Proceedings of the
44th IEEE Conference on Decision and Control, and
the European Control Conference 2005
Seville, Spain, December 12-15, 2005

Set Robust Control Invariance for Linear Discrete Time Systems

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***Abstract*— This paper introduces *set robust control invariance*, a concept that generalizes robust control invariance for systems described by difference equations to systems described by difference inclusions of special structure. The concept is useful for the analysis and synthesis of uncertain systems where a given control policy results in, for each initial state, a *tube* of trajectories rather than a single trajectory; it also reveals the properties required for the terminal constraint set in receding horizon control of constrained linear systems with bounded disturbances and shows how improved terminal sets may be constructed. The family of set robust control invariant sets is characterized and the most important members of this family, the minimal and the maximal, are identified.**

I. INTRODUCTION

Set invariance theory, excellently surveyed in [1], provides, inter alia, useful tools for the synthesis of reference governors [2] and predictive controllers [3]–[6] with guaranteed invariance, stability and convergence properties. A robust control (positively) invariant set, in particular the minimal robust positively invariant set, is also a suitable target set in robust time-optimal control [7]–[9] and plays an integral part in the novel robust predictive control methods, recently proposed in [10]–[12].

In this paper we introduce the concept of *set robust control invariance* that generalizes robust control invariance for systems described by difference equations to systems described by difference inclusions of special structure (a set $X \subset \mathbb{R}^n$ is robust control invariant for the system $x^+ = f(x, u, w)$ and constraint set $(\mathbb{X}, \mathbb{U}, \mathbb{W})$ if, $X \subseteq \mathbb{X}$ and for every $x \in X$, there exists an admissible control $u \in \mathbb{U}$ such that $f(x, u, w) \in X$ for all $x \in X$, all $w \in \mathbb{W}$). The motivation for this generalization lies in the fact that, when uncertainty is present in the controlled system, we are forced to consider a tube of trajectories (a sequence of *sets* of states) instead of a single trajectory (a sequence of states); each trajectory of the system, arising from a particular realization of the uncertainty, lies in the tube. Tubes have been extensively studied in [13]–[18], mainly in the context of constrained continuous-time systems. In practice, determination of 'exact' (or non-conservative) tubes is difficult, so 'outer bounding tubes' that bound 'exact' tubes are employed. If we wish to establish asymptotic/exponential stability of an uncertain system under receding horizon control using techniques, similar to those conventionally employed for deterministic systems [19] when only a local Control Lyapunov function is available for the terminal cost, we need to generalize appropriately the concept of (robust) control invariance. In particular, we wish to generalize the requirement in receding horizon control that, if the last element x_N of a feasible trajectory $\{x_0, x_1, \ldots, x_N\}$ lies in a terminal constraint set T, there exists an admissible control action u such that the successor element $x_N^+ = f(x_N, u)$ also lies in T. Since the elements of a tube are *sets*, it is necessary to find a set Φ of *sets* that is *set robust control invariant* for a system $x^+ = f(x, u, w)$, $w \in \mathbb{W}$ in the following sense: for any set $X \in \Phi$, there exists a control law $\theta : X \to \mathbb{U}$ such that $X^+ = f(X, \theta, \mathbb{W}) \subset Y$ for some $Y \in \Phi$, where $f(X, \theta, \mathbb{W}) \triangleq \{f(x, \theta(x), w) \mid x \in X,\ w \in \mathbb{W}\}$. Thus, if the last element X_N of a tube $\{X_0, X_1, \ldots, X_N\}$ lies in Φ, it is possible to find an admissible local control law π such that the successor element $X_N^+ \subset Y \in \Phi$ in which case the new outer-bounding tube $\{X_1, X_2, \ldots, X_N, Y\}$ has its last element lying in Φ; this is an appropriate generalization (with X_N and $\theta(\cdot)$ replacing x_N and u respectively) of the condition that the terminal constraint set in deterministic receding horizon control is control (positively) invariant. Practicality enforces the use of 'outer bounding tubes' having a simple structure; for example each element of Φ has the form $z \oplus R$.

This paper is organized as follows. Section 2 is concerned with preliminaries. Section 3 provides a characterization of a family of set robust control invariant sets. Section 4 discusses dynamical behavior of trajectories of a sequence of sets of states starting from a set of states that is element of a set robust control invariant set. Section 5 provides a set of constructive simplifications. Section 6 gives an illustrative example. Finally, Section 7 presents conclusions.

NOTATION AND BASIC DEFINITIONS: Let $\mathbb{N} \triangleq \{0, 1, 2, \ldots\}$ and $\mathbb{N}_+ \triangleq \{1, 2, \ldots\}$. Given two sets $\mathcal{U}$ and $\mathcal{V}$, such that $\mathcal{U} \subset \mathbb{R}^n$ and $\mathcal{V} \subset \mathbb{R}^n$, the Minkowski set addition is defined by $\mathcal{U} \oplus \mathcal{V} \triangleq \{u + v \mid u \in \mathcal{U},\ v \in \mathcal{V}\}$, the Minkowski (Pontryagin or geometric) set difference is: $\mathcal{U} \ominus \mathcal{V} \triangleq \{x \mid x \oplus \mathcal{V} \subseteq \mathcal{U}\}$; we employ $x \oplus X$ where X is a set to denote $\{x\} \oplus X$. Given the sequence of sets $\{\mathcal{U}_i \subset \mathbb{R}^n\}_{i=a}^{b}$, we define $\bigoplus_{i=a}^{b} \mathcal{U}_i \triangleq \mathcal{U}_a \oplus \cdots \oplus \mathcal{U}_b$. A closed hyperball in $\mathbb{R}^n$ is denoted by $\mathbb{B}_p^n(r) \triangleq \{x \in \mathbb{R}^n \mid |x|_p \le r\}$ $(r > 0)$ where given a vector $x \in \mathbb{R}^n$, $|x|_p$ denotes the vector p–norm.

II. PRELIMINARIES

We consider the following discrete-time linear time-invariant (DLTI) system:

$$x^+ = Ax + Bu + w \tag{1}$$

This research is supported by the Engineering and Physical Sciences Research Council, UK.

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where $x \in \mathbb{R}^n$ is the current state, $u \in \mathbb{R}^m$ is the current control input, x^+ is the successor state and w is the additive and bounded disturbance. The system is subject to hard constraints:

$$(x, u, w) \in \mathbb{X} \times \mathbb{U} \times \mathbb{W} \tag{2}$$

The sets $\mathbb{U}$ and $\mathbb{W}$ are compact, the set $\mathbb{X}$ is closed; each set is convex and contains the origin as an interior point. We also define the corresponding nominal system:

$$z^+ = Az + Bv, \tag{3}$$

where $z \in \mathbb{R}^n$ is the current state, $v \in \mathbb{R}^m$ the current control action and z^+ the successor state of the nominal system. We make the standing assumption that:

Assumption 1: The couple (A, B) is controllable.

We first recall a few standard definitions in set invariance theory [1].

Definition 1: A set $\Omega \subset \mathbb{R}^n$ is *control invariant* (CI) for system $x^+ = f(x, u)$ and constraint set $(\mathbb{X}, \mathbb{U})$ if $\Omega \subset \mathbb{X}$ and, for all $x \in \Omega$, there exists a $u \in \mathbb{U}$ such that $f(x, u) \in \Omega$.

A set $\Omega \subset \mathbb{R}^n$ is *robust control invariant* (RCI) for system $x^+ = f(x, u, w)$ and constraint set $(\mathbb{X}, \mathbb{U}, \mathbb{W})$ if $\Omega \subseteq \mathbb{X}$ and, for all $x \in \Omega$, there exists a $u \in \mathbb{U}$ such that $f(x, u, w) \in \Omega$ for all $w \in \mathbb{W}$.

Definition 2: A set $\Omega \subset \mathbb{R}^n$ is *positively invariant* (PI) for system $x^+ = f(x)$ and constraint set $\mathbb{X}$ if $\Omega \subset \mathbb{X}$ if, for all $x \in \Omega$, $f(x) \in \Omega$.

A set $\Omega \subset \mathbb{R}^n$ is *robust positively invariant* (RPI) for system $x^+ f(x, w)$ and constraint set $(\mathbb{X}, \mathbb{W})$ if $\Omega \subseteq \mathbb{X}$ and if, for all $x \in \Omega$, $f(x, w) \in \Omega$ for all $w \in \mathbb{W}$.

Robust control invariance is illustrated in Figure 1.

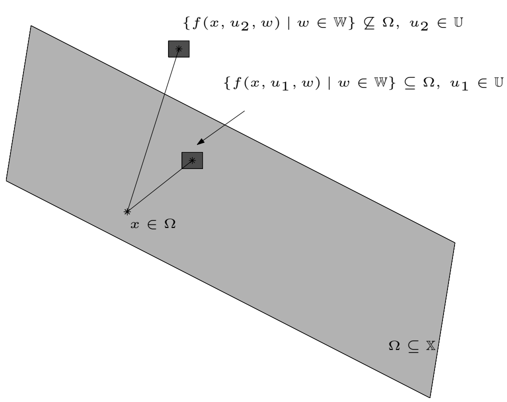

Fig. 1. Illustration of Robust Control Invariance

If a given set Ω is RCI for system (1) and constraint set $(\mathbb{X}, \mathbb{U}, \mathbb{W})$, then there exists a control law $\nu : \Omega \to \mathbb{U}$ such that the set Ω is a RPI for the system $x^+ = Ax + B\nu(x) + w$ and constraint set $(\mathbb{X}_\nu, \mathbb{W})$ where

$$\mathbb{X}_\nu \triangleq \mathbb{X} \cap \{x \mid \nu(x) \in \mathbb{U}\}. \tag{4}$$

A suitable control law $\nu(\cdot)$ is a selection from set valued map $\mathcal{U}(x)$ defined for each $x \in \Omega$ by:

$$\mathcal{U}(x) \triangleq \{u \in \mathbb{U} \mid Ax + Bu + w \in \Omega,\ \forall w \in \mathbb{W}\} \tag{5}$$

In other words, the control law $\nu(\cdot)$ is any control law satisfying:

$$\nu(x) \in \mathcal{U}(x),\ x \in \Omega \tag{6}$$

Similar statement may be made for control and positively invariant sets.

Given an initial state $x \in \Omega$, the set of possible state trajectories (each trajectory corresponding to a particular realization of the disturbance process) lies in the 'exact tube' $\{X_0, X_1, \ldots\}$ defined by the following set recursion for $i \in \mathbb{N}_+$:

$$X_i(x) \triangleq \{Ay + B\nu(y) + w \mid y \in X_{i-1}(x),\ w \in \mathbb{W}\} \tag{7}$$

with $X_0(x) \triangleq \{x\}$. The set sequence $\{X_i(x)\}$ is the exact 'tube' containing all the possible state trajectory realizations due to the uncertainty and it contains the actual state trajectory corresponding to a particular uncertainty realization. If $x \in \Omega$, the sets $X_i(x),\ i \in \mathbb{N}$ satisfy $X_i(x) \subseteq \Omega, \forall i \in \mathbb{N}$ because $x \in \Omega$ and Ω is RPI for the system $x^+ = Ax + B\nu(x) + w$ and constraint set $(\mathbb{X}_\nu, \mathbb{W})$. The shapes of the sets $X_i(x),\ i \in \mathbb{N}$ change with time i and they are, in general, complex geometrical objects (depending on the properties of the couple (A, B), control law $\nu(\cdot)$ and the geometry of constraint set $(\mathbb{X}, \mathbb{U}, \mathbb{W})$). In the sequel we demonstrate that it is possible to generate a tube (sequence of state sets) with elements of fixed, simple shape that bounds the exact tube and possesses robust control invariance properties.

First, we introduce the concepts of *set robust positive invariance* and *set robust control invariance*:

Definition 3: A set of sets Φ is *set robust positively invariant* (SRPI) for system $x^+ = f(x, w)$ and constraint set $(\mathbb{X}, \mathbb{W})$ if, for any set $X \in \Phi$, (i) $X \subseteq \mathbb{X}$ and, (ii) $X^+ = f(X, \mathbb{W}) \triangleq \{f(x, w) \mid x \in X,\ w \in \mathbb{W}\} \subseteq Y$ for some set $Y \in \Phi$.

Definition 4: A set of sets Φ is *set robust control invariant* (SRCI) for system $x^+ = f(x, u, w)$ and constraint set $(\mathbb{X}, \mathbb{U}, \mathbb{W})$ if, for any set $X \in \Phi$, (i) $X \subseteq \mathbb{X}$ and, (ii) there exists a policy $\theta_X : X \to \mathbb{U}$ such that $X^+ = f(X, \theta_X, \mathbb{W}) \triangleq \{f(x, \theta_X(x), w) \mid x \in X,\ w \in \mathbb{W}\} \subseteq Y$ for some set $Y \in \Phi$.

In the sequel we consider the simple case when each element X of Φ has the form $z \oplus R$, i.e. $\Phi \triangleq \{z \oplus R \mid z \in Z\}$ and refer to R as the shape of X. Note that any arbitrary $X \in \Phi$ satisfies that $X \subseteq Z \oplus R$ by definitions of Φ and Minkowski set addition. An appropriate illustration of set robust control invariance is given in Figure 2.

It follows from the definition of Φ and the fact that R is fixed that each $X \in \Phi$ is parameterized by z; accordingly we use X_z to denote $z \oplus R$.

III. A FAMILY OF SET ROBUST CONTROL INVARIANT SETS

As stated above, we restrict attention to a *set of sets* defined as follows:

$$\Phi \triangleq \{z \oplus R \mid z \in Z\} \tag{8}$$

where $R \subset \mathbb{R}^n$ and $Z \subset \mathbb{R}^n$. We are interested in characterizing all those Φ that are *set robust control invariant*.

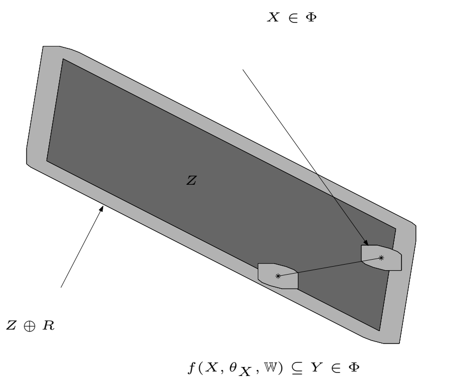

Fig. 2. Illustration of Set Robust Control Invariance

Thus the sets X and Y in definitions 3 and 4 have the form $X = z_1 \oplus R$ and $Y = z_2 \oplus R$ with $z_1, z_2 \in Z$.

Assumption 2: (i) The set R is RCI for system (1) and constraint set $(\alpha\mathbb{X}, \beta\mathbb{U}, \mathbb{W})$ where $(\alpha, \beta) \in [0,1) \times [0,1)$.

(ii) The control law $\nu : R \to \beta U$ is such that R is RPI for system $x^+ = Ax + B\nu(x) + w$ and constraint set $(\mathbb{X}_\nu, \mathbb{W})$, where $\mathbb{X}_\nu = \alpha\mathbb{X} \cap \{x \mid \nu(x) \in \beta\mathbb{U}\}$.

Note that the control law $\nu(\cdot)$ in assumption 2 (ii) exists by assumption 2 (i). Let the sets $\mathbb{U}_\nu$, $\mathbb{Z}$ and $\mathbb{V}$ be defined by:

$$\mathbb{U}_\nu \triangleq \{\nu(x) \mid x \in R\}, \tag{9}$$

$$\mathbb{Z} \triangleq \mathbb{X} \ominus R, \ \ \mathbb{V} \triangleq \mathbb{U} \ominus \mathbb{U}_\nu \tag{10}$$

Assumption 2 implies that the set $\mathbb{Z}$ and $\mathbb{V}$ are non-empty sets and contain the origin in their interiors. We also assume:

Assumption 3: (i) The set Z is a CI set for the nominal system (3) and constraint set $(\mathbb{Z}, \mathbb{V})$.

(ii) The control law $\varphi : Z \to \mathbb{V}$ is such that Z is PI for system $z^+ = Az + B\varphi(z)$ and constraint set $\mathbb{Z}_\varphi$, where $\mathbb{Z}_\varphi \triangleq \mathbb{Z} \cap \{z \mid \varphi(z) \in \mathbb{V}\}$.

Existence of the control law $\varphi(\cdot)$ in assumption 3 (ii) is guaranteed by assumption 3 (i).

We can now establish the following result:

Theorem 1: Suppose that Assumptions 2 and 3 are satisfied. Then $\Phi \triangleq \{z \oplus R \mid z \in Z\}$ is *set robust control invariant* for system $x^+ = Ax + Bu + w$ and constraint set $(\mathbb{X}, \mathbb{U}, \mathbb{W})$.

Proof: Let $X \in \Phi$, then $X = z \oplus R$ for some $z \in Z$ so that $X \subseteq Z \oplus R$. By assumption 3, $Z \subseteq \mathbb{Z}$ so that, by (10), $X \subseteq Z \oplus R \subseteq \mathbb{Z} \oplus R \subseteq \mathbb{X}$. For every $x \in X$ we have $x = z + y$, where $y = y_z(x) \triangleq x - z \in R$. Let the control law $\theta_z : z \oplus R \to \mathbb{U}$ be defined by $\theta_z(x) \triangleq \varphi(z) + \nu(y)$, $y = y_z(x)$ and let $u = \theta_z(x)$. Then the successor state to x is $x^+ \triangleq Ax + B\theta_z(x) + w = A(z+y) + B(\varphi(z) + \nu(y)) + w = z^+ + Ay + B\nu(y) + w$ where $z^+ \triangleq Az + B\varphi(z)$. It follows from assumption 3 that $z^+ \in Z$ and, by assumption 2, that $Ay + B\nu(y) + w \in R$, $\forall (y, w) \in R \times \mathbb{W}$. We conclude that $x^+ = Ax + B\theta_z(x) + w \in Y$, $\forall (y, w) \in R \times \mathbb{W}$ where $Y \triangleq z^+ \oplus R \in \Phi$. That $\theta_z(x) \in \mathbb{U}$ for all $x \in X$ follows from assumptions 2 and 3, since $\varphi(z) \in \mathbb{V} = \mathbb{U} \ominus \mathbb{U}_\nu$, $\forall z \in Z$ and $\nu(y) \in \mathbb{U}_\nu$, $\forall y \in R$ so that $\theta_z(x) \in (\mathbb{U} \ominus \mathbb{U}_\nu) \oplus \mathbb{U}_\nu \subseteq \mathbb{U}$, $\forall x \in X$ and every $X \in \Phi$. ■

The (local) control law $\theta_z : X_z \to \mathbb{U}$ where $X_z = z \oplus R \in \Phi$ is defined in the proof of Theorem 1:

$$\theta_z(x) = \varphi(z) + \nu(y_z(x)), \ \ y_z(x) \triangleq x - z \tag{11}$$

Theorem 1 exploits linearity and state decomposition as illustrated in Figure 3.

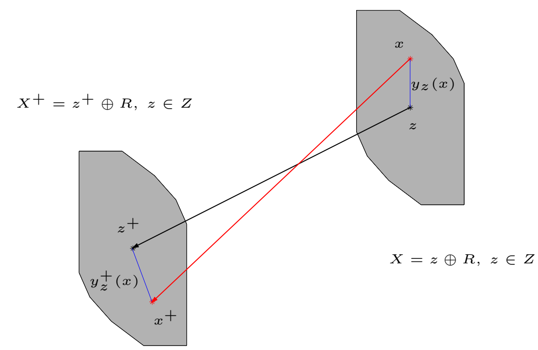

Fig. 3. Exploiting Linearity – Theorem 1

Generally, there exists an infinite number of set robust control invariant sets Φ, since, given a set R satisfying assumption 2 (i), there exists an infinite number of CI sets Z satisfying assumption 3 (i). Our attention is therefore restricted to important cases such as the minimal and the maximal set robust control invariant set for a given set R satisfying assumption 2 (i). Before proceeding we need to define the maximal control invariant set.

Definition 5: A set $\Omega_{\max} \subset \mathbb{R}^n$ is the maximal control invariant set for the system $x^+ = f(x, u)$ and constraint set $(\mathbb{X}, \mathbb{U})$ if $\Omega_{\max}$ is control invariant set for the system $x^+ = f(x, u)$ and constraint set $(\mathbb{X}, \mathbb{U})$ and $\Omega_{\max}$ contains all control invariant sets for the system $x^+ = f(x, u)$ and constraint set $(\mathbb{X}, \mathbb{U})$.

We now introduce:

Definition 6: Given a RCI set R satisfying assumption 2 (i), a set $\Phi_{\max} = \{z \oplus R \mid z \in Z\}$ is a *maximal set robust control invariant* (MSRCI) set for system $x^+ = Ax + Bu + w$ and constraint set $(\mathbb{X}, \mathbb{U}, \mathbb{W})$ if Z is the maximal control invariant set satisfying assumption 3 (i).

In order to define the minimal set robust control invariant set for a given set R satisfying assumption 2 (i), we define:

$$\rho(Z) \triangleq \sup_{z \in Z} |z|_p \tag{12}$$

Any equilibrium point for the difference equation $x^+ = f(x, u)$, $(\bar{x} = f(\bar{x}, \bar{u}))$ is a minimal, control invariant set the system $x^+ = f(x, u)$ and constraint set $(\mathbb{X}, \mathbb{U})$ providing that $(\bar{x}, \bar{u}) \in (\mathbb{X}, \mathbb{U})$; however one can always extract the equilibrium point such that $|\bar{x}|_p$ is minimal.

Definition 7: Given a RCI set R satisfying assumption 2 (i), a set $\Phi_{\min} = \{z \oplus R \mid z \in Z\}$ is a *minimal set robust control invariant* (mSRCI) set for system $x^+ = Ax + Bu + w$ and constraint set $(\mathbb{X}, \mathbb{U}, \mathbb{W})$ if the set Z

is a control invariant set satisfying assumption 3 (i) and Z minimizes $\rho(Z)$ over all control invariant sets satisfying assumption 3 (i) (i.e. Z is contained in the minimal p–norm ball).

The following observation is a direct consequence of Definitions 6 and 7.

Proposition 1: Let a set R satisfying assumption 2 (i) be given. Then: (i) the minimal set robust control invariant set is $\Phi_{\min} = 0 \oplus R$ and, (ii) the maximal set robust control invariant set is $\Phi_{\max} = \{z \oplus R \mid z \in Z_{\max}\}$, where $Z_{\max}$ is the maximal control invariant set satisfying assumption 3 (i).

Proof of this observation follows directly from Definitions 6 and 7, the facts that $\{0\}$ is a control invariant set satisfying assumption 3 (i), $\rho(\{0\}) = 0$, and $Z_{\max}$, the maximal control invariant set satisfying assumption 3 (i), exists and is unique [14], [20].

If $\mathbb{W} = \{0\}$, then $R = \{0\}$ satisfies assumption 2 (i), so that $\{0\}$ is the minimal and $Z_{\max}$ the maximal set robust control invariant set for the system $x^+ = Ax + Bu + w$ and constraint set $(\mathbb{X}, \mathbb{U}, \{0\})$. The set $Z_{\max}$, in this case, corresponds to the maximal control invariant set for the system $x^+ = Ax + Bu$ and constraint set $(\mathbb{X}, \mathbb{U})$.

IV. Dynamical Behavior of $X \in \Phi \triangleq \{z \oplus R \mid z \in Z\}$

We consider now a tube (sequence of sets) $\{X_{z_0}, X_{z_1}, X_{z_2}, \ldots\}$, $X_{z_i} \triangleq z_i \oplus R$, where the sequence $\{z_0, z_1, z_2 \ldots\}$ satisfies:

$$z_{i+1} = Az_i + B\varphi(z_i),\ z_0 \in Z. \tag{13}$$

It follows from assumption 3 and the fact $z_0 \in Z$ that that $z_i \in Z$ for all $i \in \mathbb{N}$ implying that $X_{z_i} \in \Phi$ for all $i \in \mathbb{N}$. Next we consider a sequence of sets $\{Y_i\}$ with initial set $Y_0 \subseteq X_{z_0} \in \Phi$ generated using the control policy $\pi_{z_0} \triangleq \{\theta_{z_0}, \theta_{z_1}, \theta_{z_2}, \ldots\}$ where, for each $z \in Z$, θ_z is specified as in Theorem 1 (see (11)). The sets $\{Y_i\}$ satisfy the following difference equation for $i \in \mathbb{N}_+$:

$$Y_i \triangleq \{Ax + B\theta_{z_i}(x) + w \mid x \in Y_{i-1},\ w \in \mathbb{W}\} \tag{14}$$

with initial condition $Y_0 \subseteq X_{z_0}$. The tube $\{Y_i\}$ is a forward reachable tube [14], [18]; any trajectory of the system $x^+ = Ax + Bu + w$ with initial state $x \in Y_0$, control policy π_{z_0} and admissible disturbance sequence lies in this tube.

Proposition 2: Suppose assumptions 2 and 3 hold. The sequences $\{Y_i\}$ and $\{X_{z_i}\}$ satisfy

$$Y_i \subseteq X_{z_i},\ \forall i \in \mathbb{N}. \tag{15}$$

Proof: Suppose, for some $i \in \mathbb{N}$, $Y_i \subseteq X_{z_i}$. By Theorem 1, $Y_{i+1} \subseteq X_{z_{i+1}}$. Since $Y_0 \subseteq X_{z_0}$, the desired result follows by induction. ■

We now make the assumption:

Assumption 4: The origin is exponentially stable for the system $z^+ = Az + B\varphi(z)$ with a region of attraction Z, where Z and $\varphi(\cdot)$ satisfy assumption 3.

Definition 8: A set R is robustly exponentially stable for $x^+ = f(x, w)$, $w \in \mathbb{W}$, with a region of attraction S if there exists a $c > 0$ and a $\gamma \in (0, 1)$ such that any solution $\{x_i\}$ of $x^+ = f(x, w)$ with initial state $x(0) \in S$, and admissible disturbance sequence $\{w_i\}$ ($w_i \in \mathbb{W}$ for all $i \in \mathbb{N}$) satisfies $d(x(i), R) \le c\gamma^i d(x(0), R)$ for all $i \in \mathbb{N}$.

Theorem 2: Suppose assumptions 2, 3 and 4 are satisfied and that R is compact. Then R is robustly exponentially stable for the system $x^+ = Ax + B\theta_{z_i}(x) + w$ where $\{z_i\}$ satisfies (13) with any initial state z_0 such that $x_0 \in z_0 \oplus R$. The region of attraction is $Z \oplus R$.

Proof: If $x \in X_z \triangleq z \oplus R$, $d(x, R) \le |z|$. Since $x_i \in X_{z_i}$ for all $i \in \mathbb{N}$ and $z_i \to 0$ exponentially, it follows that $d(x_i, R) \to 0$ exponentially. Hence R is robustly exponentially stable for the system $x^+ = Ax + B\theta_{z_i}(x) + w$ with a region of attraction $Z \oplus R$. ■

An illustration of Theorem 2 is given in Figure 4.

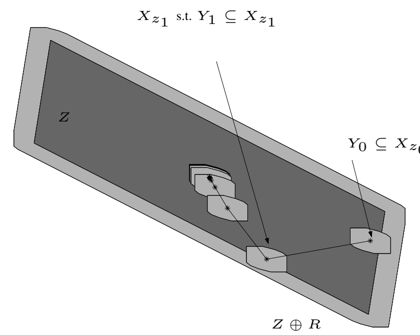

Fig. 4. Illustration of Theorem 2

V. Constructive Simplifications

Here we provide possible choices for the sets R and Z and the corresponding control laws.

A. A set R and feedback control law $\nu(\cdot)$ satisfying assumption 2

It is desirable to confine the spread of trajectories due to the bounded disturbance. This can be achieved if R is chosen to be minimal in some sense. The set R can be constructed using recent results established in [21]–[23].

1) Case I: A possible choice for $\nu(\cdot)$ is $\nu(x) = Kx$. The corresponding controlled nominal system is:

$$x^+ = A_K x + w,\ A_K \triangleq (A + BK), \tag{16}$$

where K is chosen so that all the eigenvalues of A_K are strictly inside the unit disk. In the absence of constraints, the minimal RPI (mRPI) set F_∞ (for the system (16) and constraint set $(\mathbb{R}^n, \mathbb{W})$) exists, is unique, compact and contains the origin [24, Sect. IV]; it is given by

$$F_\infty = \bigoplus_{i=0}^{\infty} A_K^i \mathbb{W}. \tag{17}$$

Thus F_∞ is mRPI for system (16) and constraint set $(\mathbb{X}_K, \mathbb{W})$, $\mathbb{X}_K \triangleq \mathbb{X} \cap \{x \mid Kx \in \mathbb{U}\}$ if $F_\infty \subseteq \mathbb{X}_K$. It is impossible, in general, to obtain an explicit characterization of the mRPI set F_∞ except when A_K is nilpotent or $A_K^s = \alpha I$ for $(\alpha, s) \in [0, 1) \times \mathbb{N}$ [9], [24]. In [21] a method for computation of an ε ($\varepsilon > 0$) outer RPI approximation of F_∞ is given:

Theorem 3: [21] If $0 \in \text{interior}(\mathbb{W})$, then for all $\varepsilon > 0$, there exists $\zeta \in [0, 1)$ and a corresponding integer s such that the following set inclusions

$$A_K^s \mathbb{W} \subseteq \zeta \mathbb{W} \text{ and } \zeta(1-\zeta)^{-1} F_s \subseteq \mathbb{B}_p^n(\varepsilon) \quad (18)$$

are true. Furthermore, if (18) is satisfied, then the set $F_{(\zeta,s)}$ defined by:

$$F_{(\zeta,s)} \triangleq (1-\zeta)^{-1} F_s \quad (19)$$

$$F_s \triangleq \bigoplus_{j=0}^{s-1} A_K^j \mathbb{W}, \; i \in \mathbb{N}_+ \quad (20)$$

is an RPI set for the system (16) and constraint set $(\mathbb{R}^n, \mathbb{W})$ such that $F_\infty \subseteq F_{(\zeta,s)} \subseteq F_\infty \oplus \mathbb{B}_p^n(\varepsilon)$.
This result can be extended to case when the origin is in the *relative interior* of $\mathbb{W}$ [23].

Thus, the first suitable candidate for a set R satisfying assumption 2 is the set $F_{(\zeta,s)}$ defined in (19) providing that $F_{(\zeta,s)} \subseteq \mathbb{X}_K$.

2) Case II: A significantly improved result appears in [22], [23] where it is shown that a linear programming problem can be posed the solution of which yields a piecewise affine control law $\nu^0(\cdot)$, parameters α^0, β^0, and a RPI set R for system $x^+ = Ax + B\nu^0(x) + w$ and constraint set $(\mathbb{X}_{\nu^0}, \mathbb{W})$ where $\mathbb{X}_{\nu^0} \triangleq \alpha^0 \mathbb{X} \cap \{x \mid \nu^0(x) \in \beta^0 \mathbb{U}\}$. If α^0 and β^0 lie in $(0, 1)$, assumption 2 is satisfied.

B. A set Z and feedback control law $\varphi(\cdot)$ satisfying assumption 3

1) Case I: Given a set R and control law $\nu(\cdot)$ satisfying assumprion 2, a set Z and control law $\varphi(\cdot)$ satisfying assumption 3 can be chosen as follows. The control law $\varphi(\cdot)$ can be chosen to be any exponentially stabilizing linear state feedback control law for system $z^+ = Az + Bv$; thus $\varphi(z) = Kz$ and a suitable choice for Z is any positively invariant set or system $z^+ = (A + BK)z$ and the (tighter) constraint set specfified in assumption 3. For any set $X \in \Phi$ ($X = X_z = z \oplus R$ for some $z \in Z$), the corresponding control law is $\theta_z(\cdot)$ is then defined by:

$$\theta_z(x) = Kz + \nu(y_z(x)), \; x \in X_z = z \oplus R \quad (21)$$

with $y_z(x) \triangleq x - z$.

2) Case II: Further simplification is obtained if the control law $\nu(\cdot)$ is linear ($\nu(x) \triangleq K_1 x$) and the set R is chosen to be robust positively invariant set for the system $x^+ = (A + BK_1)x + w$ and constraint set $(\mathbb{X}_\nu, \mathbb{W})$. In this case the control law $\varphi(\cdot)$ can be chosen to be any exponentially stabilizing linear state feedback control law ($\varphi(x) \triangleq K_2 x$) and the set Z is chosen to be positively invariant set for the system $z^+ = (A + BK_2)z$ and constraint set $\mathbb{Z}_\varphi \triangleq \mathbb{Z} \cap \{z \mid \varphi(z) \in \mathbb{U} \ominus \mathbb{U}_\nu\}$ where $\nu(\cdot)$ is the control law defined in Case I in Subsection V-A. For any set $X \in \Phi$ ($X = X_z = z \oplus R$ for some $z \in Z$), the corresponding control law is $\theta_z(\cdot)$ is:

$$\theta_z(x) = K_2 z + K_1 y_z(x), \; x \in X_z = z \oplus R \quad (22)$$

with $y_z(x) \triangleq x - z$.

3) Case III: The simplest but most conservative case is when the control law $\varphi(\cdot)$ is chosen to be identical to the control law $\nu(\cdot)$ which is chosen to be any exponentially stabilizing linear state feedback control law ($\nu(x) = Kx$). In this case the set Z can be chosen to be any positively invariant set Z for the system $z^+ = (A + BK)z$ and tighter constraint set specified in assumption 3.

For any set $X \in \Phi$ ($X = X_z = z \oplus R$ for some $z \in Z$), the corresponding control law is $\theta_z(\cdot)$ is then defined by:

$$\theta_z(x) = Kz + Ky_z(x) = Kx, \; x \in X_z = z \oplus R \quad (23)$$

with $y_z(x) \triangleq x - z$.

4) Case IV: Further improvement is obtained if the feedback control law $\varphi(\cdot)$ is obtained by exploiting the robust model predictive control schemes discussed in more detail in [11], [12] and the set Z is defined as in [11], [12].

C. Ellipsoidal Sets

The theory outlined above does not require the sets R and Z to have a particular shape but merely to possess certain properties specified in assumptions 2 and 3; these properties are illustrated in Figure 5 for ellipsoidal sets R and Z. Sets R and Z with ellipsoidal shape can be computed by solving Linear Matrix Inequalities, exploiting standard results in [13], [17], [25]–[27].

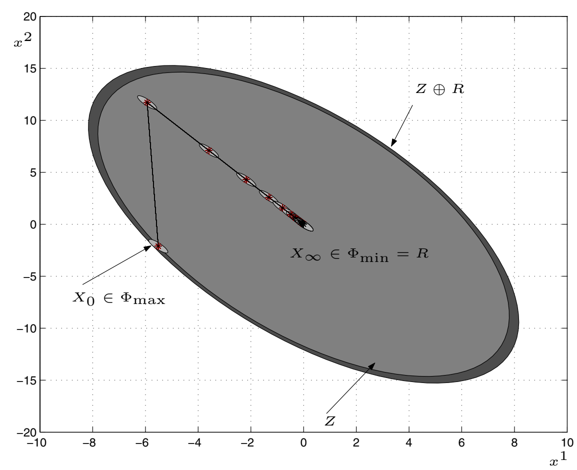

Fig. 5. Sample Set Trajectory for a set $X_0 \in \Phi$ – Ellipsoidal Sets

VI. ILLUSTRATIVE EXAMPLE

To illustrate and enable simple visualization of our results we consider a simple, second order, linear discrete time system defined by:

$$x^+ = \begin{bmatrix} 1 & 0.2 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + w, \quad (24)$$

which is subject to the control constraints:

$$u \in \mathbb{U} \triangleq \{\, u \in \mathbb{R} \mid -2 \le u \le 2\} \quad (25)$$

and state constraints:

$$x \in \mathbb{X} \triangleq \{\, x = (x^1, x^2)' \in \mathbb{R}^2 \mid -10 \le x^1 \le 1, \; -10 \le x^2 \le 10\} \quad (26)$$

while the the disturbance is bounded by $\mathbb{W}$ where:

$$\mathbb{W} \triangleq \{\ w \in \mathbb{R}^2 \mid |w|_\infty \leq 0.1\}. \tag{27}$$

We illustrate our results by considering the simplest case – case III. The local, linear control law

$$u = -[2.4\ 1.4]x \tag{28}$$

places the eigenvalues of the closed loop system to 0.2 and 0.4. The invariant set R is computed by using the methods of [21]. In Figure 6 we show the tube (trajectory of a set) $\{X_0, X_1, X_2, \dots\}$ with initial set $X_0 = z_0 \oplus R \in \Phi$ where $z_0 \in Z$ and z_0 is one of the vertices of Z. The set trajectory converges to $X_\infty \in \Phi = R$.

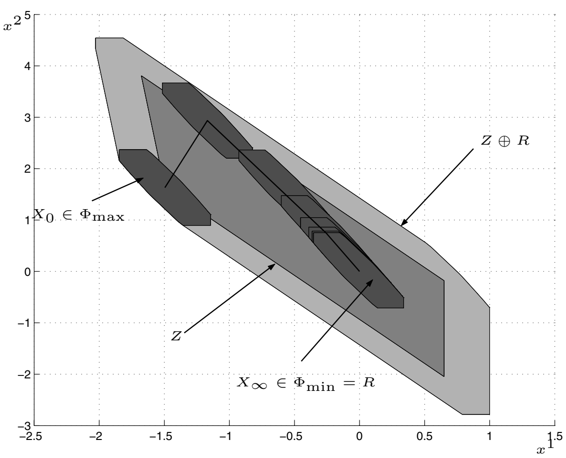

Fig. 6. Sample Set Trajectory for a set $X_0 \in \Phi_{\max}$

VII. Conclusions

In this paper we have introduced the concept of set robust control invariance that is a generalization of standard set invariance theory. A novel family of set robust control invariant sets has been characterized and the most important members of this family, the minimal and the maximal, have been identified. A set of constructive simplifications and methods have been also provided. These results are useful in the design of robust receding horizon controllers; in particular the paper shows how terminal sets of the form $Z \oplus R$ may be constructed, removing the restriction, commonly imposed, that the terminal set should be R. The concept was illustrated, for a simple case, by a numerical example.

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