

Controlling the density evolution of classical, thermodynamic and quantum systems

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Abstract—We present perturbation formulas for the (Markovian) density evolution of physical systems. Examples are provided from classical mechanics, thermodynamics and quantum mechanics. In all situations, formulas concern the relative entropy (Kullback-Leibler) distance between densities corresponding to unperturbed (uncontrolled) and perturbed (controlled) evolutions.

I. INTRODUCTION

Advances in microscopy and laser technology keep offering new opportunities and challenges to control theorists at the macromolecule and atomic scale. For instance, modern nanotechnology permits the active control of position and velocity of nanodevices through feedback. Velocity dependent feedback control (VFC) has been recently implemented to reduce thermal noise of a cantilever in atomic force microscopy (AFM) [1]. The thermodynamics of (VFC) on nanodevices has been recently studied in [2]. The *entropy production rate* is there decomposed into a *positive production rate* and an *entropy pumping rate*. The latter indicates how much entropy is pumped out or into the macromolecule by the control force. It may render the overall entropy production negative. This is at the basis of a *macromolecular cooling* mechanism [1], [3], [4]. This paper, which is a shortened version of the forthcoming journal paper [5], is devoted to a more general study of the entropy production rate in the presence of an external controlling action.

As is well-known, probabilistic models of a physical system evolution are needed whenever there is uncertainty on the initial condition (classical systems and closed quantum systems) and/or the system is coupled to a heat bath producing fluctuations (thermodynamic systems and open quantum systems). In this paper, we consider various instances of physical systems whose evolution is, in some suitable sense, “Markovian”. We show that it is possible to derive perturbation formulas in all such situations. The perturbation of the Hamiltonian may be interpreted as a *control function* which is designed by the controller in order to drive the system to a desired state. Applications include, besides active control of nanodevices [2], control of robot and vehicular motion [6], [7], molecular kinetics [8], quantum computation [9], etc.

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As “distance” between the two densities associated to the unperturbed and perturbed evolution we employ *relative entropy*. Relative entropy, also called *divergence*, *Kullback-Leibler pseudo-distance*, etc. is one of the most unifying concepts of modern science. Originating from statistical mechanics (Boltzmann, Gibbs), problems of minimization of relative entropy have been promoted to a general inference method primarily by Kullback and Jaynes [17], [18], see [19] for a survey. It plays a central role in Information Theory since Shannon’s work [10]. Important recent applications are in the area of signal processing see e.g. [20], [21] and references therein. A constrained minimization of relative entropy for distributions on path space is connected through Sanov’s theorem to a problem of large deviations of the empirical distribution [22]. Relative entropy occurs in many problems of mathematical statistics and probability such as limit theorems [24], [25]. Its non-commutative analogue also plays an important role in Quantum Information Theory see e.g. [26], [27], [28].

II. A RELATIVE ENTROPY PRODUCTION FORMULA

Consider two families of nonnegative functions on \mathbb{R}^n : $\{\rho_t; t_0 \leq t \leq t_1\}$ and $\{\tilde{\rho}_t; t_0 \leq t \leq t_1\}$. We are interested in how the relative entropy

$$\mathbb{D}(\rho_t || \tilde{\rho}_t) = \int_{\mathbb{R}^n} \log \frac{\rho_t}{\tilde{\rho}_t} \rho_t dx,$$

between ρ_t and $\tilde{\rho}_t$ evolves in time. We make the following

Assumptions:

A1 There exist measurable functions $v(x, t)$ and $\tilde{v}(x, t)$ such that $\{\rho_t; t_0 \leq t \leq t_1\}$ and $\{\tilde{\rho}_t; t_0 \leq t \leq t_1\}$ are everywhere positive C^1 solutions of

$$\frac{\partial \rho_t}{\partial t} + \nabla \cdot (v \rho_t) = 0, \quad (1)$$

$$\frac{\partial \tilde{\rho}_t}{\partial t} + \nabla \cdot (\tilde{v} \tilde{\rho}_t) = 0. \quad (2)$$

A2 For every $t \in [t_0, t_1]$

$$\lim_{|x| \rightarrow \infty} \tilde{v}(x, t) \tilde{\rho}_t(x) = 0, \quad (3)$$

$$\lim_{|x| \rightarrow \infty} v(x, t) \rho_t(x) = 0, \quad (4)$$

$$\lim_{|x| \rightarrow \infty} v(x, t) \rho_t(x) \log \frac{\rho_t}{\tilde{\rho}_t}(x) = 0 \quad (5)$$

Theorem 2.1: Suppose $\mathbb{D}(\rho_t|\tilde{\rho}_t) < \infty, \forall t \geq 0$. Assume moreover A1 and A2 above. Then

$$\frac{d}{dt}\mathbb{D}(\rho_t|\tilde{\rho}_t) = \int_{\mathbb{R}^n} \left[\nabla \log \frac{\rho_t}{\tilde{\rho}_t} \cdot (v - \tilde{v}) \right] \rho_t dx. \quad (6)$$

This result follows from a direct calculation employing integration by parts [5]. When ρ_t integrates to one and $\tilde{\rho}_t(x) \equiv 1$, we have that $-\mathbb{D}(\rho_t|\tilde{\rho}_t) = S(\rho_t)$ the entropy. Taking $\tilde{v}(x, t) \equiv 0$, the first condition in A2 is verified.

Corollary 2.2: Suppose $\{\rho_t; t_0 \leq t \leq t_1\}$ is of class C^1 and

$$\lim_{|x| \rightarrow \infty} v(x, t)\rho_t(x) = 0, \lim_{|x| \rightarrow \infty} v(x, t)\rho_t(x) \log \rho_t(x) = 0.$$

Suppose $S(\rho_t) < \infty, \forall t \in [t_0, t_1]$. Then

$$\frac{d}{dt}S(\rho_t) = - \int_{\mathbb{R}^n} [\nabla \log \rho_t \cdot v] \rho_t dx. \quad (7)$$

III. CLASSICAL MECHANICS

We apply Theorem 2.1 to a mechanical system with phase space given, for simplicity, by $\Gamma = \mathbb{R}^n \times \mathbb{R}^n$. Here $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$ are the configurational and momentum variables. Let $H(x, y, t)$ be the (smooth) Hamiltonian function. Physical trajectories of the system $t \rightarrow \{q(t), p(t)\}$ obey Hamilton's equations

$$\dot{q}(t) = \nabla_y H(q(t), p(t), t), \quad (8)$$

$$\dot{p}(t) = -\nabla_x H(q(t), p(t), t). \quad (9)$$

Γ represents *classical pure states*. We can obtain *statistical mixtures* by introducing probability measures on Γ . Let $\rho(x, y, t)$ be a probability density, and consider an *observable* $f: \Gamma \rightarrow \mathbb{R}$. Then, the expected value of f in state ρ is given by

$$\langle f \rangle_\rho = \int \int f(x, y) \rho(x, y, t) dx dy. \quad (10)$$

A standard calculation (with appropriate boundary conditions at infinity) yields Liouville's equation

$$\frac{\partial \rho_t}{\partial t} = \{H, \rho\}, \quad (11)$$

where the *Poisson bracket* is given by

$$\{H, \rho\} = \nabla_x H \cdot \nabla_y \rho - \nabla_y H \cdot \nabla_x \rho.$$

It should be observed that (11) is precisely the continuity equation of the form (1) for the system (8)-(9), where a cancellation has occurred. Consider now the *perturbed Hamiltonian*

$$\tilde{H}(x, y, t) = H(x, y, t) + \Delta H(x, y, t).$$

Let $\rho(x, y, t_0)$ be the initial state, and let $\tilde{\rho}(x, y, t)$ and $\rho(x, y, t)$ be the corresponding solutions of the Liouville equation (11) for the unperturbed and the perturbed system, respectively. We are interested in the following standard canonical perturbation theory problem: How does the change in the Hamiltonian affect motion? More explicitly:

Question: How does the relative entropy between ρ and $\tilde{\rho}$ depend on the perturbation Hamiltonian ΔH ?

Under suitable conditions at infinity corresponding to A1 and A2 above, Theorem 2.1 gives

$$\frac{d}{dt}\mathbb{D}(\rho_t|\tilde{\rho}_t) = \left\langle \left\{ \log \frac{\rho_t}{\tilde{\rho}_t}, \Delta H \right\} \right\rangle_{\rho_t}, \quad (12)$$

where, as before, $\{\cdot, \cdot\}$ stands for the Poisson bracket. Suppose now that

$$H(x, y, t) = \frac{1}{2m} y \cdot y + V(x, t)$$

and that

$$\Delta H(x, y, t) = \Delta V(x, t).$$

Then (12) gives

$$\frac{d}{dt}\mathbb{D}(\rho_t|\tilde{\rho}_t) = \left\langle -\nabla_y \log \frac{\rho_t}{\tilde{\rho}_t} \cdot \nabla_x \Delta V \right\rangle_{\rho_t}. \quad (13)$$

IV. THERMODYNAMICAL SYSTEMS

Consider now an open thermodynamical system whose macroscopic evolution is modelled by an n -dimensional diffusion process $\{x(t); t_0 \leq t\}$. The components of x form a *complete set*, i.e. all other variables have a much shorter relaxation time. It is then customary, see e.g. [11], to assume that x is a *vector Markov* diffusion process $\{x(t); t_0 \leq t\}$. Let $\bar{\rho}(x)$ be the probability density corresponding to thermodynamical equilibrium. According to Gibbs' postulate, we have

$$\bar{\rho}(x) = C \exp\left[-\frac{H(x)}{kT}\right]. \quad (14)$$

where H is the Hamiltonian function, and the (forward) Ito differential of x is

$$dx(t) = -\frac{\sigma^2}{2kT} \nabla H(x(t)) dt + \sigma dw, \quad (15)$$

where w is a standard n -dimensional Wiener process. For the ergodic properties of this class of diffusions see e.g. [12, Section 7.5]. Under reasonable assumptions, see e.g. [13], as $t \rightarrow \infty$, the density ρ_t of $x(t)$ tends to $\bar{\rho}$ in *relative entropy* and, consequently, in total variation [14]. Suppose now that the thermodynamical system is subject to other forces, so that the actual differential is

$$dx(t) = \left(-\frac{\sigma^2}{2kT} \nabla H(x(t)) + u(x(t), t) \right) dt + \sigma dw,$$

where $u(x(t), t)$ is a stochastic process representing action on the system not included in ∇H . We think of u as of a *feedback control* function designed to alter the natural evolution of the density toward equilibrium. Let ρ_t^u denote the density of the controlled process. We are interested in the evolution of $\mathbb{D}(\rho_t|\rho_t^u)$. In order to do that, we first need to recall a few basic facts about Markov diffusion processes.

Consider a vector Markov diffusion process $\{\xi(t); t_0 \leq t\}$ with (forward) Ito differential

$$d\xi(t) = b(\xi(t), t) dt + \sigma dw.$$

The probability density ρ_t of $\xi(t)$ satisfies (at least weakly) the *Fokker-Planck equation*

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (b\rho) = \frac{\sigma^2}{2} \Delta \rho. \quad (16)$$

Following Nelson [29], define the *current drift field* v by

$$v(x, t) = b(x, t) - \frac{\sigma^2}{2} \nabla \log \rho_t(x).$$

Then (16) can be rewritten in the form of (1)

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (v\rho) = 0. \quad (17)$$

We get the following corollary of Theorem 2.1.

Corollary 4.1: Let $\{x(t); t_0 \leq t\}$ and $\{\tilde{x}(t); t_0 \leq t\}$ be n -dimensional Markov diffusion processes with drifts b and \tilde{b} , respectively. Let ρ_t and $\tilde{\rho}_t$ be the corresponding densities, where ρ_0 may be different from $\tilde{\rho}_0$. Then

$$\frac{d}{dt} \mathbb{D}(\rho_t | \tilde{\rho}_t) = \int_{\mathbb{R}^n} \left(\nabla \log \frac{\rho_t}{\tilde{\rho}_t} \cdot (b - \tilde{b} - \frac{\sigma^2}{2} \nabla \log \frac{\rho_t}{\tilde{\rho}_t}) \right) \rho_t dx \quad (18)$$

If, moreover, $\tilde{b} = b$, and $\tilde{\rho}_t \equiv \bar{\rho}$ the Maxwell-Boltzmann distribution (14), then $F(\rho_t) = \mathbb{D}(\rho_t | \bar{\rho})$ is the *free energy functional* and we recover a well known result, see e.g. [11]:

Corollary 4.2: The free energy decay is given by

$$\frac{d}{dt} F(\rho_t) = -\frac{\sigma^2}{2} \int_{\mathbb{R}^n} |\nabla \log \frac{\rho_t}{\bar{\rho}}|^2 \rho_t dx. \quad (19)$$

The physical significance of these results is further discussed in [5], [36]. The results of this section may be readily extended to non-Markovian, finite-energy diffusions employing the Nelson-Föllmer kinematics [35], [23], see [5]. They also extend without too much difficulty to a large class of diffusions with constant but singular diffusion coefficient such as in the case of the Orstein-Uhlenbeck model of physical Brownian motion [29]. They may also be established for a large class of Markovian diffusion processes with local diffusion coefficient given the results in [31], [32], [33], [29], [34].

We now apply formula (18) to the evolution of thermodynamical systems described by (15). Let

$$b(x) = -\frac{\sigma^2}{2kT} \nabla H(x), \quad \tilde{b}(x, t) = b(x) + u(x, t),$$

where u is a continuous function. Then (18) reads

$$\frac{d}{dt} \mathbb{D}(\rho_t | \tilde{\rho}_t) = - \int_{\mathbb{R}^n} \left(\nabla \log \frac{\rho_t}{\tilde{\rho}_t} \cdot (u + \frac{\sigma^2}{2} \nabla \log \frac{\rho_t}{\tilde{\rho}_t}) \right) \rho_t dx. \quad (20)$$

Suppose that assumptions A1 and A2 of Section II are satisfied by $\rho_t = \bar{\rho}$ and $\tilde{\rho}_t = \rho_t^u$. We are interesting in modifying the rate at which ρ_t^u tends to the invariant density (14). Let $\alpha(t) > -\frac{\sigma^2}{2}$, and consider the feedback control

$$u(x, t) = -\alpha(t) \nabla \log \frac{\rho_t^u}{\bar{\rho}}(x). \quad (21)$$

Then, according to (20), we have

$$\frac{d}{dt} \mathbb{D}(\rho_t^u | \bar{\rho}) = - \left(\frac{\sigma^2}{2} + \alpha(t) \right) \int_{\mathbb{R}^n} |\nabla \log \frac{\rho_t^u}{\bar{\rho}}|^2 \rho_t^u dx. \quad (22)$$

Hence, the controlled diffusion still tends to the Maxwell-Boltzmann distribution but at a different, ‘‘modulated’’ rate. The controlled Fokker-Planck equation can be shown to be a *linear* equation and can be solved off-line, see [36] for details. In the Gauss-Markov case, it may be replaced by a linear matricial equation, see the next section.

V. THE GAUSS-MARKOV CASE

We now consider the situation of two centered n -dimensional Gauss-Markov processes with forward differential

$$d_+ x = Ax dt + \sigma d_+ w_+, \quad (23)$$

$$d_+ \tilde{x} = \tilde{A} \tilde{x} dt + \sigma d_+ \tilde{w}_+. \quad (24)$$

Here A and \tilde{A} are $n \times n$ continuous matrix functions. The densities at time t are completely determined by the covariance matrices $P(t) = E\{x(t)x(t)^T\}$ and $\tilde{P}(t) = E\{\tilde{x}(t)\tilde{x}(t)^T\}$. Indeed, $\rho_t(x) = C(t) \exp[-\frac{1}{2}x^T P(t)^{-1}x]$ and $\tilde{\rho}_t(x) = \tilde{C}(t) \exp[-\frac{1}{2}x^T \tilde{P}(t)^{-1}x]$, where, for instance,

$$C(t) = [(2\pi)^n \det P(t)]^{-\frac{1}{2}}.$$

The covariances satisfy the linear differential equations

$$\dot{P} = AP + PA^T + \sigma^2 I_n, \quad (25)$$

$$\dot{\tilde{P}} = \tilde{A}\tilde{P} + \tilde{P}\tilde{A}^T + \sigma^2 I_n. \quad (26)$$

Equation (18) now gives

$$\begin{aligned} \frac{d}{dt} \mathbb{D}(\rho_t | \tilde{\rho}_t) = & \int_{\mathbb{R}^n} \left[x^T (\tilde{P}(t)^{-1} - P(t)^{-1})(A - \tilde{A})x \right] \rho_t(x) dx \\ & - \frac{\sigma^2}{2} \int_{\mathbb{R}^n} x^T (\tilde{P}(t)^{-1} - P(t)^{-1})^2 x \rho_t(x) dx. \end{aligned} \quad (27)$$

Let $\tilde{A}(t) = A(t) + K(t)$, and let $u(t) = K(t)x(t)$. We think of u as a linear *feedback control* and $K(t)$ is the *gain matrix*. Then, (27) gives

Proposition 5.1:

$$\begin{aligned} \frac{d}{dt} \mathbb{D}(\rho_t | \tilde{\rho}_t) = & \int_{\mathbb{R}^n} \left[x^T (P(t)^{-1} - \tilde{P}(t)^{-1})K(t)x \right] \rho_t(x) dx \\ & - \frac{\sigma^2}{2} \int_{\mathbb{R}^n} x^T (\tilde{P}(t)^{-1} - P(t)^{-1})^2 x \rho_t(x) dx. \end{aligned}$$

Suppose now that A is a *stability matrix* (all eigenvalues in the open left half-plane), and let \bar{P} be the unique solution of the Lyapunov equation

$$0 = AP + PA^T + \sigma^2 I_n.$$

Of course, \bar{P} is given by

$$\bar{P} = \sigma^2 \int_0^\infty \exp[At] \exp[A^T t] dt.$$

Then $\bar{\rho} = \bar{C} \exp[-\frac{1}{2}x^T \bar{P}^{-1}x]$ is the invariant density for (23). Let us denote by ρ_t^u and $P^u(t)$ the density and covariance of the controlled diffusion, respectively. Let, for simplicity, α be a positive constant. Choosing the feedback gain

$$K(t) = -\alpha(\bar{P}^{-1} - P^u(t)^{-1}), \quad (28)$$

in view of Proposition 5.1, we get the relative entropy decay

$$\begin{aligned} & \frac{d}{dt} \mathbb{D}(\rho_t^u || \bar{\rho}) \\ &= - \left(\frac{\sigma^2}{2} + \alpha \right) \int_{\mathbb{R}^n} x^T (\bar{P}^{-1} - P^u(t)^{-1})^2 x \rho_t^u dx \\ &= - \left(\frac{\sigma^2}{2} + \alpha \right) \text{trace} [(\bar{P}^{-1} - P^u(t)^{-1}) P^u(t) \times \\ & \quad (\bar{P}^{-1} - P^u(t)^{-1})]. \end{aligned}$$

In view of (28), the matrix equation

$$\dot{P} = (A + K)P + P(A + K)^T + \sigma^2 I_n,$$

satisfied by $P^u(t)$ reads

$$\dot{P} = (A - \alpha \bar{P}^{-1})P + P(A^T - \alpha \bar{P}^{-1}) + (2\alpha + \sigma^2)I_n. \quad (29)$$

The linear matrix equation (29) with some appropriate initial condition P_0 may be solved off-line. Hence we can compute the feedback control law $u(t, x) = K(t)x$ with K given by (28) beforehand. If we can physically implement such a control, the controlled diffusion tends to (the same) equilibrium at a rate faster than the natural one.

VI. QUANTUM MECHANICS

In Sections II-V, we first presented an abstract result and next we applied it to particular cases to derive entropic perturbation formulas for classical uncertain systems. In the present section we show that for quantum mechanical systems analogues of these formulas can be obtained, whenever a Markovian evolution is assumed.

First we recall some basic facts from the quantum mechanical formalism. These can be found e.g. in [37], [9]. To every physical system S is associated a complex Hilbert space \mathcal{H}_S . Pure states for S , corresponding to maximal information with respect to some observable, are represented by unit vectors in \mathcal{H}_S . For the sake of simplicity, here we will consider only finite dimensional Hilbert spaces, but results hold in the general case.

We consider situations in which uncertainty on the system state affects our model. Statistical mixture of pure states can be introduced by considering the density operators ρ in \mathcal{H}_S : A density operator is a unit trace, Hermitian operator on \mathcal{H}_S . They form a convex set $\mathcal{D}(\mathcal{H}_S)$ and the extremals of $\mathcal{D}(\mathcal{H}_S)$ correspond to pure states. Physical observables are represented by Hermitian operators on \mathcal{H}_S .

Let A be an observable: The expected value of A is defined as:

$$\langle A \rangle_\rho := \text{trace} (\rho A), \quad (30)$$

where ρ is the density operator assigned to S .

Quantum analogues of entropic quantities have been considered since the very beginning of the mathematical foundation of quantum mechanics [38]. We are interested here in the quantum relative entropy, that is defined as:

$$\mathbb{D}(\rho || \tilde{\rho}) := \text{trace} (\rho (\log \rho - \log \tilde{\rho})). \quad (31)$$

As in the classical case, quantum relative entropy has the property of a pseudo-distance (see e.g. [27]).

The time evolution for the density operator of an isolated quantum system is determined by the Hamiltonian, i.e. the energy observable. The dynamical equation in this case, the Liouvillian *master equation*, is:

$$i\hbar \frac{d}{dt} \rho_t = [H, \rho_t], \quad (32)$$

where $[\cdot, \cdot]$ now denotes the commutator $[A, B] := AB - BA$ and \hbar is Plank's constant.

We remark the similarity with the classical case, that partially motivated the derivation of a quantum analogue of formula (12).

Consider the perturbed Hamiltonian:

$$\tilde{H} = H + \Delta H,$$

and the initial condition density operator ρ_{t_0} assigned to the system initial condition. Let ρ_t and $\tilde{\rho}_t$ be the solution of (32) corresponding to the unperturbed and the perturbed Hamiltonians, respectively. The idea, as in the previous sections, is to investigate how the relative entropy evolution depends on the perturbation. Using equation (32) along with some standard calculations, in [5] we obtain the following:

$$\frac{d}{dt} \mathbb{D}(\rho || \tilde{\rho}) = -\frac{i}{\hbar} \langle [\log \tilde{\rho}, \Delta H] \rangle_\rho. \quad (33)$$

The analogy with the corresponding formula in classical mechanics (12) is apparent. The perturbation can be interpreted as an additive control Hamiltonian.

VII. QUANTUM OPEN SYSTEMS

Consider a quantum system interacting with the environment in some uncontrollable way, namely an *open quantum system* [9], [39]. The most complete dynamical description of the situation should be done considering the tensor product space of both the system and the environment space. Typically, the environment has too many degrees of freedom to be modelled. Moreover, only partial information about initial states and interactions may be available. In these cases, we can still obtain a dynamical equation for the system state by simply averaging over the environment degrees of freedom [39]. If the system evolution is assumed to be Markovian, the generator of the system density operator dynamics can always be written in the following [40]:

$$\frac{d}{dt} \rho_t = -\frac{i}{\hbar} [H, \rho_t] + \mathcal{L}[\rho_t], \quad (34)$$

where H is the Hamiltonian for the system and the generator for the dissipative evolution \mathcal{L} has the form:

$$\mathcal{L}[\rho] = \frac{1}{2} \sum_k \left([L_k \rho, L_k^\dagger] + [L_k, \rho L_k^\dagger] \right). \quad (35)$$

The operators L_k can be derived under different assumptions on the couplings with the environment or on a phenomenological basis. This equation can be seen as a quantum analogue of a Fokker-Planck equation, since it describes the time evolution of the density operator in the absence of measurements.

Denote with $\bar{\rho}$ a stationary state for the evolution (34), commuting with the Hamiltonian. As an example and an interesting remark, the Gibbs state:

$$\rho_G = Z^{-1} e^{-\beta H},$$

where Z is the *partition function* and H the system Hamiltonian, is a stationary state for equation (34) when it is derived from e.g. *weak coupling* or *low density* limits [39]. These are two kind of constructive derivations of equations of the form (34) from the joint (tensor) description of the system and the environment, under suitable hypothesis on the coupling. Under these assumptions, the analogy with the thermodynamical setting we presented in Section IV is explicit. Under this assumption, from (34) we obtain for the relative entropy production [5]:

$$\frac{d}{dt} \mathbb{D}(\rho || \bar{\rho}) = \text{trace} (\mathcal{L}[\rho](\log \rho - \log \bar{\rho})). \quad (36)$$

To extend this (known) result to Hamiltonian perturbations, more suitable to control application, we impose now $\bar{\rho}$ to be a *fixed target state*, since the introduction of perturbations could in general destabilize a stationary state. In this setting, we have:

$$\begin{aligned} \frac{d}{dt} \mathbb{D}(\rho || \bar{\rho}) &= \frac{i}{\hbar} \langle [\log \bar{\rho}, \Delta H] \rangle_\rho + \\ &+ \text{trace} (\mathcal{L}[\rho](\log \rho - \log \bar{\rho})), \end{aligned} \quad (37)$$

where now ρ is undergoing a *perturbed evolution*, i.e. $H' = H + \Delta H$, and the different sign in the perturbation term accounts for this change of role. Possible application and candidates of further studies are the analysis of the convergency rate towards the equilibrium for a quantum statistical ensemble and the proof of the asymptotic stability of a target state for controlled Markovian evolutions (see e.g. the control techniques via Markovian feedback [41], [42]).

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