Proceedings of the
44th IEEE Conference on Decision and Control, and
the European Control Conference 2005
Seville, Spain, December 12-15, 2005

Stabilization of continuous-time nonlinear switched systems

Patrizio Colaneri, José C. Geromel and Alessandro Astolfi

Abstract— **The first result of this paper is a strategy for global stabilization of continuous time nonlinear switched system. The strategy is of closed loop nature (trajectory dependent) and is designed from the solution of what we call nonlinear Lyapunov-Metzler inequalities from which the stability condition is expressed. Next, results on the stabilization of nonlinear time varying polytopic systems are provided.**

I. Introduction

This paper aims at providing new results on stabilizing control synthesis for a continuous time switched nonlinear system of the following general form

$$\dot{x}(t) = f_{\sigma(t)}(x(t)) \ , \ x(0) = x_0 \tag{1}$$

defined for all $t \geq 0$ where $x(t) \in \mathbb{R}^n$ is the state, $\sigma(t)$ is the switching rule and x_0 is the initial condition. Given a set of vector fields $\{f_1(x), \cdots, f_N(x)\}$, such that $f_i(0) = 0$ for all $i = 1, \cdots, N$, two different classes of switched systems are studied. The first is characterized by the fact that the switching rule, for each $t \geq 0$, is such that

$$f_{\sigma(t)} \in \{f_1, \cdots, f_N\} \tag{2}$$

while the second one is such that, for each $t \geq 0$,

$$f_{\sigma(t)} \in co\{f_1, \cdots, f_N\} \tag{3}$$

where $co\{\cdot\}$ denotes the convex hull. It is important to make clear the basic difference between these two classes of switched systems. The model (2) naturally imposes a discontinuity on $f_{\sigma(t)}$ since this vector must jump instantaneously from f_i to f_j for some $i \neq j = 1, \cdots, N$ once switching occurs. In other words, $f_{\sigma(t)}$ is constrained to jump among the N vertices of the vector polytope $\{f_1, \cdots, f_N\}$. The model defined by (3) is more general in the sense that the interior of the same polytope is now feasible for $f_{\sigma(t)}$ and so it supports switching rules with no discontinuity with respect to time. As it will become clear in the sequel there are some important relationship between the stability conditions of both models.

This research was supported by grants from "Conselho Nacional de Desenvolvimento Científico e Tecnológico - CNPq" - Brazil, and supported by the Italian National Research Council (CNR) and by MIUR project New methods and algorithms for identification and adaptive control of technological systems

P. Colaneri is with Dipartimento di Elettronica e Informazione, Politecnico di Milano, Piazza leonardo da Vinci 32, Milano, Italy colaneri@elet.polimi.it

José C. Geromel is with DSCE / School of Electrical and Computer Engineering, UNICAMP, CP 6101, 13083 - 970, Campinas, SP, Brazil geromel@dsce.fee.unicamp.br

Alessandro Astolfi is with the Department of Electrical and Electronic Engineering, Imperial College, Exhibition Road, London SW7-2BT, United Kingdom a.astolfi@ic.ac.uk

The nonlinear switching stability condition has been spurred by reading the recent paper [21], where a method for stability analysis of switched and hybrid systems is provided by using polynomial and piecewise polynomial Lyapunov functions.

The notation used throughout is standard. Capital letters denote matrices, small letters denote vectors and small Greek letters denote scalars. For matrices or vectors ($'$) indicates transpose. The sets of real and natural numbers are denoted by $\mathbb{R}$ and $\mathbb{N}$ respectively.

II. State Switching Control

In this section we consider the system (1) where the switching rule satisfies (2). It is assumed that the state vector $x(t)$ is available for feedback for all $t \geq 0$, and our goal is to determine a function $u(\cdot) : \mathbb{R}^n \to \{1, \cdots, N\}$, such that the switching rule

$$\sigma(t) = u(x(t)) \tag{4}$$

assures that the equilibrium $x = 0$ of (1) is globally asymptotically stable. Note that we do not assume that any of the vector fields in the set $\{f_1, \cdots, f_N\}$ be either locally or globally asymptotically stable.

Let us define the simplex

$$\Lambda := \left\{ \lambda \in \mathbb{R}^N \ : \ \sum_{i=1}^{N} \lambda_i = 1 \ , \ \lambda_i \geq 0 \right\} \tag{5}$$

and the following function

$$v(x) := \min_{i=1,\cdots,N} V_i(x) \tag{6}$$

where $\{V_1, \cdots, V_N\}$ is a set of differentiable, positive definite and radially unbounded functions, which are zero at $x = 0$.

As it will be clear in the sequel, the function $v(x)$ is a candidate Lyapunov function, crucial for our purposes. However, even if the functions $V_i(x)$ are differentiable, the function $v(x)$ remains differentiable but it is not (in general) differentiable everywhere. To address this issue the set $I(x) = \{i : v(x) = V_i(x)\}$ plays a central role since $v(x)$ fails to be differentiable at all $x \in \mathbb{R}^n$ such that $\operatorname{card} I(x)$ is discontinuous [13].

Before proceeding, recall the class of Metzler matrices denoted by $\mathcal{M}$ and constituted by all matrices $\Pi \in \mathbb{R}^{N \times N}$ with elements π_{ij}, such that

$$\pi_{ij} \geq 0 \ \forall i \neq j \ , \ \sum_{i=1}^{N} \pi_{ij} = 0 \quad \forall j. \tag{7}$$

It is clear that any $\Pi \in \mathcal{M}$ presents an eigenvalue at the origin of the complex plane since $c'\Pi = 0$ where $c' = [1 \ \cdots \ 1]$. In addition, it is well known from the Frobenius-Perron's theorem that the eigenvector associated to the null eigenvalue of Π is non-negative, yielding the conclusion that there always exists $\lambda_\infty \in \Lambda$ such that $\Pi\lambda_\infty = 0$. The next theorem summarizes the main result of this section.

Theorem 1: Assume that there exist a set of functions $\{V_1, \cdots, V_N\}$, which are all differentiable, positive definite, radially unbounded and zero at zero, and a matrix $\Pi \in \mathcal{M}$ satisfying the Lyapunov-Metzler inequalities

$$\frac{\partial V_i'}{\partial x} f_i + \sum_{j=1}^{N} \pi_{ji} V_j < 0 \ , \ i = 1, \cdots, N \tag{8}$$

for all $x \neq 0$. Then, the state switching control (4) with

$$u(x(t)) = \arg \min_{i=1,\cdots,N} V_i(x(t)) \tag{9}$$

globally asymptotically stabilizes the equilibrium point $x = 0$ of the nonlinear systems (1).

Proof: To begin with, notice that the Lyapunov function (6) is differentiable, positive definite, radially unbounded and zero at $x = 0$. Moreover, the Lyapunov function (6) is not differentiable for all $x \in \mathbb{R}^n$. For this reason we need to deal with the Dini derivative (see [19])

$$D^+ v(x(t)) = \limsup_{h \to 0^+} \frac{v(x(t+h)) - v(x(t))}{h}. \tag{10}$$

Assume, in accordance with (9), that at an arbitrary $t \geq 0$, the state switching control is given by $\sigma(t) = u(x(t)) = i$ for some $i \in I(x(t))$. Hence, from (10) and the system dynamic equation (1) we have

$$\begin{aligned} D^+ v(x(t)) &= \min_{l \in I(x(t))} \frac{\partial V_l'}{\partial x} f_i \\ &\leq \frac{\partial V_i'}{\partial x} f_i, \end{aligned} \tag{11}$$

where the inequality holds from the fact that $i \in I(x(t))$. Finally, remembering that (8) is valid for $\Pi \in \mathcal{M}$ and that $V_j(x) \geq V_i(x)$ for all $j \neq i = 1, \cdots, N$ once again due to the fact that $i \in I(x(t))$, using the Lyapunov-Metzler inequalities (8) one gets

$$\begin{aligned} D^+ v(x(t)) &< -\left(\sum_{j=1}^{N} \pi_{ji} V_j(x) \right) \\ &\leq -\left(\sum_{j=1}^{N} \pi_{ji} \right) V_i(x) = 0 \end{aligned} \tag{12}$$

for all $x \neq 0$, which proves the claim. ■

Remark 1: Theorem 1 does not require that the set $\{f_1, \cdots, f_N\}$ be composed exclusively by (locally) asymptotically stable vector fields. Indeed, if a function $V_i(x)$ is locally quadratic, a necessary condition for the Lyapunov-Metzler inequalities to be feasible with respect to $\{V_1, \cdots, V_N\}$ is that the vector fields $f_i + (\pi_{ii}/2)x$ be locally asymptotically stable. Since $\pi_{ii} \leq 0$ this condition does not imply local asymptotic stability of any of the f_i's.

On the other hand, in general, the Lyapunov-Metzler inequalities imply that

$$D^+ V_i(x(t)) < |\pi_{ii}| V_i(x(t)) \quad \forall i$$

and, since the functions V_i are radially unbounded, this implies that the vector fields $\{f_1, \cdots, f_N\}$ are complete and that, along the trajectories of f_i the functions V_i are such that

$$V_i(x(t)) \leq V_i(x(0)) e^{|\pi_{ii}| t} \quad \forall i.$$

Remark 2: An interesting case occurs when all vector fields $\{f_1, \cdots, f_N\}$ are globally asymptotically stable for which the choice $\Pi = 0$ is possible and the state switching strategy proposed preserves stability. Furthermore, if the set $\{f_1, \cdots, f_N\}$ admits a unique Lyapunov function V, then the Lyapunov-Metzler inequalities admit a solution $V_1 = \cdots = V_N = V$ and $I(x(t)) = \{1, \cdots, N\}$ for all $t \geq 0$. In this classical but particular case, at any $t \geq 0$, the control law $u(x(t))$ being an arbitrary logic state $i \in \{1, \cdots, N\}$, asymptotic stability is once again guaranteed.

Remark 3: Theorem 1 also holds if the matrix Π is a function of x, *i.e.* $\Pi = \Pi(x)$, provided that, for each fixed $x \in \mathbb{R}^n$, it satisfies $\Pi(x) \in \mathcal{M}$.

Remark 4: In the literature of linear systems, the Lyapunov-Metzler inequalities, with $\Pi \in \mathcal{M}$ fixed, have been introduced in order to study the *Mean-Square* (MS) stability of Markov Jump Linear Systems (MJLS), see e.g. [4]. In that context, the Metzler matrix $\Pi \in \mathcal{M}$ is given and Π' represents the infinitesimal transition matrix of a Markov chain $\sigma(t)$ governing the dynamical system. In this respect, each component of the vector $\lambda(t) \in \Lambda$ is the probability of the Markov chain to be on the $i-th$ logical state and obeys the differential equation

$$\dot{\lambda}(t) = \Pi \lambda(t) \ , \ \lambda(0) = \lambda_0 \in \Lambda \tag{13}$$

where the eigenvector $\lambda_\infty \in \Lambda$ associated to the null eigenvalue of Π represents the stationary probability vector.

Consider now the modified Lyapunov-Metzler inequalities defined as:

$$\frac{\partial V_i'}{\partial x} f_i + \alpha \sum_{j=1}^{N} \pi_{ji} V_j < 0 \ , \ i = 1, \cdots, N \tag{14}$$

for all $x \neq 0$, where α is a positive parameter. This parameter multiplies all elements of the matrix Π, therefore the matrix $\alpha\Pi$ is still a Metzler matrix, i.e. $\alpha\Pi \in \mathcal{M}$ whenever $\Pi \in \mathcal{M}$. Notice that these new inequalities are those relative to vector fields $\{f_1/\alpha, \cdots, f_N/\alpha\}$, obtained by the time scaling $t \to t/\alpha$. If the solutions V_j exist for each $\alpha \geq 1$, then (pointwise)

$$\lim_{\alpha \to \infty} \sum_{j=1}^{N} \pi_{ji} V_j \leq 0, \quad \forall i.$$

Moreover, recalling the role of the vector λ_∞ in the Metzler matrix Π, we have

$$\sum_{i=1}^{N}\lambda_{\infty i}\sum_{j=1}^{N}\pi_{ji}V_j=0$$

It then follows that

$$\lim_{\alpha\to\infty}\Pi'\begin{bmatrix}V_1\\V_2\\\vdots\\V_N\end{bmatrix}=0$$

which implies that $\lim_{\alpha\to\infty}V_i=V$, for all $i=1,\cdots,N$. Finally, (14) yields

$$\sum_{i=1}^{N}\lambda_{\infty i}\frac{\partial V_i'}{\partial x}f_i<0,$$

and hence

$$\frac{\partial V'}{\partial x}f_{\lambda_\infty}<0,\quad f_{\lambda_\infty}=\sum_{i=1}^{N}\lambda_{\infty i}f_i.$$

This means that if (14) holds for a sufficiently large α, the "average" system characterized by the vector field f_{λ_∞} is globally asymptotically stable. This is a relevant point further, since it meets the already classical stability condition provided in [17] and [18]. To prove this fact in our present context, let us assume that there exists $\lambda_\infty\in\Lambda$ such that f_{λ_∞} is globally asymptotically stable, making possible the determination of $V>0$ satisfying the Lyapunov inequality $\frac{\partial V'}{\partial x}f_{\lambda_\infty}<0$. Hence, the switching rule (4) with

$$u(x(t))=\arg\min_{i=1,\cdots,N}\frac{\partial V'}{\partial x}f_i \tag{15}$$

makes the equilibrium point $x=0$ of the switched system (1) globally asymptotically stable. Indeed, considering the Lyapunov function $V(x)$ we have

$$\begin{aligned}\dot V(x(t)) &= \frac{\partial V'}{\partial x}f_{\sigma(t)}\\ &= \min_{i=1,\cdots,N}\frac{\partial V'}{\partial x}f_i\\ &= \min_{\lambda\in\Lambda}\frac{\partial V'}{\partial x}f_\lambda\\ &\le \frac{\partial V'}{\partial x}f_{\lambda_\infty}<0\end{aligned}$$

for all $x\neq 0$. It is important to keep in mind that the numerical determination (if any) of $\lambda\in\Lambda$ and $V>0$ such that

$$\frac{\partial V'}{\partial x}\left(\sum_{i=1}^{N}\lambda_i f_i\right)<0$$

is not a simple task even in the simplest case of linear time invariant systems.

We conclude this section introducing a guaranteed cost associated to the proposed state switching control law (9).

Lemma 1: Let $h(x)$ be a given p-valued mapping. Assume that there exist a set of functions $\{V_1,\cdots,V_N\}$, which are differentiable, positive definite, radially unbounded and zero at zero, and a matrix $\Pi\in\mathcal{M}$ satisfying the Lyapunov-Metzler inequalities

$$\frac{\partial V_i'}{\partial x}f_i+\sum_{j=1}^{N}\pi_{ji}V_j+h'h<0\ ,\ i=1,\cdots,N \tag{16}$$

for all $x\neq 0$. Then, the state switching control (4) with $u(x(t))$ given by the equation (9) globally asymptotically stabilizes the equilibrium point $x=0$ of (1) and it is such that the inequality

$$\int_0^\infty h(x)'h(x)dt<\min_{i=1,\cdots,N}V_i(x_0). \tag{17}$$

holds.

Proof: The proof has the same structure as the proof of Theorem 1. The Lyapunov function (6) and the Lyapunov-Metzler inequalities (16) yield

$$D^+v(x(t))<-h(x)'h(x)\qquad x\neq 0, \tag{18}$$

and, by integration it is readily verified that

$$\begin{aligned}v(x(t))-v(x(0)) &= \int_0^t D^+V(x(\tau))d\tau\\ &< -\int_0^t h(x)'h(x)d\tau,\end{aligned} \tag{19}$$

is valid $\forall\ t\ge 0$, proving thus the claim since, by asymptotic stability, $v(x(t))$ goes to zero as t goes to infinity. ■

III. Extended Lyapunov-Metzler inequalities

In this section we discuss a possible stabilizing switching strategy that includes the previous one as a particular case and hence may provide less conservative results. To this end, define a set of positive definite functions $\{W_1(x),\cdots,W_N(x)\}$ and the functions

$$H_i(x)=\sum_{j=1}^{N}\pi_{ji}W_j(x),\quad i=1,2,\cdots,N.$$

Due to the structure of the Metzler matrices, these functions cannot be strictly negative for all $i=1,\cdots,N$, since $\sum_{i=1}^{N}\lambda_{\infty i}H_i(x)=0$. As a result, for each $x\in\mathbb{R}^n$ the set

$$\tilde I(x)=\{i:\ H_i(x)\ge 0\}$$

is not empty, and it is possible to define the candidate Lyapunov function

$$v(x):=\min_{i\in\tilde I(x)}V_i(x) \tag{20}$$

Now, assume that there exist a set of function $\{V_1,\cdots,V_N\}$, which are differentiable, positive definite, radially unbounded functions, and zero at zero, a set of positive definite functions $\{W_1,\cdots,W_N\}$ and a matrix $\Pi\in\mathcal{M}$ satisfying the extended Lyapunov-Metzler inequalities

$$\frac{\partial V_i'}{\partial x}f_i+\sum_{j=1}^{N}\pi_{ji}W_j<0\ ,\ i=1,\cdots,N \tag{21}$$

for all $x \neq 0$. Finally, consider the switching control rule (4) with

$$u(x(t)) = \arg \min_{i \in \tilde{I}(x))} V_i(x) \tag{22}$$

Notice that the Lyapunov-Metzler inequalities (8) are recovered by imposing $W_i(x) = V_i(x)$ in the extended Lyapunov-Metzler inequalities (21). These inequalities imply the partition of the state-space into subsets where the Lyapunov function (6) is decreasing. However, this Lyapunov function is not continuous with respect to $x \in \mathbb{R}^n$ and the stabilizing property of the switching rule (22) depends on the jumps of $v(x)$ in the switching instances and by the possible presence of sliding modes. Despite this fact, inequalities (21) are easy to be handled and verified, and this is the main advantage for their use. Further research is necessary to incorporate to (22) additional constraints that imply global stability of the equilibrium point $x = 0$.

Remark 5: The role of the strictly positive parameters π_{ji} is immaterial in the case $N = 2$. Indeed, in this case, the inequalities reduce to

$$\frac{\partial \bar{V}_1'}{\partial x} f_1 + \Gamma < 0, \quad \frac{\partial \bar{V}_2'}{\partial x} f_2 + (-\Gamma) < 0 \quad x \neq 0,$$

where $\Gamma = W_2 - W_1$, $\bar{V}_1 = V_1/\pi_{21}$ and $\bar{V}_2 = V_2/\pi_{12}$.

Remark 6: The positive-definiteness assumptions in the Lyapunov-Mezler inequalities (21) (resp. (8)), can be relaxed by noting that, for all $i = 1, \cdots, N$, the i-th condition has to hold only for all $x \in \mathbb{R}^n$ such that $v(x) = V_i(x)$. Moreover, the functions V_i do not have to be positive definite, or even defined, for all $x \in \mathbb{R}^n$, provided that the function $v(x)$ is differentiable, positive definite, radially unbounded and zero at zero.

Finally it is possible to introduce, following the same rationale adopted in Lemma 1, a guaranteed cost associated to the state switching control law (22), as stated in the following Lemma.

Lemma 2: Let $h(x)$ be a given p-valued mapping. Assume that there exist a set of functions $\{V_1, \cdots, V_N\}$ which are differentiable, positive definite, radially unbounded and zero at zero, a set of positive definite functions $\{W_1, \cdots, W_N\}$ and a matrix $\Pi \in \mathcal{M}$ satisfying the Lyapunov-Metzler inequalities

$$\frac{\partial V_i'}{\partial x} f_i + \sum_{j=1}^{N} \pi_{ji} W_j + h(x)' h(x) < 0 \ , \ i = 1, \cdots, N \tag{23}$$

for all $x \neq 0$. If the state switching control (4) with $u(x(t))$ given by equation (22) globally asymptotically stabilizes the equilibrium point $x = 0$ of (1) then, the guaranteed cost

$$\int_0^\infty h(x)' h(x) dt < \min_{i \in \tilde{I}(x_0)} V_i(x_0). \tag{24}$$

holds.

IV. Stability of Time Varying Polytopic Systems

In this section we discuss the stability of systems defined by (1) which are classified in the literature as polytopic systems [5]. In this case, the very basic requirement on each trajectory of $\sigma(t)$ is that $\sigma(t) \in \Lambda$ for all $t \geq 0$. Since this property alone does not suffice to define the way $\sigma(t)$ evolves with time, we consider further that

$$\dot{\sigma}(t) = \Pi \sigma(t) \ , \ \sigma(0) = \sigma_0 \tag{25}$$

where $\Pi \in \mathbb{R}^{N \times N}$ is a Metzler matrix *a priori* known or to be determined by the designer. The rationale behind this choice follows from a well known property of this class of matrices. Whenever the initial condition $\sigma_0 \in \Lambda$ then $\sigma(t) \in \Lambda$ for all $t \geq 0$ as we have just required. From now on it is assumed that $\sigma_0 \in \Lambda$. Due to the fact that $\Pi \in \mathcal{M}$ is necessarily marginally stable, and there exists $\lambda_\infty \in \Lambda$ such that $\Pi \lambda_\infty = 0$ then $\sigma(t)$ evolves inside Λ and goes to λ_∞ as t goes to infinity. The time evolution of $\sigma(t)$ towards λ_∞ depends, of course, on each particular choice of $\Pi \in \mathcal{M}$. The results given in the sequel are based on the parameter dependent Lyapunov function

$$v(x(t)) := \left(\sum_{i=1}^{N} \sigma_i(t) V_i(x(t)) \right) \tag{26}$$

defined by an adequately determined set of functions $\{V_1, \cdots, V_N\}$, which are differentiable, positive definite, radially unbounded and zero at $x = 0$. The next theorem provides the way to determine either the Lyapunov function (26) and a sufficient condition for asymptotic stability of the considered system.

Theorem 2: Assume that there exist a set of functions $\{V_1, \cdots, V_N\}$, which are differentiable, positive definite, radially unbounded and zero at zero, a matrix $\Pi \in \mathcal{M}$, and a function $G(x, y)$ satisfying, for each x and y and for each $i = 1, 2, \cdots, N$ the following inequalities

$$\begin{aligned} 0 \quad > \quad & \frac{\partial V_i(x)'}{\partial x} y + \sum_{j=1}^{N} \pi_{ji} V_j(x) \\ & + \left(\frac{\partial G(x,y)}{\partial x} + \frac{\partial G(x,y)}{\partial y} \right)' (f_i(x) - y) \end{aligned} \tag{27}$$

for all $x, y \neq 0$. Then, provided $\sigma(t)$ is given by the rule (25), the equilibrium point $x = 0$ is a globally asymptotically stable equilibrium point of (1).

Proof: Assume that (27) holds and $\sigma(t) \in \Lambda$ for all $t \geq 0$. Multiplying each inequality by $\sigma_i(t)$, adding up for all $i = 1, \cdots, N$, and letting $y = \sum_{i=1}^{N} \sigma_i f_i(x)$ one gets

$$0 > \frac{\partial V_m(x,t)'}{\partial x} f_m(x,t) + \frac{\partial V_m(x,t)}{\partial t} \tag{28}$$

where

$$V_m(x,t) = \sum_{i=1}^{N} \sigma_i(t) V_i(x), \quad f_m(x,t) = \sum_{i=1}^{N} \sigma_i(t) f_i(x)$$

This states that the time derivative of the parameter dependent Lyapunov function (26), which is differentiable, positive definite, radially unbounded and zero at zero, is negative along all trajectories of $\dot{x}(t) = f_{\sigma(t)}(x(t))$, proving thus global asymptotical stability of (1). ■

Several remarks are in order. First notice that, as before, Theorem 2 does not require the set $\{f_1, \cdots, f_N\}$ to be composed only by asymptotically stable vector fields. Of course, this is a consequence of our previous assumption which implies that the variables $\sigma(t)$ and $\dot{\sigma}(t)$ are not independent but are coupled together by the linear model (25). The second remark comes from the fact that the inequalities (27) must be satisfied for all $x, y \neq 0$ and in particular for all $x, y \neq 0$ satisfying the additional constraint $y = f_i(x)$ for each $i = 1, \cdots, N$ implying that the inequalities

$$\frac{\partial V_i'}{\partial x} f_i + \sum_{j=1}^{N} \pi_{ji} V_j < 0 \ , \ i = 1, \cdots, N \tag{29}$$

must hold for all $x \neq 0$, which is nothing else than the stability condition provided by Theorem 1 for state switching control. The conclusion is that the stability condition of Theorem 2 is more exigent than the one of Theorem 1. This fact was expected since the set of vector fields defined in (2) is a subset of that defined in (3).

V. An illustrative example

In this section we propose an illustrative example of application of some of the proposed theoretical tools introduced so far. The purpose of this example is twofold. First, to illustrate the theory, and then, to underscore that the proposed switching law may be non-robust (this is actually true also if the underlying system is linear), hence further research is needed to derive a robustly stabilizing switching mechanism.

Consider the so-called Artstein circle [1], [10], [14], namely the system described by the equation

$$\begin{array}{rcl} \dot{x}_1 & = & (-x_1^2 + x_2^2)u \\ \dot{x}_2 & = & -2x_1 x_2 u. \end{array} \tag{30}$$

This system is asymptotically controllable and it is (robustly) asymptotically stabilizable exploiting the results in [10].

We exploit the observation in Remark 6, applied to the system (30). For that, consider the set of vector fields

$$\mathcal{F} = \{f_1, f_2\}$$

with

$$f_1 = \begin{bmatrix} -x_1^2 + x_2^2 \\ -2x_1 x_2 \end{bmatrix}$$

and $f_2 = -f_1$. Note that f_1 (resp. f_2) is obtained from system (30) setting $u = 1$ (resp. $u = -1$). Consider now the functions

$$V_+(x_1, x_2) = (x_1^2 + x_2^2)\frac{\pi \ \mathrm{sign}(x_2) - 2\arctan(x_1/x_2)}{2x_2}$$

for $x_1 \geq 0$ and

$$V_-(x_1, x_2) = (x_1^2 + x_2^2)\frac{\pi \ \mathrm{sign}(x_2) + 2\arctan(x_1/x_2)}{2x_2}$$

for $x_1 \leq 0$. Notice that

$$V_+(0, x_2) = V_-(0, x_2) = \frac{\pi}{2}|x_2|.$$

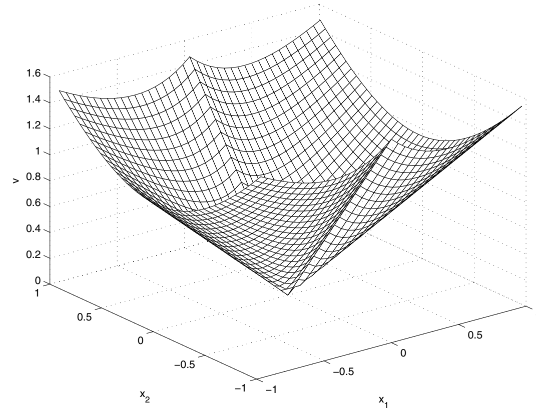

Fig. 1. The function $v(x)$ for the Artstein circle.

Let us define

$$V_1(x_1, x_2) = V_+(x_1, x_2)$$

for $x_1 \geq 0$ and define V_1 for $x_1 \leq 0$ such that the resulting function is continuous for all $x \in \mathbb{R}^2$ and $V_1(x_1, x_2) > V_-(x_1, x_2)$ for all $x_1 < 0$. Analogously, let

$$V_2(x_1, x_2) = V_-(x_1, x_2)$$

for $x_1 \leq 0$ and define V_2 for $x_1 \geq 0$ such that the resulting function is continuous for all $x \in \mathbb{R}^2$ and $V_2(x_1, x_2) > V_+(x_1, x_2)$ for all $x_1 > 0$. The function

$$v(x) = \min\{V_1(x), V_2(x)\},$$

depicted in Figure 1, is continuous, positive definite, radially unbounded and zero at zero.

For this example, and after simple computations, the conditions (21) yields

$$\begin{array}{ll} -(x_1^2 + x_2^2) + \Gamma \leq 0 & \forall x_1 \geq 0 \\ -(x_1^2 + x_2^2) - \Gamma \leq 0 & \forall x_1 \leq 0 \end{array}$$

so that the following selection

$$\Gamma(x) = \frac{1}{2}(x_1^2 - x_2^2)$$

is a consistent one.

The above discussion leads itself to the following interpretation. The control law

$$u(x) = \begin{cases} 1 & \text{if} \ \ x \geq 0 \\ -1 & \text{if} \ \ x < 0 \end{cases}$$

or, alternatively

$$u(x) = \begin{cases} 1 & \text{if} \ \ x > 0 \\ -1 & \text{if} \ \ x \leq 0 \end{cases}$$

can be shown to globally asymptotically stabilize the system (30). This is the same control law proposed in [10]. Therein (see also [14]) it is however argued that this control law is not robust against measurement noises, and a (simple) robust modification of this controller (in the spirit of the result in [11]) is proposed. Hence, the results presented in this paper, and in its linear counterparts, see [20], have to be understood as first steps toward a general (robust) stabilization theory for switched systems. We believe that this theory could be developed exploiting the results in this paper and the results in [11], [10] and [12].

VI. Conclusion

In this paper we have introduced stability conditions for switched systems. They have been used for control synthesis of state dependent (closed loop) switching rules using nonlinear Lyapunov-Metzler inequalities. The determination of a guaranteed cost associated to each control strategy has been addressed. The relationship between switched systems and time varying polytopic systems stability has been investigated, yielding useful mathematical properties for both classes of dynamical systems.

Various issues deserve more attention. The first is related to the development of numerical algorithms for the solution of the introduced nonlinear Lyapunov-Metzler inequalities . The second one is the possible generalization of the stability conditions to cope with an optimal guaranteed cost. Taking into account the nonlinear nature of the involved stability conditions, this point constitutes a real theoretical challenge. Finally, the crucial and difficult issue of robust stability requires an in-depth investigation. These issues are being currently studied.

References

[1] Z. Artstein, Stabilization with relaxed controls, Nonlinear Anal. TMA, 7 (1983), pp. 1163–1173.

[2] S. P. Boyd, L. El Ghaoui, E. Feron and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*, SIAM, Philadelphia, 1994.

[3] Branicky, M. S., “Multiple Lyapunov functions and other analysis tools for switched and hybrid systems”, *IEEE Trans. Automat. Contr.*, vol. 43, pp. 475-482, 1998.

[4] Feng, X., Loparo, K. A., Ji, Y., and Chizeck, H. J., “Stochastic stability properties of jump linear systems”, *IEEE Trans. Automat. Contr.*, vol. 37, pp. 38-53, 1992.

[5] Geromel, J. C., Peres, P. L. D., and Bernussou, J., “On a convex parameter space method for linear control design of uncertain systems”, *SIAM J. Control Optim.*, vol. 29, pp. 381-402, 1991.

[6] Geromel, J. C, de Oliveira, M. C, and Hsu, L., “LMI characterization of structural and robust stability”, *Linear Algebra and its Applications*, vol. 285, pp. 69-80, 1998.

[7] Hespanha, J. P., “Uniform stability of switched linear systems : extensions of LaSalle’s principle”, *IEEE Trans. Automat. Contr.*, vol. 49, pp. 470-482, 2004.

[8] Hockerman-Frommer, J., Kulkarni, S. R., and Ramadge, P. J., “Controller switching based on output predictions errors”, *IEEE Trans. Automat. Contr.*, vol. 43, pp. 596-607, 1998.

[9] Johansson, M., and Rantzer, A., “Computation of piecewise quadratic Lyapunov functions for hybrid systems”, *IEEE Trans. Automat. Contr.*, vol. 43, pp. 555-559, 1998.

[10] C. Prieur, A Robust Globally Asymptotically Stabilizing Feedback: The Example of the Artstein’s Circles, A. Isidori, F. Lamnabhi-Lagarrigue, W. Respondek, eds., in *Nonlinear Control in the Year 2000*, Volume 2 (NCN), Lecture Notes in Control and Information Sciences, Vol. 258, Springer Verlag, London (2000), pp. 279-300.

[11] C. Prieur, Uniting local and global controllers with robustness to vanishing noise, *Math. Control Signals Systems*, 14 (2001), pp. 143-172.

[12] C. Prieur and A. Astolfi, Robust stabilization of chained systems via hybrid control, *IEEE Trans. Automat. Control*, 48 (2003), pp. 1768-1772.

[13] R. Rockafellar, *Convex Analysis*, Princeton Press, 1970.

[14] E.D. Sontag, Clocks and insensitivity to small measurement errors. ESIAM: COCV, `www.emath.fr/cocv/` 4 (1999), pp. 537-557.

[15] Xu, X., and Antsaklis, P. J., “Optimal control of switched systems based on parameterization of the switching instants”, *IEEE Trans. Automat. Contr.*, vol. 49, pp. 2-16, 2004.

[16] Ye, H., Michel, A. N., and Hou, L., “Stability theory for hybrid dynamical systems”, *IEEE Trans. Automat. Contr.*, vol. 43, pp. 461-474, 1998.

[17] D. Liberzon, *Switching in Systems and Control*, Birkhauser, 2003.

[18] Zhao, J., and Dimirovski, G. M., “Quadratic stability of a class of switched nonlinear systems”, *IEEE Trans. Automat. Contr.*, vol. 49, pp. 574-578, 2004.

[19] Krishna M. Garg, “ Theory of Differentiation : A Unified Theory of Differentiation Via New Derivate Theorems and New Derivatives”, Wiley-Interscience, 1998.

[20] J.C. Geromel, P. Colaneri, ”Stabilization of continuous-time switched systems”, Proc. *16th IFAC World Congress*, Prague, 2005.

[21] S. Prajna, A. Papachristodoulou. ”Analysis of switched and hybrid systems”, Proc. *American Control Conference*, Denver, 2003.