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Establishing iISS Property of Interconnected Systems via Parametrization of Supply Rates

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Abstract— **This paper deals with the problem of establishing integral Input-to-State Stability(iISS) of interconnected nonlinear systems. Recently, the iISS-ISS small-gain theorem has been developed to cover iISS components to which the popular ISS small-gain theorem is not applicable. This paper focuses on a small-gain-like 'condition' in the iISS-ISS small-gain 'theorem'. The purpose is to expand the capability of the iISS-ISS small-gain 'condition' by introducing new flexibility in choosing supply rates. A novel technique of parametrization of supply rates provides us with many supply rates with which a fixed single iISS-ISS small-gain condition can establish the iISS property of the interconnection.**

I. INTRODUCTION

The problem of establishing stability properties of interconnected systems has attracted a lot of attention in the field of nonlinear systems control for many decades. Classical methods which can be found in [1], [2], [3] have become popular. Recently, the ISS small-gain theorem[4], [5] has been found useful in dealing with an important class of essential nonlinearities described by the input-to-state stability(ISS) property[6], and widely exploited in stabilization of nonlinear systems. There are, however, systems for which ISS is too strong requirement. The integral input-to-state stability(iISS) property introduced by [7], [8] covers nonlinearities much broader and stronger than the ISS. More recently, the spirit of the ISS small-gain theorem has been smoothly extended to interconnected systems involving the iISS property in [9] by the author. The new stability criterion deals with iISS components which do not have finite ISS nonlinear gain in a global sense. For brevity, the author refers to the new criterion as the iISS-ISS small-gain theorem although it is not completely described in terms of gain. The technique proposed in [9] also covers cascade systems as special cases, and it generalizes a result in [10] developed for a class of cascades from a different angle.

The iISS-ISS small-gain theorem developed in [9] applies to the interconnected system of the form

$$\Sigma_1 : \ \dot{x}_1 = f_1(x_1,u_1,r_1), \quad u_1 = x_2 \tag{1}$$

$$\Sigma_2 : \ \dot{x}_2 = f_2(x_2,u_2,r_2), \quad u_2 = x_1 \tag{2}$$

It assumes the existence of $\mathbf{C}^1$ functions V_i and continuous functions $\underline{\alpha}_i, \bar{\alpha}_i, \alpha_1 \in \mathscr{K}_\infty$, $\alpha_2, \sigma_i, \sigma_{ri} \in \mathscr{K}$ such that

$$\underline{\alpha}_i(|x_i|) \leq V_i(x_i) \leq \bar{\alpha}_i(|x_i|) \tag{3}$$

$$\frac{\partial V_i}{\partial x_i} f_i(x_i,u_i,r_i) \leq -\alpha_i(|x_i|)+\sigma_i(|u_i|)+\sigma_{ri}(|r_i|) \tag{4}$$

hold. This assumption indicates that Σ_2 is only iISS, while Σ_1 is ISS. The iISS-ISS small-gain theorem is written as

$$\max_{w\in[0,s]} \frac{[c_2\sigma_2 \circ \Gamma_1(w)]^k}{\sigma_1(w)} \leq \frac{[\alpha_2 \circ \bar{\alpha}_2^{-1} \circ \underline{\alpha}_2(s)]^k}{\sigma_1(s)}, \ \forall s \in \mathbb{R}_+ \tag{5}$$

$$\Downarrow$$

(1)-(2) is iISS with respect to (r_1,r_2) and (x_1,x_2).

where $k>0$, $c_i>1$, $i=1,2$, and Γ_1 given by

$$\Gamma_1(s) = \underline{\alpha}_1^{-1} \circ \bar{\alpha}_1 \circ \alpha_1^{-1} \circ c_1\sigma_1(s)$$

is the nonlinear gain of Σ_1 [11], [12]. The iISS-ISS small-gain theorem supposes that the supply rates(See [1] for terminology) $-\alpha_i(|x_i|)+\sigma_i(|u_i|)+\sigma_{ri}(|r_i|)$ are given *a priori*, as does the ISS small-gain theorem. This paper pays particular attention to the fact that the small-gain *condition* (5) never leads us to the stability if one fails to find supply rates that are accepted by Σ_i, $i=1,2$ as in (4), and satisfy the condition (5) at the same time. Selecting supply rates fulfilling these two requirements is not an easy task.

This paper develops a useful tool for accomplishing the task. For this purpose, this paper employs the unique idea of *parametrization of supply rates* which originates from [13]. Answers to the following problem are given in this paper.

Parametrization of supply rates: Suppose that a supply rate of Σ_2 is fixed *a priori*. Find a set of *multiple* supply rates for Σ_1 with which the iISS property of the interconnected system can be proved under a fixed *single* iISS-ISS small-gain *condition* (5).

The previous study presented in [13] deals with only ISS systems. This paper extends it to iISS systems on the basis of the recent result of the iISS-ISS small-gain theorem[9]. The results of this paper provide us with more chances to come at a supply rate that fit a system.

II. ILLUSTRATIVE EXAMPLE

This section explains the goal of this paper using a simple example. Consider an interconnected system defined by

$$\Sigma_1 : \ \dot{x}_1 = -\mu x_1^4 + x_1^2\left(\frac{x_2}{x_2+1}\right)^2, \ \mu=1.1, \ x_1(0)\in\mathbb{R}_+ \tag{6}$$

$$\Sigma_2 : \ \dot{x}_2 = f(x_2,x_1,r_2), \qquad x_2(0)\in\mathbb{R}_+, \ r_2(t)\in\mathbb{R}_+ \tag{7}$$

where x_1, x_2 and r_2 are scalar variables. The set $\mathbb{R}_+$ denotes the interval $[0,\infty)$. Assume

$$f_2(0,0,0)=0, \quad f_2(0,x_1,r_2)\geq 0, \ \forall x_1\in\mathbb{R}_+, r_2\in\mathbb{R}_+$$

so that the interconnected system has an equilibrium at the origin $x=[x_1,x_2]^T=0$ for $r_2=0$, and all trajectories remain

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in the positive cone $x(t) = [x_1(t), x_2(t)]^T \in \mathbb{R}_+^2$ for all $t \in \mathbb{R}_+$. Supposed that we do not have information about f_2 except the existence of a continuously differentiable radially unbounded function $V_2 : x_2 \in \mathbb{R}_+ \to \mathbb{R}_+$ satisfying

$$\frac{dV_2(x_2)}{dt} \le -\frac{x_2}{x_2+1} + x_1 + r_2^2 \tag{8}$$

along the trajectories of Σ_2. This paper addresses the question of whether the interconnected system is iISS with respect to input r_2 and the state $x(t) = [x_1(t), x_2(t)]^T$.

The assumption (8) only ensures that Σ_2 is iISS with respect to input (x_1, r_2) and state x_2 since V_2 is an iISS Lyapunov function[7]. The system Σ_1 is ISS with respect to input x_2 and state x_1, and $V_1 = x_1$ is an ISS Lyapunov function[6], [11]. Since the system Σ_2 is not ISS, we are not able to resort to the ISS small-gain theorem[4], [5].

A stability theorem which is recently developed in [9] and called the iISS-ISS small-gain theorem is 'applicable' to the interconnection involving the iISS property. The time-derivative of $V_1 = x_1$ along the trajectories of (6) satisfies

$$\begin{aligned} \dot{V}_1(x_1) \quad &\le \quad \begin{cases} -\mu x_1^4 + a_1^{-1} x_1^4 & \text{for } x_1 \ge \sqrt{a_1}\frac{x_2}{x_2+1} \\ -\mu x_1^4 + a_1 \left(\frac{x_2}{x_2+1}\right)^4 & \\ & \text{for } x_1 \le \sqrt{a_1}\frac{x_2}{x_2+1} \end{cases} \\ &\le \quad -\alpha_1(s) + \sigma_1(s) \\ \alpha_1(s) &= (\mu - a_1^{-1}) s^4, \ \sigma_1(s) = a_1 \left(\frac{s}{s+1}\right)^4 \end{aligned} \tag{9}$$

for any $a_1 > 0$. A nonlinear gain function of Σ_1 with respect to input x_2 and state x_1 is obtained from (9) as

$$\Gamma_{1,a_1}(s) = (1+\varepsilon_1) \left(\frac{a_1}{\mu - a_1^{-1}}\right)^{1/4} \left(\frac{s}{s+1}\right)$$

for any $\varepsilon_1 > 0$ [11], [12]. The minimum of the gain function is achieved with $a_1 = 2/\mu$ as follows:

$$\Gamma_1(s) = (1+\varepsilon_1)\sqrt{\frac{2}{\mu}\frac{s}{s+1}}$$

Write (8) as follows:

$$\begin{aligned} \dot{V}_2(x_2) &\le -\alpha_2(x_2) + \sigma_2(x_1) + \sigma_{r2}(r_2) \\ \alpha_2(s) &= \frac{s}{s+1}, \ \sigma_2(s) = s, \ \sigma_{r2}(s) = s^2 \end{aligned} \tag{10}$$

We obtain

$$\sigma_2 \circ \Gamma_1(s) = (1+\varepsilon_1)\sqrt{\frac{2}{\mu}\frac{s}{s+1}}, \quad \frac{[\sigma_2(s)]^k}{\alpha_1(s)} = \frac{2s^{k-4}}{\mu}$$

Thus, there exist $\varepsilon_1, \varepsilon_2 > 0$ such that

$$(1+\varepsilon_2)\sigma_2 \circ \Gamma_1(s) \le \alpha_2(s), \quad \forall s \in \mathbb{R}_+ \tag{11}$$

holds for $\mu > 2$. There also exists $k > 0$ such that

$$\frac{[\sigma_2(s)]^k}{\alpha_1(s)} \text{ is non-decreasing.} \tag{12}$$

According to the iISS-ISS small-gain theorem[9], the contraction condition (11) with the help of the non-decreasing property (12) could imply iISS of the interconnected system if $\mu > 2$ held. However, the theorem does not guarantee the stability for $\mu = 1.1$ given in (6).

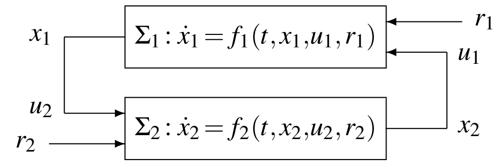

Fig. 1. Feedback interconnected system Σ

If we could use

$$\hat{V}_1(x_1) = \int_0^{V_1(x_1)} \frac{1}{s^2} ds \tag{13}$$

we would obtain

$$\begin{aligned} \dot{\hat{V}}_1(x_1) &= -\mu x_1^2 + \left(\frac{x_2}{x_2+1}\right)^2 \\ &= -\alpha_1(s) + \sigma_1(s), \ \alpha_1(s) = \mu s^2, \ \sigma_1(s) = \left(\frac{s}{s+1}\right)^2 \\ \Gamma_1(s) &= (1+\varepsilon_1)\sqrt{\frac{1}{\mu}\frac{s}{s+1}} \end{aligned} \tag{14}$$

along the trajectories of (6). Then, there exist $\varepsilon_1, \varepsilon_2 > 0$ and $k > 0$ such that the contraction condition (11) holds for $\mu > 1$, and the property (12) holds. The function in (13) is, however, not integrable, so that $\hat{V}_1(x_1)$ in (13) is not qualified as a Lyapunov function. In addition, a function of the form

$$\hat{V}_1(x_1) = \int_0^{V_1(x_1)} \beta(s) ds$$

with a positive-valued function $\beta(s)$ decreasing faster than or as fast as $1/s^2$ toward ∞ is not radially unbounded, so that it cannot be used for proving global properties[14]. Indeed, the gain in (14) cannot be justified by (13). Thus, the iISS-ISS small-gain theorem does not bring the stability of the interconnected system to us.

This example suggests that the supply rate in (9) for Σ_1 is not a right choice for establishing the stability of the interconnected system. The iISS-ISS small-gain theorem is surely better than the ISS small-gain theorem in the sense that it covers iISS systems. However, we cannot appreciate it truly unless we select a successful supply rate for an individual system. It is also indicated that direct use of the technique of changing supply rates introduced in [14] does not help much for this example. This paper will demonstrate that the iISS-ISS small-gain 'condition' held for the ficticious functions in (14) is able to lead us to the iISS property of the interconnected system. In other words, the usage of the ficticious functions chosen in (14) will be justified without relying on the forbidden function in (13).

III. Parametrization of supply rates

Consider the nonlinear interconnected system Σ shown in Fig.1. Suppose that subsystems Σ_1 and Σ_2 are described by

$$\Sigma_1 : \ \dot{x}_1 = f_1(t, x_1, u_1, r_1) \tag{15}$$

$$\Sigma_2 : \ \dot{x}_2 = f_2(t, x_2, u_2, r_2) \tag{16}$$

These two systems are connected each other through $u_1 = x_2$ and $u_2 = x_1$. Assume that $f_1(t,0,0,0) = 0$ and $f_2(t,0,0,0) = 0$ hold for all $t \in [t_0,\infty)$, $t_0 \geq 0$. The functions f_1 and f_2 are supposed to be piecewise continuous in t, and locally Lipschitz in the other arguments. The state vector of the interconnected system Σ is $x = [x_1^T, x_2^T]^T \in \mathbb{R}^n$ where $x_i \in \mathbb{R}^{n_i}$ is the state of Σ_i. The exogenous inputs $r_1 \in \mathbb{R}^{m_1}$ and $r_2 \in \mathbb{R}^{m_2}$ are packed into a single vector $r = [r_1^T, r_2^T]^T \in \mathbb{R}^m$. We make the following assumption.

Assumption 1: There exist a $\mathbf{C}^1$ function $V_2 : \mathbb{R}_+ \times \mathbb{R}^{n_2} \to \mathbb{R}_+$, continuous functions $\alpha_2, \sigma_2, \sigma_{2r} \in \mathcal{K}$ such that

$$\underline{\alpha}_2(|x_2|) \leq V_2(t,x_2) \leq \bar{\alpha}_2(|x_2|) \tag{17}$$

$$\frac{\partial V_2}{\partial t} + \frac{\partial V_2}{\partial x_2} f_2(t,x_2,u_2,r_2) \leq -\alpha_2(|x_2|) + \sigma_2(|u_2|) + \sigma_{r2}(|r_2|) \tag{18}$$

hold for all $x_2 \in \mathbb{R}^{n_2}$, $u_2 \in \mathbb{R}^{n_1}$, $r_2 \in \mathbb{R}^{m_2}$ and $t \in \mathbb{R}_+$ with some $\underline{\alpha}_2$, $\bar{\alpha}_2 \in \mathcal{K}_\infty$.

The following are the main results which offer techniques of *parametrization of supply rates* for interconnected systems involving iISS properties.

Theorem 1: Suppose that the system Σ_2 satisfies Assumption 1. Assume that real numbers $c_i > 1$, $i = 1,2$, and $k > 0$ and functions

$$\alpha_1 \in \mathcal{K}_\infty, \quad \sigma_1 \in \mathcal{K} \setminus \mathcal{K}_\infty \tag{19}$$

satisfy

$$\max_{w\in[0,s]} \frac{[c_2\sigma_2 \circ \underline{\alpha}_1^{-1} \circ \bar{\alpha}_1 \circ \alpha_1^{-1} \circ c_1\sigma_1(w)]^k}{\sigma_1(w)} \leq \frac{[\alpha_2 \circ \bar{\alpha}_2^{-1} \circ \underline{\alpha}_2(s)]^k}{\sigma_1(s)}, \ \forall s \in \mathbb{R}_+ \tag{20}$$

If there exist a continuous function $\hat{\lambda} : \mathbb{R}_+ \to \mathbb{R}$ and a $\mathbf{C}^1$ function $V_1 : \mathbb{R}_+ \times \mathbb{R}^{n_1} \to \mathbb{R}$ such that

$$\hat{\lambda}(s) > 0, \ \forall s \in (0,\infty) \tag{21}$$

$$\lim_{s\to 0^+} \frac{[\sigma_2 \circ \underline{\alpha}_1^{-1}(s)]^k}{[\alpha_1 \circ \bar{\alpha}_1^{-1}(s)]\hat{\lambda}(s)} < \infty \tag{22}$$

$$\underline{\alpha}_1(|x_1|) \leq V_1(t,x_1) \leq \bar{\alpha}_1(|x_1|) \tag{23}$$

$$\frac{\partial V_1(t,x_1)}{\partial t} + \frac{\partial V_1(t,x_1)}{\partial x_1} f_1(t,x_1,u_1,r_1) \leq \hat{\lambda}(V_1(t,x_1))\left[-\alpha_1(|x_1|) + \sigma_1(|u_1|) + \sigma_{r1}(|r_1|)\right] \tag{24}$$

hold for all $x_1 \in \mathbb{R}^{n_1}$, $u_1 \in \mathbb{R}^{n_2}$, $r_1 \in \mathbb{R}^{m_1}$ and $t \in \mathbb{R}_+$ with some $\underline{\alpha}_1$, $\bar{\alpha}_1 \in \mathcal{K}_\infty$ and some $\sigma_{r1} \in \mathcal{K}$, then the interconnected system Σ is iISS with respect to input r and state x.

Theorem 2: Suppose that the system Σ_2 satisfies Assumption 1. Assume that real numbers $c_i > 1$, $i = 1,2$, and $k > 0$ and functions

$$\alpha_1 \in \mathcal{K}_\infty, \quad \sigma_1 \in \mathcal{K}_\infty \tag{25}$$

satisfy (20). If there exist a continuous function $\hat{\lambda} : \mathbb{R}_+ \to \mathbb{R}$ and a $\mathbf{C}^1$ function $V_1 : \mathbb{R}_+ \times \mathbb{R}^{n_1} \to \mathbb{R}$ such that

$$\int_1^\infty \max_{w\in[0,s]} \left\{ \frac{[\sigma_2 \circ \underline{\alpha}_1^{-1}(w)]^k}{\alpha_1 \circ \bar{\alpha}_1^{-1}(w)} \right\} \frac{1}{\hat{\lambda}(s)} ds = \infty \tag{26}$$

and (21)-(24) hold for all $x_1 \in \mathbb{R}^{n_1}$, $u_1 \in \mathbb{R}^{n_2}$, $r_1 \in \mathbb{R}^{m_1}$ and $t \in \mathbb{R}_+$ with some $\underline{\alpha}_1$, $\bar{\alpha}_1 \in \mathcal{K}_\infty$ and some $\sigma_{r1} \in \mathcal{K}$, then the interconnected system Σ is iISS with respect to input r and state x.

The system Σ_2 satisfying Assumption 1 is iISS and it does not require the ISS property. The condition (20) is referred to as the iISS-ISS small-gain condition in [9]. Theorem 1 incorporates new flexibility $\hat{\lambda}$ into the stability test based on the iISS-ISS small-gain *condition*. The function $\hat{\lambda}$ in (24) provides an adaptable parameter in selecting a supply rate of Σ_1 to establish the iISS property of the interconnected system. The flexible function $\hat{\lambda}$ allows us to *scale* an initial supply rate $-\alpha_1(|x_1|) + \sigma_1(|u_1|) + \sigma_{r1}(|r_1|)$ which is chosen such that the iISS-ISS small-gain condition is fulfilled. The freedom of $\hat{\lambda}$ can be utilized to have (24) fulfilled by a given system Σ_1. The flexibility of $\hat{\lambda}$ in the supply rate offers more chances to come at a supply rate that fit the system Σ_1.

The iISS property of interconnected systems can be established by Theorem 1 and 2 without constructing Lyapunov functions of the closed-loop systems. When we need a Lyapunov function, a formula is available in the next section.

Remark 1: Theorem 1 and 2 include the iISS-ISS small-gain theorem developed in [9] as a special case. In fact, when we pick $\hat{\lambda}(s) = 1$, the condition (22) and (26) are always satisfied. To see this, we suppose that (20) holds. The left hand side of (20) is non-decreasing due to the maximization. The right hand side takes finite positive value at all $s \in (0,\infty)$. These two facts imply

$$\frac{[c_2\sigma_2 \circ \underline{\alpha}_1^{-1} \circ \bar{\alpha}_1 \circ \alpha_1^{-1}(s)]^k}{s} < \infty, \quad \forall s \in [0,h)$$

for some $h > 0$. In the case of $\sigma_1 \in \mathcal{K}_\infty$ we can use $h = \infty$. Thus, (22) is fulfilled for any constant $\hat{\lambda}$. On the other hand, that constraint (26) is fulfilled for any constant $\hat{\lambda}$ since

$$\frac{[\sigma_2 \circ \underline{\alpha}_1^{-1}(w)]^k}{\alpha_1 \circ \bar{\alpha}_1^{-1}(w)} > 0, \quad w \in (0,\infty)$$

is implied by σ_2 ,α_1, $\underline{\alpha}_1^{-1}$, $\bar{\alpha}_1^{-1} \in \mathcal{K}$.

Remark 2: Theorem 1 and Theorem 2 allow the functions $\hat{\lambda}$ to be much more flexible than ones we can obtained from techniques in [14], [12]. Indeed, the function $\int_0^{V_1} 1/\hat{\lambda}(s)ds$ does not guaranteed to be integrable and radially unboundeded. Theorem 1 and Theorem 2 are developed without using $\int_0^{V_1} 1/\hat{\lambda}(s)ds$ in the construction of Lyapunov functions. This paper extends the idea proposed in [13] to iISS systems.

Remark 3: The assumption (20) can be replaced by simpler assumptions. In fact, if there exists a constant $k > 0$ such that at least one of

$$\frac{[\sigma_2 \circ \underline{\alpha}_1^{-1}(s)]^k}{\alpha_1 \circ \bar{\alpha}_1^{-1}(s)} \text{ is non-decreasing} \tag{27}$$

$$\frac{[\alpha_2 \circ \bar{\alpha}_2^{-1}(s)]^k}{\sigma_1 \circ \underline{\alpha}_2^{-1}(s)} \text{ is non-decreasing} \tag{28}$$

holds and

$$c_2\sigma_2 \circ \underline{\alpha}_1^{-1} \circ \bar{\alpha}_1 \circ \alpha_1^{-1} \circ c_1\sigma_1(s) \leq \alpha_2 \circ \bar{\alpha}_2^{-1} \circ \underline{\alpha}_2(s), \ \forall s \in \mathbb{R}_+ \tag{29}$$

is satisfied, the assumption (20) is fulfilled.

Remark 4: An easy way to pick an initial supply rate $-\alpha_1(|x_1|)+\sigma_1(|u_1|)+\sigma_{r1}(|r_1|)$ fulfilling the iISS-ISS small-gain condition (20) is to take copies of functions in the supply rates of Σ_2. If we choose σ_1 and α_1 such that

$$\alpha_1(s)=v_1\sigma_2(s), \quad \sigma_1(s)=v_2\alpha_2(s), \quad v_i>0 \tag{30}$$

holds, the condition (20) is reduced to (29) and the two constraints (22) and (26) can be replaced by

$$\lim_{s\to 0^+}\frac{[\sigma_2\circ\underline{\alpha}_1^{-1}(s)]^{k-1}}{\hat{\lambda}(s)}<\infty$$

$$\int_1^\infty\frac{[\sigma_2\circ\underline{\alpha}_1^{-1}(s)]^{k-1}}{\hat{\lambda}(s)}ds=\infty$$

respectively. These facts can be verified from Remark 6.

Remark 5: It is worth noting that the condition (22) is satisfied for all $k>l$ if it is satisfied for $k=l$. The conditions (26) and (20) have the same property. It is also worth noticing that there always exists $k>0$ such that (22) holds if each function appearing in (22) satisfies a Lipschitz condition of some order at $s=0$. There always exists $k>0$ such that (26) holds if each function there satisfies a Lipschitz condition of some order toward $s\to\infty$.

IV. ILLUSTRATIVE EXAMPLE CONTINUED

Using the development in the previous section, we can give an affirmative answer to our question posed in Section II. For the simple choice $V_1=x_1$, we obtain

$$\dot{V}_1=x_1^2\left\{-\mu x_1^2+\left(\frac{x_2}{x_2+1}\right)^2\right\}$$

which fulfills (24) with

$$\hat{\lambda}(s)=s^2,\ \alpha_1(s)=\mu s^2,\ \sigma_1(s)=\left(\frac{s}{s+1}\right)^2,\ r_1(t)\equiv 0 \tag{31}$$

Thus, the inequality (29) and the property (27) become

$$c_2\sigma_2\circ\alpha_1^{-1}\circ c_1\sigma_1(s)\leq\alpha_2(s),\ \forall s\in\mathbb{R}_+ \tag{32}$$

and (12), respectively. It is verified that (12) holds for $k\geq 2$, and there exist $c_1,c_2>1$ such that (32) holds for $\mu>1$. Theorem 1 concludes that the interconnected system (6)-(7) is iISS with respect to input r_2 and state $x(t)=[x_1(t),x_2(t)]^T$.

It is important that Theorem 1 and Theorem 2 enable us to obtain the iISS property without constructing a Lyapunov function of the overall system explicitly. It is, however, possible to construct a Lyapunov function explicitly whenever one prefers it. For instance, if Σ_2 achieves (8) with $V_2(x_2)=x_2$, an iISS Lyapunov function of the interconnected system is

$$V_{cl}(x_1,x_2)=x_1+\int_0^{V_2(x_2)}2\left(\frac{s}{s+1}\right)^3ds \tag{33}$$

which is obtained with $q=4$ from a formula presented in the next section. Indeed, the time-derivative of this function V_{cl} along the trajectories of (6)-(7) is computed as

$$\begin{aligned}\dot{V}_{cl}(x_1,x_2)\leq&-\mu x_1^4+x_1^2\left(\frac{x_2}{x_2+1}\right)^2\\&-2\left(\frac{x_2}{x_2+1}\right)^4+2x_1\left(\frac{x_2}{x_2+1}\right)^3+2\left(\frac{x_2}{x_2+1}\right)^3r_2^2\\\leq&-\rho_I(x_1,x_2)+br_2^8\end{aligned} \tag{34}$$

It can be verified that (34) holds for some positive definite function $\rho_I(x_1,x_2)$ and some positive constant b if and only if $\mu>1$ holds. Since $V_{cl}(x_1,x_2)$ defined by (33) is positive definite and radially unbounded, the fulfillment of (34) ensures the iISS property of (6)-(7).

V. PROOFS

A. A key lemma

Lemma 1: Suppose that continuous functions $V_i:(t,x_i)\in\mathbb{R}_+\times\mathbb{R}^{n_i}\to\mathbb{R}_+$, $i=1,2$, satisfy

$$\underline{\alpha}_i(|x_i|)\leq V_i(t,x_i)\leq\bar{\alpha}_i(|x_i|),\ \forall x_i\in\mathbb{R}^{n_i},t\in\mathbb{R}_+ \tag{35}$$

for some $\underline{\alpha}_i$, $\bar{\alpha}_i\in\mathscr{K}_\infty$. Let ρ_i, $i=1,2$, be

$$\rho_i(x_i,u_i,r_i)=-\alpha_i(|x_i|)+\sigma_i(|u_i|)+\sigma_{ri}(|r_i|) \tag{36}$$

consisting of continuous functions satisfying

$$\alpha_1\in\mathscr{K}_\infty,\quad \alpha_2\in\mathscr{K}, \tag{37}$$

$$\sigma_1\in\mathscr{K},\quad \sigma_2\in\mathscr{K}, \tag{38}$$

$$\sigma_{r1}(s)\geq 0,\quad \sigma_{r2}(s)\geq 0\quad \forall s\in\mathbb{R}_+ \tag{39}$$

If there exist $c_i>1$, $i=1,2$ and $k>0$ such that (20) holds, there exists a continuous function $\rho_e(x,r)$ of the form

$$\rho_e(x,r)=-\alpha_{cl}(|x|)+\sigma_{cl}(|r|) \tag{40}$$

$$\alpha_{cl}\in\mathscr{K},\qquad \sigma_{cl}(s)\geq 0,\ \forall s\in\mathbb{R}_+ \tag{41}$$

such that

$$\begin{aligned}&\lambda_1(V_1(t,x_1))\rho_1(x_1,x_2,r_1)\\&\quad+\lambda_2(V_2(t,x_2))\rho_2(x_2,x_1,r_2)\leq\rho_e(x,r_1,r_2),\\&\quad\forall x_1\in\mathbb{R}^{n_1},x_2\in\mathbb{R}^{n_2},r_1\in\mathbb{R}^{m_1},r_2\in\mathbb{R}^{m_2},t\in\mathbb{R}_+\end{aligned} \tag{42}$$

holds with non-decreasing continuous functions $\lambda_i:s\in\mathbb{R}_+\to\mathbb{R}_+$, $i=1,2$ satisfying

$$\lambda_1(s)=\max_{w\in[0,s]}c_1c_2^q\delta^{\frac{q}{q+1}}\frac{[\sigma_2\circ\underline{\alpha}_1^{-1}(w)]^q}{\alpha_1\circ\bar{\alpha}_1^{-1}(w)}\quad\forall s\in[0,d) \tag{43}$$

$$\lambda_1(s)\geq\max_{w\in[0,s]}c_1c_2^q\delta^{\frac{q}{q+1}}\frac{[\sigma_2\circ\underline{\alpha}_1^{-1}(w)]^q}{\alpha_1\circ\bar{\alpha}_1^{-1}(w)}\quad\forall s\in[d,\infty) \tag{44}$$

$$d=\lim_{s\to\infty}\bar{\alpha}_1\circ\alpha_1^{-1}\circ c_1\sigma_1(s)$$

$$\lambda_2(s)=q[\delta^{\frac{1}{q+1}}\alpha_2\circ\bar{\alpha}_2^{-1}(s)]^{q-1}\ \forall s\in\mathbb{R}_+ \tag{45}$$

where δ and q are any constants satisfying

$$1>\delta>0,\quad c_2^q>[\delta(c_1-1)]^{-1},\quad q\geq k,\quad q>1 \tag{46}$$

Furthermore, if σ_{r1} and σ_{r2} are class $\mathscr{K}$ functions, the inequality (42) holds with a class $\mathscr{K}$ function σ_{cl}.

It is worth noting that there always exist δ and q such that (46) holds. In the case of $\sigma_1 \in \mathscr{K}_\infty$, the function $\lambda_1(s)$ is given completely for all $s \in \mathbb{R}_+$ by (43), In the case of $\sigma_1 \in \mathscr{K} \setminus \mathscr{K}_\infty$, the segment of $\lambda_1(s)$ on the interval $s \in [d, \infty)$ can be any non-decreasing continuous curve satisfying (44) and connected continuously to (43) at $s = d$. The function $\lambda_1(s)$ given as (43) fulfills $\lim_{s\to 0^+} \lambda_1(s) < \infty$, which is guaranteed by (20). In fact, the left hand side of (20) is a non-decreasing continuous function due to the maximization. The right hand side of (20) takes finite positive value at all $s \in (0, \infty)$. In this situation, the inequality of (20) implies

$$\lim_{s\to 0^+} \frac{[\sigma_2 \circ \underline{\alpha}_1^{-1}(s)]^k}{\alpha_1 \circ \bar{\alpha}_1^{-1}(s)} < \infty \tag{47}$$

Hence, the functions $\lambda_i(s)$, $i = 1, 2$ given in (43)-(45) are non-decreasing continuous and $\lim_{s\to 0^+} \lambda_i(s) < \infty$ is satisfied.

Lemma 1 is proved as follows. Define $p > 1$ as

$$1 = (1/p) + (1/q)$$

Due to (46) and $c_2 > 1$, there exists $\mu > 1$ such that

$$\left(\frac{c_2}{\mu}\right)^q \geq \frac{1}{\delta(c_1 - 1)} \tag{48}$$

holds. If $\sigma_{r1}(s)$ is not identically zero, pick $\bar{\delta}$ satisfying

$$\delta^{\frac{1}{q+1}} < \bar{\delta} < 1 \tag{49}$$

There exists $\tau_r > 1$ such that

$$1 - \frac{1}{c_1} - \frac{1}{\tau_r} \geq \bar{\delta}\left(1 - \frac{1}{c_1}\right) \tag{50}$$

is satisfied. If $\sigma_{r1}(s) \equiv 0$ holds, let $\bar{\delta} = 1$ and the rest of the proof does not require τ_r. If $\sigma_{r2}(s)$ is not identically zero, there exists $\mu_r > 1$ such that

$$1 \geq \frac{1}{\mu^p} + \frac{1}{\mu_r^p} \tag{51}$$

holds. In the case of $\sigma_{r2}(s) \equiv 0$, the parameter μ_r does not appear in the rest of the proof. Define the following class $\mathscr{K}$ functions.

$$\theta_1(s) = \bar{\alpha}_1 \circ \alpha_1^{-1} \circ c_1 \sigma_1(s)$$
$$\theta_{r1}(s) = \bar{\alpha}_1 \circ \alpha_1^{-1} \circ \tau_r \sigma_{r1}(s)$$

Suppose that $\lambda_1, \lambda_2 : s \in \mathbb{R}_+ \to \mathbb{R}_+$ satisfy (43)-(45) which are non-decreasing continuous functions. Combining calculations in individual cases separated by $\alpha_1(|x_1|) \geq c_1\sigma_1(|x_2|)$, $\alpha_1(|x_1|) < c_1\sigma_1(|x_2|)$, $\alpha_1(|x_1|) \geq \tau_r\sigma_{r1}(|r_1|)$ and $\alpha_1(|x_1|) < \tau_r\sigma_{r1}(|r_1|)$, we obtain

$$\begin{aligned}&\lambda_1(V_1(t,x_1))\{-\alpha_1(|x_1|) + \sigma_1(|x_2|) + \sigma_{r1}(|r_1|)\}\\ &\leq \bar{\delta}\left(-1 + \frac{1}{c_1}\right)\lambda_1(V_1(t,x_1))\alpha_1(|x_1|)\\ &\qquad + \lambda_1(\theta_1(|x_2|))\sigma_1(|x_2|) + \lambda_1(\theta_{r1}(|r_1|))\sigma_{r1}(|r_1|)\end{aligned}$$

Using Young's inequality

$$xy \leq \frac{1}{p}\left|\frac{x}{a}\right|^p + \frac{1}{q}|ay|^q \quad ,\forall x, y \in \mathbb{R}, a \neq 0$$

we obtain

$$\begin{aligned}&\lambda_2(V_2(t,x_2))\{-\alpha_2(|x_2|) + \sigma_2(|x_1|) + \sigma_{r2}(|r_2|)\}\\ &\leq -\lambda_2(V_2(t,x_2))\alpha_2(|x_2|) + q\left[\frac{1}{p}\left(\frac{1}{q\mu}\lambda_2(V_2(t,x_2))\right)^p + \right.\\ &\left.\frac{\mu^q}{q}\sigma_2(|x_1|)^q + \frac{1}{p}\left(\frac{1}{q\mu_r}\lambda_2(V_2(t,x_2))\right)^p + \frac{\mu_r^q}{q}\sigma_{r2}(|r_2|)^q\right]\end{aligned}$$

Define $\alpha_{cl}(s)$ and $\sigma_{cl}(s)$ as follows:

$$\begin{aligned}\alpha_{cl}(s) &= \min_{s=|x|}\left\{(\bar{\delta} - \delta^{\frac{1}{q+1}})\frac{c_1 - 1}{c_1}\lambda_1(\underline{\alpha}_1(|x_1|))\alpha_1(|x_1|)\right.\\ &\qquad \left. + (1 - \delta^{\frac{1}{q+1}})\lambda_2(\underline{\alpha}_2(|x_2|))\alpha_2(|x_2|)\right\}\\ \sigma_{cl}(s) &= \max_{s=|r|}\{\lambda_1(\theta_{r1}(|r_1|))\sigma_{r1}(|r_1|) + \mu_r^q\sigma_{r2}(|r_2|)^q\}\end{aligned}$$

From (49) it follows that $\alpha_{cl} \in \mathscr{K}$. Moreover, the function σ_{cl} satisfies $\sigma_{cl}(s) \geq 0$ for all $s \in \mathbb{R}_+$. It is class $\mathscr{K}$ if $\sigma_{ri} \in \mathscr{K}$ holds for $i = 1, 2$. The inequality (42) is achieved if the pair of λ_1 and λ_2 solves

$$\begin{aligned}&-\delta^{\frac{1}{q+1}}\frac{c_1 - 1}{c_1}\lambda_1(s)\alpha_1(\bar{\alpha}_1^{-1}(s))\\ &\qquad + \mu^q[\sigma_2(\underline{\alpha}_1^{-1}(s))]^q \leq 0, \ \forall s \in \mathbb{R}_+ \end{aligned}\tag{52}$$

$$\begin{aligned}&\frac{1}{p}\left(\frac{1}{q}\right)^{p-1}\lambda_2(s)^p - \delta^{\frac{1}{q+1}}\lambda_2(s)\alpha_2(\bar{\alpha}_2^{-1}(s))\\ &\qquad + \lambda_1(\theta_1(\underline{\alpha}_2^{-1}(s)))\sigma_1(\underline{\alpha}_2^{-1}(s)) \leq 0, \ \forall s \in \mathbb{R}_+\end{aligned}\tag{53}$$

The inequalities (50) and (51) are used to obtain (53). The inequality (52) holds if and only if

$$\frac{\mu^q c_1[\sigma_2(\underline{\alpha}_1^{-1}(s))]^q}{\delta^{\frac{1}{q+1}}(c_1 - 1)\alpha_1(\bar{\alpha}_1^{-1}(s))} \leq \lambda_1(s), \quad \forall s \in \mathbb{R}_+ \tag{54}$$

is achieved. Using λ_2 given by (45) in (53), we obtain

$$\lambda_1(\theta_1(s))\sigma_1(s) \leq [\delta^{\frac{1}{q+1}}\alpha_2(\bar{\alpha}_2^{-1}(\underline{\alpha}_2(s)))]^q, \ \forall s \in \mathbb{R}_+$$

Hence, the pair of (52) and (53) holds if λ_1 satisfies (54) and

$$\lambda_1(s)\sigma_1(\theta_1^{-1}(s)) \leq [\delta^{\frac{1}{q+1}}\alpha_2(\bar{\alpha}_2^{-1}(\underline{\alpha}_2(\theta_1^{-1}(s))))]^q \quad \forall s \in [0, d) \tag{55}$$

The choice of λ_1 given in (43)-(44) fulfills (54) since we have (48). The function λ_1 satisfies (55) if

$$\max_{w\in[0,s]}\frac{[c_2\sigma_2(\underline{\alpha}_1^{-1}(\theta_1(w)))]^q}{\alpha_1(\bar{\alpha}_1^{-1}(\theta_1(w)))} \leq \frac{[\alpha_2(\bar{\alpha}_2^{-1}(\underline{\alpha}_2(s)))]^q}{c_1\sigma_1(s)} \quad \forall s \in \mathbb{R}_+ \tag{56}$$

The inequality (20) implies (29), so that

$$\max_{w\in[0,s]}\left[c_2\sigma_2 \circ \underline{\alpha}_1^{-1} \circ \bar{\alpha}_1 \circ \alpha_1^{-1} \circ c_1\sigma_1(w)\right] \leq \alpha_2 \circ \bar{\alpha}_2^{-1} \circ \underline{\alpha}_2(s)$$

holds. The inequality (20) implies that (20) still holds even if k is replaced by $q > k$. Therefore, the condition (20) ensures (56). Hence, the functions λ_1 and λ_2 given in (43)-(44) and (45) achieve (52) and (53).

Remark 6: In the case that

$$\alpha_1(s) = \nu_1\sigma_2(s), \quad \sigma_1(s) = \nu_2\alpha_2(s), \quad \nu_i > 0 \tag{57}$$

holds, the condition (20) and the formulas (43)-(44) can be replaced by (29) and

$$\lambda_1(s) = \frac{c_1 c_2^q \delta^{\frac{q}{q+1}}}{\nu_1}[\sigma_2 \circ \underline{\alpha}_1^{-1}(s)]^{q-1},\ \forall s \in [0,d) \quad (58)$$

$$\lambda_1(s) \geq \frac{c_1 c_2^q \delta^{\frac{q}{q+1}}}{\nu_1}[\sigma_2 \circ \underline{\alpha}_1^{-1}(s)]^{q-1},\ \forall s \in [d,\infty) \quad (59)$$

respectively. These facts are true since

$$-\delta^{\frac{1}{q+1}}\frac{c_1-1}{c_1}\lambda_1(s)\nu_1 + \mu^q[\sigma_2(\underline{\alpha}_1^{-1}(s))]^{q-1} \leq 0, \forall s \in \mathbb{R}_+$$

can replace (52).

B. Proof of Theorem 1

Since $k > 0$ satisfies (22), the function

$$\frac{1}{\hat{\lambda}(s)} \cdot \max_{w\in[0,s]} \frac{[\sigma_2 \circ \underline{\alpha}_1^{-1}(w)]^l}{\alpha_1 \circ \bar{\alpha}_1^{-1}(w)} \quad (60)$$

is guaranteed to be continuous on $\mathbb{R}_+$ for all $l \geq k$. Suppose that q and δ satisfying (46). Let $\eta(s)$ denote

$$\eta(s) = \max_{w\in[0,s]} c_1 c_2^q \delta^{\frac{q}{q+1}} \frac{[\sigma_2 \circ \underline{\alpha}_1^{-1}(w)]^q}{\alpha_1 \circ \bar{\alpha}_1^{-1}(w)}$$

which is non-decreasing and continuous on $\mathbb{R}_+$. Define $\lambda_1(s)$ by (43) and

$$\lambda_1(s) = \max\left\{\frac{\eta(d)}{\hat{\lambda}(d)} \max_{w\in[d,s]}\{\hat{\lambda}(w)\}, \eta(s)\right\},\ \forall s \in [d,\infty) \quad (61)$$

Then, the function $\lambda_1(s)$ is continuous on $\mathbb{R}_+$ and meets (44). Define $\lambda_0(s)$ by

$$\lambda_0(s) = \lambda_1(s)/\hat{\lambda}(s) \quad (62)$$

The function is continuous on $\mathbb{R}_+$ and satisfies

$$\lim_{s\to 0^+} \lambda_0(s) < \infty, \quad \lim_{s\to\infty} \lambda_0(s) > 0$$
$$0 < \lambda_0(s) < \infty, \quad \forall s \in (0,\infty)$$

Since $\alpha_2 \circ \bar{\alpha}_2^{-1} \in \mathcal{K}$ and $q > 1$ hold, the function $\lambda_2(s)$ given by (45) is a class $\mathcal{K}$ function. Therefore, the $\mathbf{C}^1$ function

$$V_{cl}(t,x) = \int_0^{V_1(t,x_1)} \lambda_0(s)ds + \int_0^{V_2(t,x_2)} \lambda_2(s)ds \quad (63)$$

satisfies

$$\underline{\alpha}_{cl}(|x|) \leq V_{cl}(t,x) \leq \bar{\alpha}_{cl}(|x|),\ \forall x \in \mathbb{R}^n, t \in \mathbb{R}_+ \quad (64)$$

for some $\underline{\alpha}_{cl}$, $\bar{\alpha}_{cl} \in \mathcal{K}_\infty$. Due to (62) and Lemma 1, there exist $\alpha_{cl}, \sigma_{cl} \in \mathcal{K}$ such that

$$\frac{dV_{cl}}{dt} \leq -\alpha_{cl}(|x|) + \sigma_{cl}(|r|),\ \forall x \in \mathbb{R}^n, r \in \mathbb{R}^m, t \in \mathbb{R}_+$$

holds along the trajectories of the interconnected system Σ.

C. Proof of Theorem 2

Define $\lambda_0(s)$ as (62) where $\lambda_1(s)$ is determined completely by (43) due to $\sigma_1 \in \mathcal{K}_\infty$. The requirements (22) and (26) guarantee that the $\mathbf{C}^1$ function V_{cl} given by (63) satisfies (64) for some $\underline{\alpha}_{cl}$, $\bar{\alpha}_{cl} \in \mathcal{K}_\infty$. The rest is the same as Theorem 1.

VI. Concluding remarks

This paper has expanded the capability of the iISS-ISS small-gain condition by introducing the technique of parametrization of supply rates. The parametrization has provided a set of supply rates with which a fixed single small-gain condition can establish iISS property of an interconnected system. A first attempt to introduce a parametrization technique for iISS systems was made by a preceding paper[15]. The reformulation done by this paper has enabled us to achieve the following significant points which are not possible in the previous work[15].

- This paper has dealt with iISS property of interconnected systems which is stronger than the asymptotic stability in [15].
- This paper has introduced more flexibility into the parametrization than the previous paper [15]. More precisely, we do not have to take copies of functions in the supply rate of Σ_2 in choosing a supply rate of Σ_1.
- The results of this paper are exactly identical to the iISS-ISS small-gain theorem when the free parameter $\hat{\lambda}(s)$ is fixed as $\hat{\lambda} = 1$. In contrast, the previous results in [15] do not agree with the iISS-ISS small-gain theorem even if $\hat{\lambda} = 1$ is taken. This paper has succeeded in putting freedom into the iISS-ISS small-gain theorem literally.

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