\mathcal{D} = \{s \in [0,1] \mid z'(s) = f(s,z(s),\bar{u}(s)),\ (12) \text{ holds true}\}$$

and fix $0 \le t \le 1$. We claim that there exist $w,\ v \in U(t)$ such that

$$\begin{gathered}H(t,z(t),p(t)+\psi(t-)) = \\ \langle p(t)+\psi(t-), f(t,z(t),w)\rangle - L(t,z(t),w) \ \text{ if } \ t > 0,\end{gathered}$$

$$H(t, z(t), p(t) + \psi(t+)) =$$

$$\langle p(t) + \psi(t+), f(t, z(t), v)\rangle - L(t, z(t), v) \text{ if } t < 1$$

and that $f(t, z(t), w) = f(t, z(t), v)$, $L(t, z(t), w) = L(t, z(t), v)$ whenever $t \in (0, 1)$.

Indeed fix $t > 0$. Let $\mathcal{D} \ni t_i \mapsto t-$ be such that $z'(t_i)$ converge to some $\zeta$. Then, by continuity of $H$ and closedness of $graph(F)$, $\zeta = f(t, z(t), w)$ for some $w \in U(t)$ and

$$H(t, z(t), p(t) + \psi(t-)) =$$

$$\langle p(t) + \psi(t-), f(t, z(t), w)\rangle - L(t, z(t), w).$$

Then $H'_p(t, z(t), p(t) + \psi(t-)) = f(t, z(t), w)$, because $H(t, z(t), \cdot)$ is differentiable. Consequently,

$$\text{Limsup}_{s \to_{\mathcal{D}} t-} \{z'(s)\} = \{H'_p(t, z(t), p(t)+\psi(t-))\}. \quad (17)$$

According to Lemma 2.3 applied with $A = [0, 1]\backslash\mathcal{D}$,

$$\langle \psi(t) - \psi(t-), f(t, z(t), w)\rangle \geq 0.$$

Similarly if $t < 1$ and $\mathcal{D} \ni t_i \mapsto t+$ are such that $z'(t_i)$ converge to some $\eta$, then for some $v \in U(t)$, $f(t, z(t), v) = H'_p(t, z(t), p(t) + \psi(t+))$ and

$$H(t, z(t), p(t) + \psi(t+)) =$$

$$\langle p(t) + \psi(t+), f(t, z(t), v)\rangle - L(t, z(t), v).$$

Consequently,

$$\text{Limsup}_{s \to_{\mathcal{D}} t+} \{z'(s)\} = \{H'_p(t, z(t), p(t)+\psi(t+))\}. \quad (18)$$

By Theorem 2.4, if $0 < t < 1$, then

$$H(t, z(t), p(t) + \psi(t)) = H(t, z(t), p(t) + \psi(t-)) =$$

$$\langle p(t) + \psi(t), f(t, z(t), w)\rangle - \langle \psi(t) - \psi(t-), f(t, z(t), w)\rangle$$
$$-L(t, z(t), w) \leq H(t, z(t), p(t) + \psi(t)).$$

This implies that $\langle \psi(t) - \psi(t-), f(t, z(t), w)\rangle = 0$ and $H(t, z(t), p(t) + \psi(t)) = \langle p(t) + \psi(t), f(t, z(t), w)\rangle - L(t, z(t), w)$. Since $\psi(t+) = \psi(t)$ for $t \in (0, 1)$, we deduce from (6) that $f(t, z(t), w) = f(t, z(t), v)$, $L(t, z(t), w) = L(t, z(t), v)$ and (16) follows.

By [4], $\partial z(t) = co\,\text{Limsup}_{s \to t,\, s \in \mathcal{D}}\{z'(s)\}$. So (16), (17) and (18) imply that $\partial^* z(t)$ is a singleton for all $t \in [0, 1]$. Hence $z$ is differentiable and $z'$ is continuous on $[0, 1]$. Let $V(t) \subset U(t)$ be such that for any $u \in V(t)$, if $t \in [0, 1)$ then $H(t, z(t), p(t) + \psi(t+)) = \langle p(t) + \psi(t+), f(t, z(t), u)\rangle - L(t, z(t), u)$ and if $t = 1$, $H(1, z(1), p(1) + \psi(1-)) = \langle p(1) + \psi(1-), f(1, z(1), u)\rangle - L(1, z(1), u)$. Then $V$ is measurable and has closed nonempty images. Consider a measurable selection $u(t) \in V(t)$ for all $t \in [0, 1]$ such that $u = \bar{u}$ a.e. Then $H'_p(t, z(t), p(t) + \psi(t+)) = f(t, z(t), u(t))$ and $z'(t) = f(t, z(t), u(t))$ for all $t \in [0, 1]$, and for every $t \in (0, 1)$, $\langle \psi(t) - \psi(t-), z'(t)\rangle = 0$. Since $z([0, 1]) \subset K$, by (13), $\langle \psi(0+), z'(0)\rangle \leq 0$, $\langle \psi(1) - \psi(1-), z'(1)\rangle \geq 0$. □

Theorems 2.4, 2.5 do not exclude discontinuity of $\psi$. Still Theorem 2.5 implies $C^1-$regularity of an optimal solution.

*Corollary 2.6:* Under all the assumptions of Theorem 2.5, suppose in addition that for every $t \in (0, 1)$ and $p,\ q \in \mathbf{R}^n$, satisfying $p - q \in N_K(z(t))$ and $H(t, z(t), p) = H(t, z(t), q)$, $H'_p(t, z(t), p) = H'_p(t, z(t), q) = z'(t)$ we have $p = q$. Then $\psi$ is continuous in $(0, 1)$.

In particular, if $H'_p(t, z(t), \cdot)$ is strictly monotone in the directions normal to $K$ at $z(t)$ : for every $t \in (0, 1)$ and all $p \neq q \in \mathbf{R}^n$ such that $p - q \in N_K(z(t))$ and $H(t, z(t), p) = H(t, z(t), q)$ we have

$$\left\langle H'_p(t, z(t), p) - H'_p(t, z(t), q), p - q\right\rangle > 0,$$

then $\psi$ is continuous in $(0, 1)$.

*Example 1.* Let $K$ be a closed sleek subset of $\mathbf{R}^n$, $K_1$ be a closed subset of $\mathbf{R}^n \times \mathbf{R}^n$, $d : [0, 1] \times \mathbf{R}^n \to \mathbf{R}^n$, $g : [0, 1] \times \mathbf{R}^n \to \mathbf{R}^{n \times m}$, $L : [0, 1] \times \mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}$ be continuous, $\varphi : \mathbf{R}^n \to \mathbf{R}$ be locally Lipschitz. Assume that $L(t, x, \cdot)$ is convex and satisfies the Tonelli condition

$$L(t, x, u) \geq \Theta(|u|), \ \ \forall\, t \in [0, 1],\ \forall\, x \in K,$$

where $\Theta : \mathbf{R}_+ \to \mathbf{R}_+$ has a superlinear growth.

Set $U(t) = \mathbf{R}^m$, $f(t, x, u) = d(t, x) + g(t, x)u$ and consider the associated constrained Bolza problem (1) - (3). The Hamiltonian $H$ is defined by

$$H(t, x, p) = \max_{u \in \mathbf{R}^m} (\langle p, d(t, x) + g(t, x)u\rangle - L(t, x, u)).$$

Let $(z, \bar{u})$ be a trajectory/control pair of (1), (2). Assume that for all $t \in [0, 1]$, $L(t, z(t), \cdot)$ is differentiable and that $\frac{\partial L}{\partial u}(t, z(t), \cdot)$ is monotone in the following sense:

$$u \neq v \in \mathbf{R}^n \implies$$
$$\langle L'_u(t, z(t), u) - L'_u(t, z(t), v), u - v\rangle > 0.$$

We claim that $H$ and $H'_p$ are continuous on $graph(z) \times \mathbf{R}^n$. Indeed, fix $t \in [0, 1]$ and $p_1,\ p_2 \in \mathbf{R}^n$. By Tonelli's condition, there exist $u_i \in \mathbf{R}^m$ such that

$$H(t, z(t), p_i) = \langle p_i, d(t, z(t))+g(t, z(t))u_i\rangle - L(t, z(t), u_i),$$

$i = 1, 2$. Then $L'_u(t, z(t), u_i) = g(t, z(t))^* p_i$ and by the monotonicity assumption this implies that $u_1 = u_2$ whenever $p_1 = p_2$. Hence for every $p \in \mathbf{R}^n$ there exists exactly one $u(t, p)$ satisfying $H(t, z(t), p) = \langle p, d(t, z(t)) + g(t, z(t))u(t, p)\rangle - L(t, z(t), u(t, p))$. From (6) we deduce that $H(t, z(t), \cdot)$ is differentiable and $H'_p(t, z(t), p) = d(t, z(t)) + g(t, z(t))u(t, p)$ for all $t \in [0, 1]$ and $p \in \mathbf{R}^n$. Using the Tonelli condition and continuity of $d$, $g$, $L$, it is not difficult to show that $u(\cdot, \cdot)$ is continuous on $[0, 1] \times \mathbf{R}^n$. So also $H$ and $H'_p$ are continuous on $graph(z) \times \mathbf{R}^n$.

Assume in addition that for every $t \in [0, 1]$ such that $z(t) \in \partial K$ we have

$$N_K(z(t)) \cap kernel(g(t, z(t))^*) = \{0\}. \quad (19)$$

Notice that this implies that for a constant $\rho > 0$ and all $t \in [0, 1]$ and $n \in N_K(z(t))$, $|g(t, z(t))^* n| \geq \rho|n|$. We claim that for every $t \in [0, 1]$ and all $p_1 \neq p_2 \in \mathbf{R}^n$ such that $p_1 - p_2 \in N_K(z(t))$ we have $u(t, p_1) \neq u(t, p_2)$. Indeed, assume that for some $p_i,\ i = 1,\ 2$, $u(t, p_1) = u(t, p_2)$ and $p_1 - p_2 \in N_K(z(t))$. Since $u(t, p_i)$ maximizes the function $\mathbf{R}^m \ni u \mapsto \langle p_i, g(t, z(t))u\rangle - L(t, z(t), u)$, we deduce that

$g(t, z(t))^*(p_1 - p_2) = 0$ and, by (19), $p_1 = p_2$ proving our claim.

The above implies that the assumption of Corollary 2.6 holds true. Therefore, if $(z, \bar{u})$ satisfies the normal constrained maximum principle with some $p,\ \psi$, then, by Corollary 2.6, $\psi$ is continuous on $(0, 1)$.

If moreover for all $r > 0$ there exists $c_r,\ k_r > 0$ such that for all $t \in [0, 1]$ and all $u, v \in rB$

$$\begin{array}{l} L'_u(t, z(t), \cdot) \text{ is } c_r - \text{Lipschitz on } B(0, r) \\ \langle L'_u(t, z(t), u) - L'_u(t, z(t), v), u - v\rangle \geq k_r|u - v|^2 \end{array} \quad (20)$$

then a stronger monotonicity condition holds true: for every $r > 0$, there exists a constant $l_r > 0$ such that for all $p,\ q \in rB$ satisfying $p - q \in N_K(z(t))$, we have

$$\left\langle H'_p(t, z(t), p) - H'_p(t, z(t), q), p - q\right\rangle \geq l_r|p - q|^2. \quad (21)$$

*Example 2.* This example corresponds to the control system studied in [15], where the authors considered a different set of constraints (time dependent inequality constraints).

Let $d,\ g,\ L,\ K,\ K_1,\ \varphi$ be as in Example 1, but this time for every $t \in [0, 1]$, $U(t)$ is a closed convex subset of $\mathbf{R}^m$ and the set-valued map $t \rightsquigarrow U(t)$ is lower semicontinuous and has closed graph. Then the Hamiltonian $H$ is given by

$$H(t, x, p) = \max_{u \in U(t)} (\langle p, d(t, x) + g(t, x)u\rangle - L(t, x, u)).$$

Assume that a trajectory/control pair $(z, \bar{u})$ satisfies the normal constrained maximum principle with some $p,\ \psi$ and that $L'_u(t, z(t), \cdot)$ satisfies the monotonicity assumption of Example 1 on $\mathbf{R}^m$.

Then for all $p \in \mathbf{R}^n$ there exists exactly one element $u(t, p) \in U(t)$ such that $H(t, z(t), p) = \langle p, d(t, z(t)) + g(t, z(t))u(t, p)\rangle - L(t, z(t), u(t, p))$. Using the Tonelli condition and continuity of $d,\ g,\ L$ it is not difficult to show that $u(\cdot, \cdot)$ is continuous on $[0, 1] \times \mathbf{R}^n$. So also $H$ and $H'_p$ are continuous on $graph(z) \times \mathbf{R}^n$. Since for almost all $t \in [0, 1]$, $z'(t) = H'_p(t, z(t), p(t) + \psi(t))$, from continuity of $H'_p$ on $graph(z) \times \mathbf{R}^n$ and boundedness of $p,\ \psi,\ z$ it follows that $z' \in L^\infty$ and so $z$ is Lipschitz continuous. By Theorem 2.5, $g(t, z(t))u(t, p(t) + \psi(t)) = g(t, z(t))u(t, p(t) + \psi(t-))$ for all $t \in (0, 1)$ and $z \in C^1$. Using arguments of convex analysis we deduce that $u(t, p(t) + \psi(t)) = u(t, p(t) + \psi(t-))$ for all $t \in (0, 1)$. So $u(\cdot, p(\cdot) + \psi(\cdot))$ is continuous on $(0, 1)$.

Set $u_0(t) = u(t, p(t) + \psi(t))$ for all $t \in (0, 1)$ and $u_0(0) = u(0, p(0) + \psi(0+))$, $u_0(1) = u(1, p(1) + \psi(1-))$ ($u_0(\cdot)$ corresponds to the control $u(\cdot)$ from the statement of Theorem 2.5). Notice that if $u(t, p) \neq u(t, q)$ whenever $0 \neq p - q \in N_K(z(t))$, then $\psi$ continuous on $(0, 1)$ by Corollary 2.6.

Assume (19) and that for all $t \in [0, 1]$

$$g(t, z(t))^*(N_K(z(t))) \cap span(N_U(u_0(t))) = \{0\}. \quad (22)$$

In the difference with Example 1, the condition imposed on $N_K(z(t))$ depends on the control $u_0(t)$.

Assumptions (19) and (22) together are of the same nature as Hypothesis (H6) in [15]. To prove continuity of $\psi$ fix $t \in (0, 1)$ such that $z(t) \in \partial K$.

Let $p,\ \ q \in \mathbf{R}^m$ be such that $p - q \in N_K(z(t))$. If $H'_p(t, z(t), p) = H'_p(t, z(t), q) = z'(t)$, then $g(t, z(t))u(t, p) = g(t, z(t))u(t, q) = g(t, z(t))u_0(t)$ and thus $u(t, p) = u(t, q) = u_0(t)$. This implies that

$$g(t, z(t))^*(p - q) \in span(N_U(u_0(t))).$$

From (22) we deduce that $p = q$. Therefore, by Corollary 2.6, $\psi$ is continuous on $(0, 1)$.

If moreover (20) holds true and for some $\varepsilon > 0$ and cones $C_\varepsilon,\ C_0$ defined by

$$C_\varepsilon := \cup_{u \in U(t) \cap B(u_0(t), \varepsilon)} N_{U(t)}(u)$$

$$C_0 := \cup_{t \in [0,1]} N_{U(t)}(u_0(t))$$

and for every $t \in [0, 1]$ we have

$$g(t, z(t))^*(N_K(z(t))) \cap \overline{C_\varepsilon - C_0} = \{0\}, \quad (23)$$

then a strong monotonicity condition holds true for the Hamiltonian $H$. Namely for every $r > 0$, there exists a constant $l_r > 0$ such that for all $t \in [0, 1]$ and $q \in rB$ satisfying $H'_p(t, z(t), q) = z'(t)$ and all $p - q \in N_K(z(t)) \cap \varepsilon B$

$$\left\langle H'_p(t, z(t), p) - H'_p(t, z(t), q), p - q\right\rangle \geq l_r|p - q|^2.$$

## III. ABSOLUTE CONTINUITY OF ADJOINT STATES AND APPLICATIONS

Let $Q \subset \mathbf{R}^n$ be a closed set. We say that its boundary $\partial Q \in C^{1,1}_{loc}$ if for every $x \in \partial Q$ there exists $\delta > 0$ such that the signed distance defined by

$$h(x) = \begin{cases} -dist(x, \partial Q) & \forall\, x \in Q \\ dist(x, \partial Q) & \text{otherwise} \end{cases}$$

is of class $C^{1,1}$ on $x + \delta B$. By [7] this is equivalent to the assumption : $\partial Q$ is a $C^{1,1}-$manifold. In this section we assume that the set of state constraints $K$ satisfies the following requirements:

$$K = \cap_{j=1}^m K_j,\ K_j \text{ is closed},\ \partial K_j \in C^{1,1}_{loc} \quad (24)$$

and

$$0 \notin co\{n_j(x) \mid j \in I(x)\},\ \ \forall\, x \in \partial K, \quad (25)$$

where $n_j(x)$ denotes the outward unit normal to $K_j$ at $x \in \partial K_j$ and $I(x)$ denotes the set of all indices that are active at $x$, i.e. $j \in I(x)$ if and only if $x \in \partial K_j$. Notice that (25) implies that for every $r > 0$ there exists $\rho_r > 0$ such that

$$\min_{v \in B} \max_{j \in I(x)} \langle n_j(x), v\rangle \leq -\rho_r,\ \ \forall\, x \in \partial K \cap rB. \quad (26)$$

Thus assumption (25) and [2, Chapter 4] imply that $K$ is sleek, $T_K(x) = \bigcap_{j=1}^m T_{K_j}(x)$ and for every $j \in I(x)$, $T_{K_j}(x) = \{v \mid \langle n_j(x), v\rangle \leq 0\}$, while for every $j \notin I(x)$, $T_{K_j}(x) = \mathbf{R}^n$ and

$$N_K(x) = \Sigma_{j \in I(x)} N_{K_j}(x) = \Sigma_{j=1}^m N_{K_j}(x). \quad (27)$$

Let $(z, \bar{u})$ be trajectory/control pair of (1), (2) and $F$ be defined by (15).

*Theorem 3.1:* Assume that $graph(F)$ is closed and let $(z, \bar{u})$ satisfy the normal constrained maximum principle with some $p$, $\psi$ and state constraints $K$ be as in (24), (25). Define $\Gamma := graph(z) \times \mathbf{R}^n$ and assume that $H$ is continuous on $\Gamma$, $H'_p$ is locally Lipschitz on $\Gamma$ and for every $r > 0$ there exists $k_r > 0$, $\rho > 0$ such that for all $t \in [0,1]$ and $q \in rB$ satisfying $H'_p(t, z(t), q) = z'(t)$ we have for all $p - q \in N_K(z(t)) \cap \rho B$

$$\left\langle H'_p(t, z(t), p) - H'_p(t, z(t), q), p - q \right\rangle \geq k_r |p - q|^2.$$

Then $\psi$ is absolutely continuous on $(0,1)$ and $z'$ is absolutely continuous on $[0,1]$. Furthermore, if $p(\cdot)$ is Lipschitz, then $\psi$ is Lipschitz on $(0,1)$ and $z'$ is Lipschitz on $[0,1]$.

The proof of the above theorem is very long and uses several ideas from [13]. It will appear in [10].

Theorem 3.1 can be used to study regularity of optimal controls and to obtain necessary optimality conditions in the form of variational inequalities.

*Corollary 3.2:* Under all the assumptions of Theorem 3.1 suppose that the supremum in (5) is attained by exactly one $u(t, x, p) \in U(t)$ and that $u(\cdot, \cdot, \cdot)$ is locally Lipschitz on $\Gamma$. Then there exists an absolutely continuous selection $u_{ac}(t) \in U(t)$ such that $u_{ac}(t) = \bar{u}(t)$ almost everywhere. Furthermore, if $p(\cdot)$ is Lipschitz on $[0,1]$, then $u_{ac}$ may be taken Lipschitz.

*Remark 3.3:* The problem investigated in [11] is so that $u(\cdot, \cdot, \cdot)$ is locally Lipschitz.

*Proof —* Set $u_{ac}(t) = u(t, z(t), p(t) + \psi(t))$ for $t \in (0,1)$, $u_{ac}(0) = u(0, z(0), p(0) + \psi(0+))$, $u_{ac}(1) = u(1, z(1), p(1) + \psi(1-))$ . Then $u_{ac}$ is absolutely continuous (respectively Lipschitz if $p$ is Lipschitz). From (12) we deduce that $u_{ac}(t) = \bar{u}(t)$ for a.e. $t \in [0,1]$ $\square$.

*Theorem 3.4:* Let a trajectory/control pair $(z, \bar{u})$ satisfy the normal constrained maximum principle $i)$, $ii)$ from the introduction with some $p$, $\psi$, where we assumed that $\varphi$ is differentiable at $(z(0), z(1))$ and $(L, f)(t, \cdot, \bar{u}(t))$ is differentiable at $z(t)$ for almost all $t \in [0,1]$. Under all the assumptions of Theorem 3.1 there exists an absolutely continuous mapping $q : [0,1] \to \mathbf{R}^n$ such that (8) - (10) hold true.

*Proof —* Set $q(t) = p(t) + \psi(t)$ for $t \in (0,1)$, $q(0) = p(0) + \psi(0+)$ and $q(1) = p(1) + \psi(1-)$. Then $q$ is continuous at the end points and therefore, by Theorem 3.1, it is absolutely continuous. From (12) we deduce (9) and from (11), (13) we obtain (10).

To prove (8) denote by $S^{n-1}$ the unit sphere in $\mathbf{R}^n$. Since $K$ is sleek, the set-valued map $s \rightsquigarrow N_K(z(s)) \cap S^{n-1}$ is upper semicontinuous. Fix $t \in [0,1)$ such that $\psi$ is differentiable at $t$. We claim that

$$\psi'(t) \in N_K(z(t)). \tag{28}$$

Indeed if $z(t) \in Int(K)$, then $\psi'(t) = 0 \in N_K(z(t))$. Assume next that $z(t) \in \partial K$ and define for all $\varepsilon > 0$ the convex cone

$$\Gamma(\varepsilon) = \bigcup_{\lambda \geq 0} \lambda \overline{co} \left( N_K(z(t)) \cap S^{n-1} + \varepsilon B \right).$$

By (25) the normal cone $N_K(z(t))$ is pointed, that is $0 \notin \overline{co} \left( N_K(z(t)) \cap S^{n-1} \right)$. Thus $\Gamma(\varepsilon)$ is closed when $\varepsilon > 0$ is small enough and $\cap_{\varepsilon > 0} \Gamma(\varepsilon) = N_K(z(t))$. Fix a sufficiently small $\varepsilon > 0$ and let $\delta > 0$ be such that for every $s \in [t, t + \delta]$, $N_K(z(s)) \cap S^{n-1} \subset N_K(z(t)) \cap S^{n-1} + \varepsilon B$. Define $\lambda(s) = |\nu(s)|$,

$$n(s) := \begin{cases} \frac{\nu(s)}{|\nu(s)|} & \text{if } \nu(s) \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Since $\nu(s) \in N_K(z(s)) \cap B$ $\mu-$a.e., for all $0 < h < \delta$

$$\psi(t+h) - \psi(t) = \int_{(t,t+h]} \nu(s) d\mu(s) =$$
$$\int_{(t,t+h]} n(s) \lambda(s) d\mu(s) \in \int_{(t,t+h]} \Gamma(\varepsilon) d\mu(s) \subset \Gamma(\varepsilon)$$

Dividing by $h$ and taking the limit yields $\psi'(t) \in \Gamma(\varepsilon)$. Since $\varepsilon > 0$ is arbitrary, we proved (28).

Let $t \in (0,1)$ be such that (4) holds true at $t$ and $\psi$ is differentiable at $t$. Then the equality

$$q'(t) = p'(t) + \psi'(t)$$

and (28) imply (8). $\square$

*Remark 3.5:* The proof that an optimal trajectory satisfies the constrained maximum principle for differentiable $\varphi$, $(L, f)(t, \cdot, \bar{u}(t))$ can be found for instance in [6].

## References


[1] ARUTYANOV A. V. & ASEEV S. M. (1997) *Investigation of the degeneracy phenomenon of the maximum principle for optimal control problems with state constraints*, SIAM J. Control Optim., 35, 930-952.

[2] AUBIN J.-P. & FRANKOWSKA H. (1990) SET-VALUED ANALYSIS, Birkhäuser, Boston.

[3] CANNARSA P. & FRANKOWSKA H. (1991) *Some characterizations of optimal trajectories in control theory*, SIAM J. Control Optim., 29, 1322-1347.

[4] CLARKE F.H. (1983) OPTIMIZATION AND NONSMOOTH ANALYSIS, Wiley-Interscience, New York.

[5] CERNEA A. & FRANKOWSKA H. (2004) *The connection between the maximum principle and the value function for optimal control problems under state constraints*, Proceedings of 43rd IEEE CDC, December 14-17, 2004, Paradise Island, Bahamas, pp. 893-899.

[6] CERNEA A. & FRANKOWSKA H. (to appear) *A connection between the maximum principle and dynamic programming for constrained control problems*, SIAM J. Control Optim.

[7] DELFOUR M. & ZOLESIO J.-P. (1994) *Shape analysis via oriented distance functions*, J. Func. Analysis, 123, 129-201.

[8] DONTCHEV A. I. & HAGER W. W. (1998) *A new approach to Lipschitz continuity in state constrained optimal control,* Systems and Control Letters, 35, 137-143.

[9] DUBOVITSKII A. Y. & MILYUTIN A. A., *Extremal problems with constraints*, USSR Comput. Math. and Math. Physiscs, **5** (1965), 1-80.

[10] FRANKOWSKA H. (to appear) *Regularity of Minimizers and of Adjoint States in Optimal Control under State Constraints*, J. Convex Analysis.

[11] GALBRAITH G. N. & VINTER R. B. (2003) *Lipschitz continuity of optimal controls for state constrained problems,* SIAM J. Control Optim., 42, 1727-1744.

[12] GAMKRELIDZE R. V. (1960) *Optimal processes with bounded phase coordinates*, Izv. Akad. Nauk, USSR Sec. Mat., 24, 315 - 356.

[13] HAGER W. W. (1979) *Lipschitz continuity for constrained processes,* SIAM J. Control Optim., 17, 321-338.

[14] MALANOWSKI K. M. (1978) *On the regularity of solutions to optimal control problems for systems linear with respect to control variable*, Arch. Auto i Telemech., 23, 227-241.

[15] SHVARTSMAN I. A. & VINTER R. B. (to appear) *Regularity properties of optimal controls for problems with time-varying state and control constraints,*

[16] VINTER R. B. (2000) OPTIMAL CONTROL, Birkhäuser, Boston, Basel, Berlin.