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Robust Time Optimal Obstacle Avoidance Problem for Constrained Discrete Time Systems

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Abstract— **This paper presents algorithms for the computation of the set of states that can be robustly steered in a finite number of steps via state feedback control to a given target set while avoiding pre–specified zones or obstacles. The paper therefore extends standard results in (robust) time optimal control. A general procedure is given for the case when the system is discrete-time, nonlinear and time-invariant, and subject to constraints on the state and input. Furthermore, the paper shows how the necessary set computations may be performed using polyhedral algebra, linear programming and computational geometry software, when the system is piecewise affine with additive state disturbances.**

I. Introduction

The importance of the obstacle avoidance problem is stressed in a seminal plenary lecture by A.B. Kurzhanski [1], while a more detailed discussion is given in [2]. In these papers, the obstacle avoidance problem is considered in a *continuous time* framework and when the system is *deterministic* (disturbance free case). The solution to this reachability problem is obtained by specifying an equivalent dynamic optimization problem. The set of states that can be steered to a given target set, while satisfying state and control constraints and avoiding obstacles, is characterized as the set of states belonging to an appropriate level set of the value function of the dynamic optimization problem (obtained by solving a Hamilton-Jacobi-Bellman equation). Additional results related to the time optimal, optimal and suboptimal obstacle avoidance problem can be found in [3]–[7]. These papers also provide a set of examples of the obstacle avoidance problem – including, for example, aircraft trajectory planning with collision avoidance and robot path planning. It is remarked in these papers that control under the avoidance constraints raises interesting problems. Most existing results treat the deterministic case when external disturbances are not present. The computational demands and complexity of the (robust) time optimal control obstacle avoidance problem are amplified if the system is subject to additive, bounded disturbances.

The main purpose of this paper is to demonstrate that the obstacle avoidance problem in the discrete time setup has considerable structure, even when the disturbances are present; this structure permits, in some cases, the derivation of an efficient algorithm based on set computations and polyhedral algebra. Our results extend standard ideas in robust time-optimal control [8]–[12].

This paper is organized as follows. Section 2 is concerned with preliminaries and necessary computational geometry tools. Section 3 characterizes the solution for robust time optimal obstacle avoidance problem. Section 4 provides a set of specific results for the cases when the controlled system is piecewise affine discrete time. Section 5 presents a method for selecting feedback robust time optimal controllers. Section 6 gives an interesting example. Finally, Section 7 presents conclusions and indicates possible extensions. A more detailed exposition of the results stated in this paper can be found in [13].

Notation and Basic Definitions: Let $\mathbb{N} \triangleq \{0, 1, 2, \ldots\}$, $\mathbb{N}_+ \triangleq \{1, 2, \ldots\}$; for a $q \in \mathbb{N}_+$ let $\mathbb{N}_q \triangleq \{0, 1, \ldots, q\}$ and $\mathbb{N}_q^+ \triangleq \{1, \ldots, q\}$. Given two sets $\mathcal{U}$ and $\mathcal{V}$, such that $\mathcal{U} \subset \mathbb{R}^n$ and $\mathcal{V} \subset \mathbb{R}^n$, the Minkowski (vector) sum is defined by $\mathcal{U} \oplus \mathcal{V} \triangleq \{u+v \mid u \in \mathcal{U},\ v \in \mathcal{V}\}$, the Pontryagin (geometric) set difference is: $\mathcal{U} \ominus \mathcal{V} \triangleq \{x \mid x \oplus \mathcal{V} \subseteq \mathcal{U}\}$. Given a set $\mathcal{U} \subseteq \mathbb{R}^n$, $2^{\mathcal{U}}$ denotes the power set (set of all subsets) of $\mathcal{U}$. A *polyhedron* is the (convex) intersection of a finite number of open and/or closed half-spaces. A *polytope* is a compact (i.e. closed and bounded) polyhedron. A closed (An open) *polygon* is the (possibly non-convex) union of a finite number of polytopes (polyhedra).

II. Preliminaries

A. Problem Formulation and Preliminary Definitions

We consider the discrete-time, time-invariant system:

$$x^+ = f(x, u, w) \tag{1}$$

where $x \in \mathbb{R}^n$ is the current state, $u \in \mathbb{R}^n$ is the current control input and x^+ is the successor state; the bounded disturbance w is known only to that extent that it belongs to the compact set $\mathbb{W} \subset \mathbb{R}^p$ that contains the origin in its interior. The function $f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}^n$ is assumed to be continuous.

The system is subject to hard state and input constraints:

$$(x, u) \in \mathbb{X} \times \mathbb{U} \tag{2}$$

where $\mathbb{X}$ and $\mathbb{U}$ are closed and compact sets respectively, each containing the origin in its interior. Additionally it is required that the state trajectories avoid a predefined *open* set $\mathbb{Z}$ introducing an additional state constraint

$$x \notin \mathbb{Z}. \tag{3}$$

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The set $\mathbb{Z}$ is in general specified as the union of a finite number of open sets:

$$\mathbb{Z} \triangleq \bigcup_{j \in \mathbb{N}_q} \mathbb{Z}_j, \tag{4}$$

where $q \in \mathbb{N}$ is a finite integer, and the sets $\mathbb{Z}_j$ are open sets.

The problems considered in this paper are: (i) Characterization of the set of states that can be robustly steered to a given compact target set $\mathbb{T}$ in minimal time while satisfying the state and control constraints (2) and (3), for all admissible disturbance sequences, and (ii) Synthesis of a robust time – optimal control strategy.

We treat the general case in Section III and then provide a detailed analysis in Section IV for the case when the system being controlled is piecewise affine, the corresponding constraints sets $\mathbb{X}, \mathbb{U}$ in (2) are, respectively, closed polygonic and polytopic set and $\mathbb{Z}$ in (3) is an open polygon.

We recall a few standard definitions in the set invariance theory [14].

Definition 1: A set $\Omega \subset \mathbb{R}^n$ is a *robust control invariant* (RCI) set for the system $x^+ = f(x, u, w)$ and constraint set $(\mathbb{X}, \mathbb{U}, \mathbb{W})$ if $\Omega \subseteq \mathbb{X}$ and for all $x \in \Omega$ there exists a $u \in \mathbb{U}$ such that $f(x, u, w) \in \Omega$ for all $w \in \mathbb{W}$.

A set $\Omega \subset \mathbb{R}^n$ is a *control invariant* (CI) set for the system $x^+ = f(x, u)$ and constraint set $(\mathbb{X}, \mathbb{U})$ if $\Omega \subseteq \mathbb{X}$ and for all $x \in \Omega$ there exists a $u \in \mathbb{U}$ such that $f(x, u) \in \Omega$.

Given a control law $\nu : \mathbb{R}^n \to \mathbb{R}^m$, let:

$$\mathbb{X}_\nu \triangleq \mathbb{X} \cap \{x \mid \nu(x) \in \mathbb{U}\} \tag{5}$$

Definition 2: A set $\Omega \subset \mathbb{R}^n$ is a *robust positively invariant* (RPI) set for system $x^+ = f(x, \nu(x), w)$ and constraint set $(\mathbb{X}_\nu, \mathbb{W})$ if $\Omega \subseteq \mathbb{X}_\nu$ and $f(x, \nu(x), w) \in \Omega,\ \forall w \in \mathbb{W}$, $\forall x \in \Omega$.

A set $\Omega \subset \mathbb{R}^n$ is a *positively invariant* (RPI) set for system $x^+ = f(x, \nu(x))$ and constraint set $\mathbb{X}_\nu$ if $\Omega \subseteq \mathbb{X}_\nu$ and $f(x, \nu(x)) \in \Omega,\ \forall x \in \Omega$.

We also need the following definition:

Definition 3: The set Ω is robust asymptotically (finite-time) attractive, for the controlled system $x^+ = f(x, \nu(x), w)$, with domain of attraction Ψ if, for all $x(0) \in \Psi$, $d(x(i), \Omega) \to 0$ as $i \to \infty$ (there exists a time I such that $x(i) \in \Omega$ for all $i \geq I$) for all admissible disturbance sequences.

B. Necessary computational geometry tools

We now present some tools that are required for set computations with polygons, also used in [12], [15]. For computation of the set difference of two polyhedra the reader is referred to [16]. The first two results show how the set difference of a polygon and a polyhedron (or a polygon) may be computed:

Proposition 1: Let $\mathcal{C} \triangleq \bigcup_{j=1}^{p} \mathcal{C}_j$ be a polygon, where all the $\mathcal{C}_j,\ j \in \mathbb{N}_q^+$, are non-empty polyhedra. If $\mathcal{A}$ is a non-empty polyhedron, then $\mathcal{C} \setminus \mathcal{A} = \bigcup_{j=1}^{p} (\mathcal{C}_j \setminus \mathcal{A})$ is a polygon.

Proposition 2: Let the sets $\mathcal{C} \triangleq \bigcup_{j=1}^{p} \mathcal{C}_j$ and $\mathcal{D} \triangleq \bigcup_{k=1}^{q} \mathcal{D}_k$ be polygons, where all the $\mathcal{C}_j,\ j \in \mathbb{N}_p^+$, and $\mathcal{D}_k,\ k \in \mathbb{N}_q^+$, are non-empty polyhedra. If $\mathcal{E}_0 \triangleq \mathcal{C}$ and $\mathcal{E}_k \triangleq \mathcal{E}_{k-1} \setminus \mathcal{D}_k,\ k \in \mathbb{N}_q^+$ then $\mathcal{C} \setminus \mathcal{D} = \mathcal{E}_q$ is a polygon.

The reader is referred to [15] for proofs and comments on computational efficiency.

An efficient algorithm for computing the Pontryagin (Minkowski (geometric) Set) difference of a polygon and a polytope is discussed next. If $\mathcal{A}$ and $\mathcal{B}$ are two subsets of $\mathbb{R}^n$ it is known (see for instance [17], [18]) that $\mathcal{A} \ominus \mathcal{B} = [\mathcal{A}^c \oplus (-\mathcal{B})]^c$. The following algorithm implements the computation of the Pontryagin difference of a polygon $\mathcal{C} \triangleq \cup_{j \in \mathbb{N}_p^+} \mathcal{C}_j$, where $\mathcal{C}_j, j \in \mathbb{N}_p^+$ are polytopes in $\mathbb{R}^n$, and a polytope $\mathcal{B} \subset \mathbb{R}^n$.

Algorithm 2.1 ($\mathcal{C} \ominus \mathcal{B}$):

1) Input: polygon $\mathcal{C}$, polytope $\mathcal{B}$
2) $\mathcal{H} = \text{convh}(\mathcal{C})$
3) $\mathcal{D} = \mathcal{H} \ominus \mathcal{B}$
4) $\mathcal{E} = \mathcal{H} \setminus \mathcal{C}$
5) $\mathcal{F} = \mathcal{E} \oplus (-\mathcal{B})$
6) $\mathcal{G} = \mathcal{D} \setminus \mathcal{F}$
7) Output: polygon $\mathcal{G}$

Proposition 3: [12], [19] If the input to Algorithm 2.1 is a polygon $\mathcal{C}$ and a polytope $\mathcal{B}$, then the output is the polygon $\mathcal{G} = \mathcal{C} \ominus \mathcal{B}$.

Algorithm 2.1 is illustrated on sample polygons in Figures 1(a) to 1(f). *The Pontryagin set difference $\mathcal{C} \ominus \mathcal{B}$ is not necessarily equal to $\cup_{j \in \mathbb{N}_p^+} (\mathcal{C}_j \ominus \mathcal{B})$; in general $\cup_{j \in \mathbb{N}_p^+} (\mathcal{C}_j \ominus \mathcal{B}) \subseteq \mathcal{C} \ominus \mathcal{B}$ (equality holds only in a limited number of cases).* Algorithm 2.1 for computation of the Pontryagin difference is conceptually similar to that proposed in [17], [18]. However, computing the convex hull in the first step significantly reduces (in general) the number of sets obtained at step 3, which in turn results in fewer Minkowski set additions. Since computation of Minkowski set addition is expensive, a reasonable runtime improvement is expected. In principle, computation of the convex hull can be replaced by computation of any convex set containing the polygon $\mathcal{C}$. Necessary computations can be efficiently implemented by using standard computational geometry software such as [20]–[22].

III. Robust Time Optimal Obstacle Avoidance Problem – General Case

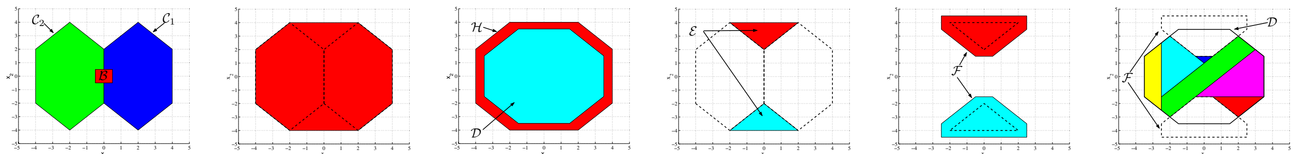
The state constraints, specified in (2) – (4) may be converted in a single, non-convex, state constraint $x \in \mathbb{X}_\mathbb{Z}$ where:

$$\mathbb{X}_\mathbb{Z} \triangleq \mathbb{X} \setminus \mathbb{Z} = \mathbb{X} \setminus \bigcup_{j \in \mathbb{N}_q} \mathbb{Z}_j \tag{6}$$

If $\mathbb{Z} \subseteq \text{interior}(\mathbb{X})$, $\mathbb{X}_\mathbb{Z}$ is a non–empty and closed set. Additionally, if the set $\mathbb{Z}$ is an open polygon and $\mathbb{X}$ is a closed polygonic set then the set $\mathbb{X}_\mathbb{Z}$ is a closed polygon by Proposition 2. In this case, it follows by Proposition 2 that the set $\mathbb{X}_\mathbb{Z}$ can be expressed by:

$$\mathbb{X}_\mathbb{Z} \triangleq \bigcup_{j \in \mathbb{N}_r} \mathbb{X}_{\mathbb{Z}j}, \tag{7}$$

where r is a finite integer and the sets $\mathbb{X}_{\mathbb{Z}j},\ j \in \mathbb{N}_r$ are constituent polytopes of the polygon $\mathbb{X}_\mathbb{Z}$.



(a) $\bigcup_j \mathcal{C}_j$ & $\mathcal{B}$. (b) $\mathcal{H} = \text{convh}(\mathcal{C})$ (c) $\mathcal{D} = \mathcal{H} \ominus \mathcal{B}$. (d) $\mathcal{E} = \mathcal{H} \setminus \mathcal{C}$ (e) $\mathcal{F} = \mathcal{E} \oplus (-\mathcal{B})$ (f) $\mathcal{G} = \mathcal{D} \setminus \mathcal{F}$

Fig. 1. Graphical Illustration of Algorithm 2.1.

In order to have a well–defined problem we make the following standing assumption:

Assumption 1: The sets $\mathbb{X}, \mathbb{T}, \mathbb{Z}$ satisfy that (i) $\mathbb{Z} \subseteq \text{interior}(\mathbb{X})$ and (ii) $\mathbb{T} \subseteq \mathbb{X}_{\mathbb{Z}} = \mathbb{X} \setminus \mathbb{Z}$.

Let $\pi \triangleq \{\mu_i(\cdot),\ i \in \mathbb{N}_{N-1}\}$, where, for each $i \in \mathbb{N}_{N-1}$, $\mu_i(\cdot) :\ \mathbb{X}_{\mathbb{Z}} \to \mathbb{U}$, denote a control policy (sequence of control laws). Also, let $\phi(i; x, \pi, \mathbf{w})$ denote the solution to (1) at time instant i, given the initial state x (at time 0), control policy π, and an admissible disturbance sequence $\mathbf{w} \triangleq \{w_0, w_1, w_2, \ldots\}$ ($w_i \in \mathbb{W}, \forall i \in \mathbb{N}$).

The robust time–optimal obstacle avoidance problem $\mathbb{P}(x)$ is defined, as usual in robust time–optimal control problems by:

$$N^0(x) \triangleq \inf_{\pi, N} \{N \mid (\pi, N) \in \Pi_N(x) \times \mathbb{N}_{N_{max}}\}, \quad (8)$$

where $N_{\max} \in \mathbb{N}$ is an upper bound on the horizon and $\Pi_N(x)$, the set of admissible control policies, is defined as follows:

$$\Pi_N(x) \triangleq \{\pi \mid (x_i, u_i) \in \mathbb{X}_{\mathbb{Z}} \times \mathbb{U},\ \forall i \in \mathbb{N}_{N-1}, \\ x_N \in \mathbb{T},\ \forall \mathbf{w}\} \quad (9)$$

where, for each $i \in \mathbb{N}$, $x_i \triangleq \phi(i; x, \pi, \mathbf{w})$ and $u_i \triangleq \mu_i(\phi(i; x, \pi, \mathbf{w}))$. The solution is sought in the class of the state feedback control laws because of the additive disturbance. The solution to $\mathbb{P}(x)$ is

$$\left(\pi^0(x), N^0(x)\right) \triangleq \arg \inf_{\pi, N} \{N \mid (\pi, N) \in \Pi_N(x) \times \mathbb{N}_{N_{max}}\}. \quad (10)$$

The value function of problem $\mathbb{P}(x)$ satisfies $N^0(x) \in \mathbb{N}_{N_{max}}$ and, for any integer i, the robust controllable set $X_i \triangleq \{x \mid N^0(x) \leq i\}$ is the set of initial states that can be *robustly* steered (for all $\mathbf{w}$) to the target set $\mathbb{T}$ in i steps or less while satisfying all state and control constraints, and avoiding the obstacles, for all admissible disturbance sequences. Hence $N^0(x) = i$ for all $x \in X_i \setminus X_{i-1}$.

The robust controllable sets $\{X_i\}$ and the associated robust time-optimal control laws $\kappa_i : X_i \to 2^{\mathbb{U}}$ can be computed by the following standard recursion [23]:

$$X_i \triangleq \{x \in \mathbb{X}_{\mathbb{Z}} \mid \exists u \in \mathbb{U} \text{ s.t. } f(x, u, \mathbb{W}) \subseteq X_{i-1}\} \quad (11)$$
$$\kappa_i(x) \triangleq \{u \in \mathbb{U} \mid f(x, u, \mathbb{W}) \subseteq X_{i-1}\},\ \forall x \in X_i \quad (12)$$

for $i \in \mathbb{N}_{N_{max}}$ with the boundary condition $X_0 = \mathbb{T}$ and where $f(x, u, \mathbb{W}) \triangleq \{f(x, u, w) \mid w \in \mathbb{W}\}$.

We now introduce the following assumption:

Assumption 2: (i) The set $\mathbb{T}$ is a robust control invariant set for system (1) and constraint set $(\mathbb{X}_{\mathbb{Z}}, \mathbb{U}, \mathbb{W})$.
(ii) The control law $\nu :\ \mathbb{X}_{\mathbb{Z}} \to \mathbb{U}$ is such that $\mathbb{T}$ is RPI for system (1) and constraint set $(\mathbb{X}_\nu, \mathbb{W})$, where $\mathbb{X}_\nu \triangleq \mathbb{X}_{\mathbb{Z}} \cap X_\nu$ and X_ν is defined by:

$$X_\nu \triangleq \{x \mid \nu(x) \in \mathbb{U}\}. \quad (13)$$

The control law $\nu(\cdot)$ in Assumption 2(ii) exists by Assumption 2(i).

Since Assumption 2 implies that $X_0 = \mathbb{T}$ is a RCI set for system (1) and constraint set $(\mathbb{X}_{\mathbb{Z}}, \mathbb{U}, \mathbb{W})$ we consider the following time-invariant (and set valued) control law $\kappa^0 : X_{N_{\max}} \to 2^{\mathbb{U}}$ defined, for all $i \in \mathbb{N}_{N_{max}}$, by

$$\kappa^0(x) \triangleq \begin{cases} \kappa_i(x), & \forall x \in X_i \setminus X_{i-1},\ i \geq 1 \\ \nu(x), & \forall x \in X_0 \end{cases} \quad (14)$$

The control law $\kappa^0 : X_{N_{\max}} \to 2^{\mathbb{U}}$ robustly steers any $x \in X_i$ to X_0 in i steps or less to X_0, while satisfying state and control constraints and avoiding the obstacles, and thereafter maintains the state in X_0. We now recall a standard result in robust time–optimal control [10]:

Proposition 4: Suppose that Assumption 2 holds and let $X_0 \triangleq \mathbb{T}$ where $\mathbb{T}$ satisfies Assumption 2(i), then the set sequence $\{X_i\}$ computed using the recursion (11) is a non-decreasing sequence of RCI sets for system (1) and constraint set $(\mathbb{X}_{\mathbb{Z}}, \mathbb{U}, \mathbb{W})$, i.e. $X_i \subseteq X_{i+1} \subseteq \mathbb{X}_{\mathbb{Z}}$ for all $i \in \mathbb{N}_{N_{max}}$ and X_i is a RCI set for system (1) and constraint set $(\mathbb{X}_{\mathbb{Z}}, \mathbb{U}, \mathbb{W})$ for all $i \in \mathbb{N}_{N_{max}}$.

The following property of the set-valued control law $\kappa^0(\cdot)$ defined in (14) follows directly from the construction of $\kappa^0(\cdot)$:

Theorem 1: Suppose that Assumption 2 holds and let $X_0 \triangleq \mathbb{T}$ where $\mathbb{T}$ satisfies Assumption 2(i). The target set $X_0 \triangleq \mathbb{T}$ is robustly finite-time attractive for the closed-loop system $x^+ \in f(x, \kappa^0(x), \mathbb{W})$ with a region of attraction $X_{N_{\max}}$.

It is clear that the solution of the robust time optimal obstacle avoidance problem requires efficient computational algorithms for performing the set operations in (6), (11) and (12). A set of necessary computational procedures is given by propositions 1– 3. Our next step is to demonstrate that in certain relevant and important cases is possible to employ standard computational geometry software (polyhedral algebra) in order to characterize the sets sequence $\{X_i\}$ and the corresponding set valued control laws $\{\kappa_i(\cdot)\}$.

IV. Robust Time Optimal Obstacle Avoidance Problem – Piecewise Affine Discrete Time Systems

In this section we treat the case when the system defined in (1) is piecewise affine:

$$\begin{aligned} x^+ = f(x,u,w) = f_l(x,u,w),\ \forall (x,u) \in P_l, \\ f_l(x,u,w) \triangleq A_l x + B_l u + c_l + w,\ \forall l \in \mathbb{N}_t^+ \end{aligned} \tag{15}$$

The function $f(\cdot)$ is assumed to be continuous and the polytopes P_l, $i \in \mathbb{N}_t^+$, have disjoint interiors and cover the region $\mathbb{Y} \triangleq \mathbb{X} \times \mathbb{U}$ of state/control space of interest so that $\bigcup_{k\in\mathbb{N}_t^+} P_k = \mathbb{Y} \subseteq \mathbb{R}^{n+m}$ and $\text{interior}(P_k) \bigcap \text{interior}(P_j) = \emptyset$ for all $k \neq j,\ k,j \in \mathbb{N}_t^+$. The set of sets $\{P_k \mid k \in \mathbb{N}_q^+\}$ is a polytopic partition of $\mathbb{Y}$.

Our assumptions on the constraint sets are that the sets $\mathbb{X}$ and $\mathbb{U}$ are polygonic and a polytopic, respectively, and the disturbance set $\mathbb{W}$ is polytopic; each of the sets contains the origin as an interior point. The set $\mathbb{Z}$ is an open polygon.

In this case, the standard recursion for the computation of the robustly controllable sets $\{X_i\}$ and the associated robust time-optimal control laws $\kappa_i : X_i \to 2^{\mathbb{U}}$ (11) and (12) is:

$$X_i \triangleq \{x \in \mathbb{X}_{\mathbb{Z}} \mid \exists u \in \mathbb{U} \text{ s.t. } f(x,u,\mathbb{W}) \subseteq X_{i-1}\} \tag{16}$$

$$\kappa_i(x) \triangleq \{u \in \mathbb{U} \mid f(x,u,\mathbb{W}) \subseteq X_{i-1}\},\ \forall x \in X_i \tag{17}$$

for each $i \in \mathbb{N}_{N_{max}}$ with boundary condition $X_0 = \mathbb{T}$.

Our next step is to provide a detailed characterization of the sets $\{X_i\}$ under assumption that the set $X_0 = \mathbb{T}$ is a polygon:

$$\begin{aligned} X_i &\triangleq \{x \in \mathbb{X}_{\mathbb{Z}} \mid \exists u \in \mathbb{U} \text{ s.t. } f(x,u,\mathbb{W}) \subseteq X_{i-1}\} && (18) \\ &= \{x \in \mathbb{X}_{\mathbb{Z}} \mid \exists u \in \mathbb{U} \text{ s.t. } f(x,u,0) \in X_{i-1} \ominus \mathbb{W}\} && (19) \end{aligned}$$

In going from (18) to (19) we have used the fact that $f(x,u,w) = f(x,u,0) + w$ for the system defined in (15). We proceed by exploiting the definition of $f(\cdot)$:

$$\begin{aligned} X_i &= \bigcup_{l\in\mathbb{N}_t^+} \{x \in \mathbb{X}_{\mathbb{Z}} \mid \exists u \in \mathbb{U} \text{ s.t. } (x,u) \in P_l \\ &\qquad f_l(x,u,0) \in X_{i-1} \ominus \mathbb{W}\} \\ &= \bigcup_{l\in\mathbb{N}_t^+} \{x \in \bigcup_{j\in\mathbb{N}_r} \mathbb{X}_{\mathbb{Z}j} \mid \exists u \in \mathbb{U} \text{ s.t. } (x,u) \in P_l \\ &\qquad A_l x + B_l u + c_l \in X_{i-1} \ominus \mathbb{W}\} \\ &= \bigcup_{(j,l)\in\mathbb{N}_r\times\mathbb{N}_t^+} \{x \in \mathbb{X}_{\mathbb{Z}j} \mid \exists u \in \mathbb{U} \text{ s.t. } (x,u) \in P_l \\ &\qquad A_l x + B_l u + c_l \in X_{i-1} \ominus \mathbb{W}\} \end{aligned} \tag{20}$$

It follows from Proposition 3 that, since X_i, $i \in \mathbb{N}_{N_{max}}$, is a polygon (polytope), so is $Y_i \triangleq X_{i-1} \ominus \mathbb{W},\ i \in \mathbb{N}_{N_{max}}^+$. Hence, for each $i \in \mathbb{N}_{N_{max}}^+$, $Y_i \triangleq X_{i-1} \ominus \mathbb{W}$ may be expressed in terms of its constituent polytopes $Y_{(i,k)}, k \in \mathbb{N}_{q_i}$ for some finite integer q_i, by $Y_i = \bigcup_{k\in\mathbb{N}_{q_i}} Y_{(i,k)}$. Exploiting this fact, it follows from (20) that:

$$\begin{aligned} X_i &= \bigcup_{(j,l)\in\mathbb{N}_r\times\mathbb{N}_t^+} \{x \in \mathbb{X}_{\mathbb{Z}j} \mid \exists u \in \mathbb{U} \text{ s.t. } (x,u) \in P_l, \\ &\qquad A_l x + B_l u + c_l \in Y_i\} \\ &= \bigcup_{(j,l)\in\mathbb{N}_r\times\mathbb{N}_t^+} \{x \in \mathbb{X}_{\mathbb{Z}j} \mid \exists u \in \mathbb{U} \text{ s.t. } (x,u) \in P_l, \\ &\qquad A_l x + B_l u + c_l \in \bigcup_{k\in\mathbb{N}_{q_i}} Y_{(i,k)}\} \end{aligned}$$

so that

$$\begin{aligned} X_i &= \bigcup_{(j,l,k)\in\mathbb{N}_r\times\mathbb{N}_t^+\times\mathbb{N}_{q_i}} \{x \in \mathbb{X}_{\mathbb{Z}j} \mid \exists u \in \mathbb{U} \text{ s.t. } (x,u) \in P_l, \\ &\qquad A_l x + B_l u + c_l \in Y_{(i,k)}\} \\ &= \bigcup_{(j,l,k)\in\mathbb{N}_r\times\mathbb{N}_t^+\times\mathbb{N}_{q_i}} X_{(i,j,l,k)} \text{ where} \\ X_{(i,j,l,k)} &\triangleq \{x \in \mathbb{X}_{\mathbb{Z}j} \mid \exists u \in \mathbb{U} \text{ s.t. } (x,u) \in P_l, \\ &\qquad A_l x + B_l u + c_l \in Y_{(i,k)}\} \end{aligned} \tag{21}$$

and index set $(i,j,l,k) \in \mathbb{N}_{N_{max}} \times \mathbb{N}_r \times \mathbb{N}_t^+ \times \mathbb{N}_{q_i}$ is such that i indexes time, j indexes the j^{th} constituent polytope of non–convex state constraints $\mathbb{X}_{\mathbb{Z}}$ (7), l indexes the polytope P_l in (x,u) space in which $f(x,u,0) = A_l x + B_l u + c_l$ is affine, and (i,k) specifies the k^{th} constituent polytope in $Y_i \triangleq X_{i-1} \ominus \mathbb{W}$.

A similar argument shows that for all $(i,j,l,k) \in \mathbb{N}_{N_{max}} \times \mathbb{N}_r \times \mathbb{N}_t^+ \times \mathbb{N}_{q_i}$:

$$\kappa_{(i,j,l,k)}(x) \subseteq \kappa_i(x),\ \forall x \in X_{(i,j,l,k)}, \tag{22}$$

where

$$\begin{aligned} \kappa_{(i,j,l,k)}(x) \triangleq \{u \in \mathbb{U} \mid (x,u) \in P_l, \\ A_l x + B_l u + c_l \in Y_{(i,k)}\},\ \forall x \in X_{(i,j,l,k)}, \end{aligned} \tag{23}$$

with $X_{(i,j,l,k)}$ defined in (21). For every $x \in X_i$ let:

$$\mathbb{N}_i(x) \triangleq \{(j,l,k) \in \mathbb{N}_r \times \mathbb{N}_t^+ \times \mathbb{N}_{q_i} \mid x \in X_{(i,j,l,k)}\}, \tag{24}$$

so that:

$$\kappa_i(x) = \bigcup_{(j,l,k)\in\mathbb{N}_i(x)} \kappa_{(i,j,l,k)}(x),\ \forall x \in X_i. \tag{25}$$

We observe that at time i it is necessary to consider those integer triplets $(j,l,k) \in \mathbb{N}_r \times \mathbb{N}_t^+ \times \mathbb{N}_{q_i}$ for which $X_{(i,j,l,k)} \neq \emptyset$. The set $X_{(i,j,l,k)}$ specified by (21) is easily computed by the standard computational software, since:

$$\begin{aligned} X_{(i,j,l,k)} &= \text{Proj}_X Z_{(i,j,l,k)} \text{ where} \\ Z_{(i,j,l,k)} &\triangleq \{(x,u) \in \left(\mathbb{X}_{\mathbb{Z}j} \times \mathbb{U}\right) \bigcap P_l \mid \\ &\qquad A_l x + B_l u + c_l \in Y_{(i,k)}\} \end{aligned} \tag{26}$$

If the Assumption 2 (with $f(\cdot)$ defined in (15)) holds the results of Proposition 4 and Theorem 1 are directly applicable to this relevant case. A final conclusion, for the case when the considered system is piecewise affine, is that if the target set $\mathbb{T}$ is a RCI polygon, the set sequence $\{X_i\}$ is also a RCI sequence of polygons.

V. Selection of Control Laws $\kappa_{(i,j,l,k)}(\cdot)$

For any $i \in \mathbb{N}_{N_{max}}$ an appropriate selection of the control laws $\kappa_{(i,j,l,k)}(\cdot)$ can be obtained by employing the *parametric mathematical programming* as we briefly demonstrate next. For each $i \geq 1,\ i \in \mathbb{N}_{N_{max}}$ let $V_i^{(p,l)}(x,u)$ be any linear or quadratic (strictly convex) function in (x,u), for instance:

$$V_i^{(p,l)}(x,u) \triangleq |A_l x + B_l u + c_l|_Q^2 \tag{27}$$

Consider the piecewise affine case and an appropriate way of selecting the feedback control law $\kappa_{(i,j,l,k)}(\cdot)$. Since $Z_{(i,j,l,k)}$ defined in (26) is a polyhedral set and since $V_i^{(p,l)}(x,u)$ is a linear or a quadratic (strictly convex) function it follows that for each $i \geq 1,\ i \in \mathbb{N}_{N_{max}}$ the optimization problem $\mathbb{P}_i^l(x)$:

$$\theta^0_{(i,j,l,k)}(x) \triangleq \arg\inf_u \{V_i^{(p,l)}(x,u) \mid (x,u) \in Z_{(i,j,l,k)}\} \tag{28}$$

is a *parametric* linear/quadratic problem. As is well know [16], [24]–[26], the solution takes the form of a piecewise affine function of the state x; for all $x \in X_{(i,j,l,k)} = \text{Proj}_X Z_{(i,j,l,k)}$:

$$\theta^0_{(i,j,l,k)}(x) = S_{(i,j,l,k,h)}x + s_{(i,j,l,k,h)},\ x \in R_{(i,j,l,k,h)},\quad h \in \mathbb{N}_{l_i} \tag{29}$$

where l_i is a finite integer and the union of polyhedral sets $R_{(i,j,l,k,h)}$ partition the set $X_{(i,j,l,k)}$, i.e. $X_{(i,j,l,k)} = \bigcup_{h \in \mathbb{N}_{l_i}} R_{(i,j,l,k,h)}$.

If we define $(i,j,l,k)^0(x)$ by:

$$(i,j,l,k)^0(x) \triangleq \arg\inf_{(i,j,l,k)} \{i \mid x \in X_{(i,j,l,k)},\ (i,j,l,k) \in \mathbb{N}_{N_{max}} \times \mathbb{N}_r \times \mathbb{N}_t^+ \times \mathbb{N}_{q_i}\} \tag{30}$$

it follows that

$$\theta^0_{(i,j,l,k)^0(x)}(x) \in \kappa_{(i,j,l,k)}(x) \subseteq \kappa_i(x), \tag{31}$$

for all $x \in X_{(i,j,l,k)}$, all $i \geq 1$, and all $(i,j,l,k) \in \mathbb{N}_{N_{max}} \times \mathbb{N}_r \times \mathbb{N}_t^+ \times \mathbb{N}_{q_i}$.

VI. Illustrative Example

Our illustrative example is the second order unstable linear system:

$$x^+ = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u + w \tag{32}$$

where

$$w \in \mathbb{W} \triangleq \{w \in \mathbb{R}^2 \mid |w|_\infty \leq 1\}.$$

The following set of 'standard' state and control constraints is required to be satisfied:

$$\begin{aligned} \mathbb{X} &= \{x \mid |x|_\infty \leq 15\}, \\ \mathbb{U} &= \{u \mid |u| \leq 4\} \end{aligned} \tag{33}$$

The obstacle configuration, state constraints and target set are shown in Figure 2.

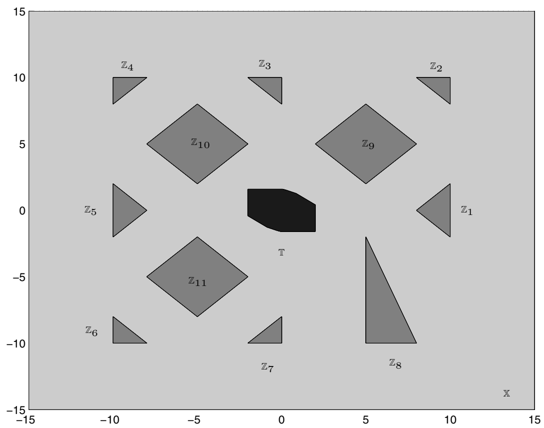


Fig. 2. Obstacles, State Constraints and Target Set

The target set is robust control invariant and is computed by method of [19], [27]. The set X_0 is shown together with the RCI set sequence $\{X_i\},\ i \in \mathbb{N}_3$, computed by using equations (20) – (21) (appropriately modified/simplified for the linear case – for more details see [13]), in Figure 3. In

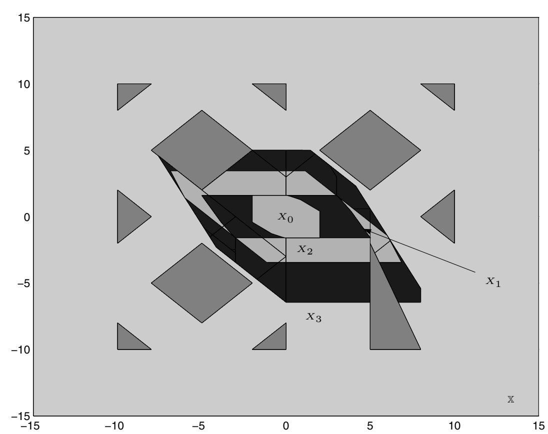


Fig. 3. RCI Set Sequence $\{X_i\},\ i \in \mathbb{N}_3$

Figure 4 we show the sets $\{X_i\},\ i \in \mathbb{N}_3$ for the case when $\mathbb{W} = \{0\}$.

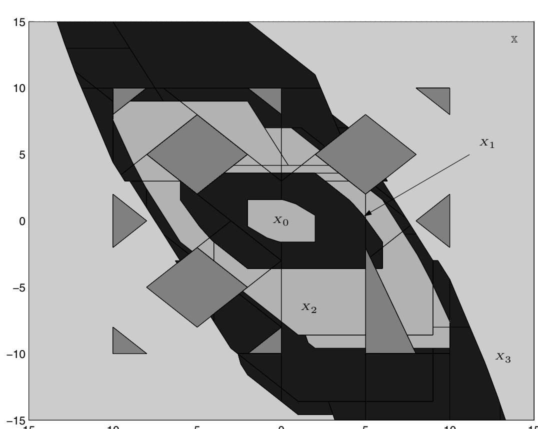


Fig. 4. CI Set Sequence $\{X_i\},\ i \in \mathbb{N}_3$

VII. Conclusions and Current Research

Our results provide an exact solution of the robust obstacle avoidance problem for constrained discrete time systems. A complete characterization of the solution is given for piecewise affine discrete time systems. The basic set structure employed is a polygon. Complexity of the solution may be considerable but the main advantage is that, the exact solution is provided and the resultant computations can be performed by using polyhedral algebra. The results in this paper are also applicable to the case when $\mathbb{W} = \{0\}$ and when system being controlled is linear/affine. The proposed algorithms can be implemented by using standard computational geometry software [20], [22].

The results can be extended to address the optimal control for obstacle avoidance problem with linear/quadratic performance index. It is also possible to address the case when the obstacles are given as a time varying set. An obvious extension of the results reported in [28], [29] is possible. This extension would allow for construction of an robust control invariant tube by solving an appropriately specified optimal control problem for nominal system with the restricted constraints. The complexity of this problem would be approximately equal to that required for optimal control in the deterministic case, i.e. the resultant optimal control problem could be posed as a mixed integer quadratic/linear programming problem; a set of preliminary ideas is reported in [30]. The proposed robust time optimal control scheme guarantees robust obstacle avoidance at discrete moments; this constraint should be additionally robustified, when the scheme is implemented to the continuous-time dynamical systems.

In conclusion, the robust time optimal avoidance problem is addressed and a set of computational procedures is derived for the relevant cases when the system being controlled is piecewise affine. The method was illustrated by a numerical example.

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