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On Uniform Global Asymptotic Stability Nonlinear Discrete-Time Systems

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***Abstract*—The paper presents new characterizations of the uniform global asymptotic stability (UGAS) for nonlinear and time-varying discrete-time systems. Under mild assumptions, it is shown that weak zero-state detectability (WZSD) is equivalent to UGAS for globally uniformly stable systems. On the other hand, WZSD is further simplified by employing the notion of reduced limiting systems. Then, a second characterization of UGAS is proposed in terms of the detectability condition of the reduced limiting systems associated with the original system. As a result, we derive a generalized, discrete-time version of the well-known Krasovskii-LaSalle theorem but for time-varying, not necessarily periodic, systems.**

I. Introduction

THE purpose of this paper is to study the uniform global asymptotic stability (for short, UGAS) for nonlinear time-varying discrete-time systems with or without an output-dominant perturbation. Our research is motivated by control engineering applications in computer controlled systems [2] and, in particular, by sampled-data nonlinear control and stabilization of nonlinear discrete-time systems (see, for instance, [8]-[10] and numerous references therein). Other related but independent work includes recent studies on the derivation of stability criteria for nonlinear and non-autonomous discrete-time systems [9, 10].

In this paper, we exploit the theory of limiting equations originally proposed by Arstein [1] for nonautonomous ordinary differential equations, and establish a new set of stability results for discrete-time systems described by nonautonomous ordinary difference equations. Our main purpose is to develop two new characterizations of UGAS.

The first characterization of UGAS is based on the finding that, under some mild assumptions, weak-zero state detectability is equivalent to UGAS for globally stable discrete-time systems. The second characterization of UGAS says that to test UGAS for a time-varying discrete-time system it suffices to verify a detectability condition for its reduced limiting systems. It is shown how the first characterization of UGAS can serve as a fundamental tool to extend the classical Lyapunov Direct Method for stability analysis of discrete-time systems. More interestingly, the second characterization of UGAS is a valuable tool for practical applications. Through the use of limiting systems, the checking UGAS of the original system is reduced down to the test of some detectability properties for its reduced limiting systems, which are often reminiscent of "zero-dynamics" of the original system. To save space, some results are only stated without proofs. Readers can contact the authors for the detailed proofs.

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II. Preliminaries

In this section, we give several basic notions and results that are instrumental for the development of our main results in the rest of this paper. Throughout this paper, we denote $Z_+ = \aleph \cup \{0\}$, $Z_+^m = \{ n \in \aleph \cup \{0\} \mid n \ge m \}$, $\forall m \in Z_+$, $|v| = \sqrt{v_1^2 + v_2^2 + \cdots + v_p^2}$, $\forall v = (v_1, v_2, \cdots, v_p) \in \Re^p$ and $B_r^p = \{ x \in \Re^p \mid |x| \le r \}$ for any $r > 0$. Whenever there is no confusion, we simply denote $B_r := B_r^p$.

A. Limiting solutions and detectability

We study an uncertain, nonlinear, time-varying, discrete-time system described by

$$x(n+1) = f(n, x(n)) + \Delta f(n, x(n)) \tag{1}$$

$$y(n) = h(n, x(n)) \tag{2}$$

where $x \in \Re^p$ is a state vector, $y \in \Re^q$ is an output vector, and f, Δf and h are all functions defined on $Z_+ \times \Re^p$ with $f(n,0) = \Delta f(n,0) = 0$, $h(n,0) = 0, \forall n \in Z_+$. System (1)-(2) is assumed in the output-injection form in the sense that $|\Delta f(n,x)| \le \lambda_r(h(n,x))$, $\forall n \in Z_+$, $\forall |x| \le r$, for any $r > 0$ and some continuous function $\lambda_r : B_r^q \to [0, \infty)$ with $\lambda_r(0) = 0$. A function $x : Z_+^m \to \Re^p$ for some $m \in Z_+$ is said to be a solution of (1) starting from $n = m$ if $x(n+1) = f(n, x(n)) + \Delta f(n, x(n))$, $\forall n \in Z_+^m$. To simplify the whole discussions, we also assume that h is continuous at every x, uniformly in n, i.e., for each $x \in \Re^p$ and each $\varepsilon > 0$, there exists a $\delta(\varepsilon, x) > 0$ so that $|h(n,y) - h(n,x)| < \varepsilon$, $\forall n \in Z_+$, $\forall |y - x| < \delta$.

To make our work self-contained, let us first recall some definitions of stability. Consider system (1). The origin is said to be uniformly stable (US) if, for any $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that for all solutions $x : Z_+^m \to \Re^p$ of (1) satisfying $|x(m)| < \delta$, we have $|x(n)| < \varepsilon, \forall n \geq m$. It is said that the solutions of (1) are globally uniformly bounded (GUB) if, for any $r > 0$ there exists a $\tilde{r} = \tilde{r}(r) > 0$ such that for all solutions $x : Z_+^m \to \Re^p$ of (1) satisfying $|x(m)| \leq r$, we have $|x(n)| \leq \tilde{r}, \forall n \geq m$. Moreover, the origin is said to be uniformly globally stable (UGS) when the origin is US and all solutions of (1) are GUB. It is said to be uniformly globally asymptotically stable (UGAS) if, additionally, it is uniformly globally attractive, i.e., for any $\varepsilon > 0$ and $r > 0$ there exists a $T = T(\varepsilon, r) > 0$ such that every solution $x : Z_+^m \to B_r$ of (1) satisfies $|x(n)| < \varepsilon, \forall n \geq m + T$.

In the following, we introduce the discrete-time variant of the notion of limiting solutions originally introduced for continuous-time non-autonomous systems (see, e.g., [1]). In essence, it describes the limit behavior of solutions as initial time instants approach to infinity.

Definition 1. A function $\overline{x} : Z_+ \to X$ is called a limiting solution of (1) with respect to an unbounded sequence $\{ t_k \}$ in Z_+, if there exist a $r > 0$ and a sequence $\{x_k : Z_+^{t_k} \to B_r\}$ of solutions of (1) such that for each $n \in Z_+$, the associated sequence $\{x_k(n + t_k)\}$ converges to $\overline{x}(n)$.

The following lemma guarantees the existence of limiting solutions. It is a direct consequence of Arzela-Ascoli lemma [4]. Thus, detailed proofs are omitted.

Lemma 1. Consider a system of the form (1)-(2). Let *r* be any positive constant. Then, for any unbounded sequence $\{ t_m \}$ in Z_+ and any sequence $\{x_m : Z_+^{t_m} \to B_r\}$ of solutions of (1), there exist a function $\overline{x} : Z_+ \to \Re^p$ and a subsequence $\{ m_k \}$ of $\{ m \}$ such that for each $n \in Z_+$, the sequence $\{x_{m_k}(n + t_{m_k})\}$ converges to $\overline{x}(n)$, i.e., $\overline{x}$ is a limiting solution of (1) with respect to $\{ t_{m_k} \}$. ■

In the following, let us state a necessary condition for UGAS in terms of limiting solutions.

(C1) $\lim_{n\to\infty} \overline{x}(n) = 0$ for any limiting solution $\overline{x}$ of (1).

By employing the uniformly attractive property of UGAS, it is not difficult to establish the following result which will be used in the characterization of UGAS in next section.

Lemma 2. Consider a system of the form (1)-(2). Then, (C1) is a necessary condition of UGAS. ■

Now, we define a detectability condition in terms of limiting solutions. It plays a central role in our new characterizations of UGAS in nonautonomous discrete-time systems.

Definition 2. System (1)-(2) is weakly zero-state detectable (WZSD) if, for any limiting solution $\overline{x}$ of (1) with respect to an unbounded sequence $\{ t_k \}$ in Z_+ that satisfies the following equation

$$\lim_{k\to\infty} h(n + t_k, \overline{x}(n)) = 0, \forall n \in Z_+, \quad (3)$$

we have

$$\inf_{n \in Z_+} |\overline{x}(n)| = 0. \quad (4)$$

Remark 1. Notice that under uniform stability, (4) can be replaced by $\lim_{n\to\infty} \overline{x}(n) = 0$. However, (4) is more convenient in practical applications [6].

B. Limiting functions: definition and basic properties

In this subsection, a definition of limiting functions for discrete-time functions will be proposed. First, let us introduce the concept of limiting functions associated with a discrete-time function.

Definition 3. Let $g : Z_+ \times \Re^p \to \Re^{\hat{p}}$ be a function. An unbounded sequence $\gamma = \{t_k\}$ in Z_+ is said to be an admissible sequence associated with g if there exists a function $g_\gamma : Z_+ \times \Re^p \to \Re^{\hat{p}}$ such that the associated sequence $\{\hat{g}_k : (n, x) \mapsto g(n + t_k, x)\}$ converges pointwise to g_γ on $Z_+ \times \Re^p$. The function g_γ is uniquely determined and called as the limiting function of *g* associated with γ. ■

For simplicity, we denote $\Lambda(g)$ the set of all admissible sequences associated with a function g. Recall that a function $g : Z_+ \times \Re^p \to \Re^{\hat{p}}$ is said to be uniformly bounded if, for any $r > 0$ there exists a $M(r) > 0$ so that $|g(n, x)| \leq M$ for all $|x| \leq r$ and all n in Z_+. We first give an interesting result about the uniformly bounded property. It can be proven by employing the connected property of Euclidean spaces and the continuity of *g*. To save space, a detailed proof is omitted.

Lemma 3. Suppose $g : Z_+ \times \Re^p \to \Re^{\hat{p}}$ is a function continuous at any *x*, uniformly in *n*. Suppose $\{g(n, 0)\}$ is bounded. Then, g is uniformly bounded. ■

In the theory of real analysis, it is well-known that every continuous function on $\Re^p$ is also uniformly continuous on every compact subset of $\Re^p$ [4]. In the following, we state an analog result for the function $g : Z_+ \times \Re^p \to \Re^{\hat{p}}$ continuous at any *x*, uniformly in *n*. Since the proof is similar, it is omitted here. To state such a result, let us say that a function $g : Z_+ \times \Re^p \to \Re^{\hat{p}}$ is uniformly continuous on a compact $K \subset \Re^p$, uniformly in *n*, if for any $\varepsilon > 0$, there exists a $\delta(\varepsilon, K)$ such that for all $x, \hat{x} \in K$ with $|x - \hat{x}| < \delta$, we have $|g(n, x) - g(n, \hat{x})| < \varepsilon, \forall n \in Z_+$.

Lemma 4. Suppose $g : Z_+ \times \Re^p \to \Re^{\hat{p}}$ is a function continuous at any *x*, uniformly in *n*. Then, for any compact $K \subset \Re^p$, it is uniformly continuous on *K*, uniformly in *n*. ■

Now, we would like to show that under the assumption of Lemma 3, the function g can be served as a discrete-time version of asymptotically almost periodic (AAP) functions [6]. More explicitly, the following result can be derived from Arzela-Ascoli lemma [4].

Proposition 1. Suppose $g: Z_+ \times \Re^p \to \Re^{\hat{p}}$ is a function continuous at any x, uniformly in n and $\{g(n,0)\}$ is bounded. Then, for any unbounded sequence $\{t_m\}$ in Z_+ there exist a subsequence $\{m_k\}$ of $\{m\}$ and a function $\bar{g}: Z_+ \times \Re^p \to \Re^{\hat{p}}$ such that $\bar{g}$ is a limiting function of g associated with $\{t_{m_k}\}$. Moreover, $\bar{g}$ is also continuous at any x, uniformly in n. ■

The following result is a direct consequence of Proposition 1.

Corollary 1. Suppose $a: Z_+ \to \Re^{\hat{p}}$ is a bounded function. Then, for any unbounded sequence $\{t_m\}$ in Z_+ there exists a subsequence $\{m_k\}$ of $\{m\}$ such that $\gamma = \{t_{m_k}\} \in \Lambda(a)$. ■

Before closing this section, let us propose a simple result related to the limiting functions of a composition function. The proof follows readily from the continuity property and thus is omitted.

Lemma 5. Let $F: \Re^{\hat{p}} \to \Re^{\hat{q}}$ be a continuous function and $g: Z_+ \times \Re^p \to \Re^{\hat{p}}$ be a function continuous at any x, uniformly in n and $\{g(n,0)\}$ is bounded. Then, the composition function $F \circ g$ of F and g is a function continuous at any x, uniformly in n and $\{F \circ g(n,0)\}$ is bounded. Moreover, for any $\gamma \in \Lambda(g)$, we have $\gamma \in \Lambda(F \circ g)$ and $(F \circ g)_\gamma = F \circ g_\gamma$. ■

III. Main Results

A. First Characterization of UGAS

In this subsection, we propose a new criterion for UGAS of time-varying systems. It is a generalization of the well-known Lyapunov theorem. Moreover, certain converse results will also be given. This way, a new characterization for UGAS can be proposed based on our approaches.

First, let us consider the following condition which, roughly speaking, states that the output energy eventually approaches to zero as the time goes to infinity.

(C2) For each $k \in \aleph$ and each $r > 0$, there exists a $M(k,r) \in Z_+^k$ such that, for any $n_0 \in Z_+$ and any solution $x: Z_+^{n_0} \to B_r$ of (1),

$$\min_{k \le m \le M} \sum_{n=m+n_0}^{m+n_0+k-1} \left| h(n, x(n)) \right|^2 < 1/k. \tag{5}$$

Based on (C2) and WZSD, we can obtain the following result guaranteeing UGAS.

Proposition 2. Consider a system of the form (1)-(2) where h is continuous at any x, uniformly in n. Suppose the origin is uniformly globally stable. Assume further that (C2) holds and the system is WZSD. Then, the origin is UGAS.

Proof. The theorem will be proven by contradiction. Suppose the origin is not UGAS. Then, there exist a $\varepsilon_0 > 0$ and a $r_0 > 0$ such that for each $m \in \aleph$, there exist some $t_m \in Z_+$, $s_m \in Z_+^m$ (thus, $s_m \ge m$) and a solution $x_m: Z_+^{t_m} \to \Re^p$ of (1) satisfying $|x_m(t_m)| \le r_0$ and $|x_m(s_m + t_m)| \ge \varepsilon_0$. Since the solutions are globally uniformly bounded, there is also a $\tilde{r}_0 > 0$ so that x_m lies within the compact set $B_{\tilde{r}_0}$. Notice that, due to the fact $|x_m(s_m + t_m)| \ge \varepsilon_0$, uniform Lyapunov stability of the origin implies that there exists a $\delta_0(\varepsilon_0) > 0$ such that $|x_m(n + t_m)| \ge \delta_0$ for all $m \in \aleph$ and all $0 \le n \le s_m$. For each $k \in \aleph$, let $M(k, \tilde{r}_0) \ge k$ be the positive integer given in (C2). Now, choose a subsequence $\{m_k\}$ of $\{m\}$ satisfying $m_k \ge k + M(k, \tilde{r}_0)$. In the following, let k be any positive integer. By (C2), there exists a $\tilde{t}_k \in Z_+$, with $k \le \tilde{t}_k \le M(k, \tilde{r}_0)$, so that

$$\begin{aligned} &\sum_{n=0}^{k-1} \left| h(n + \tilde{t}_k + t_{m_k}, x_{m_k}(n + \tilde{t}_k + t_{m_k})) \right|^2 \\ &= \min_{k \le m \le M} \sum_{\bar{n}=m+t_{m_k}}^{m+t_{m_k}+k-1} \left| h(\bar{n}, x_{m_k}(\bar{n})) \right|^2 < 1/k. \end{aligned} \tag{6}$$

Let $\hat{t}_k = \tilde{t}_k + t_{m_k} \ge k$ and $\hat{x}_k(n) = x_{m_k}(n), \forall n \in Z_+^{\hat{t}_k}$. Then, $\hat{x}_k$ is also a solution of (1) lying within the compact set $B_{\tilde{r}_0}$ and (6) can be rewritten as

$$\sum_{n=0}^{k-1} \left| h(n + \hat{t}_k, \hat{x}_k(n + \hat{t}_k)) \right|^2 < 1/k. \tag{7}$$

Since $\tilde{t}_k \le M(k, \tilde{r}_0)$, $m_k \ge k + M(k, \tilde{r}_0)$ and $s_{m_k} \ge m_k$, we have

$$0 \le n + \tilde{t}_k \le k + M(k, \tilde{r}_0) \le m_k \le s_{m_k}, \quad \forall\, 0 \le n \le k.$$

This results in

$$\left| \hat{x}_k(n + \hat{t}_k) \right| = \left| x_{m_k}(n + \tilde{t}_k + t_{m_k}) \right| \ge \delta_0, \ \forall\, 0 \le n \le k, \tag{8}$$

by the choice of δ_0. Notice that, $\{\hat{t}_k\}$ is an unbounded sequence in Z_+. According to Lemma 1, there exist a subsequence $\{k_{\bar{m}}\}$ of $\{k\}$ and a function $\bar{x}: Z_+ \to \Re^p$ such that for each $n \in Z_+$, the sequence $\{\hat{x}_{k_{\bar{m}}}(n + \hat{t}_{k_{\bar{m}}})\}$ converges to $\bar{x}(n)$, i.e., $\bar{x}$ is a limiting solution of (1) with respect to $\{\hat{t}_{k_{\bar{m}}}\}$. For simplicity, let $\bar{t}_{\bar{m}} = \hat{t}_{k_{\bar{m}}}$ and $\bar{x}_{\bar{m}}(n) = \hat{x}_{k_{\bar{m}}}(n), \forall n \in Z_+^{\bar{t}_{\bar{m}}}, \forall \bar{m} \in \aleph$. Then, for each $\bar{m} \in \aleph$ $\bar{x}_{\bar{m}}$ is a solution of (1) lying within the compact set $B_{\tilde{r}_0}$ and $\{\bar{t}_{\bar{m}}\}$ is an unbounded sequence in Z_+. Notice that, for each $n \in Z_+$, the inequality $k_{\bar{m}} - 1 \ge n$ holds for large enough $\bar{m}$.

In view of this fact and by using (7), the following inequalities can be derived:

$$\lim_{\bar m\to\infty}\left|h(n+\bar t_{\bar m},\bar x(n))\right|^2=\lim_{\bar m\to\infty}\left|h(n+\bar t_{\bar m},\bar x_{\bar m}(n+\bar t_{\bar m}))\right|^2$$

$$\le\lim_{\bar m\to\infty}\sum_{\bar n=0}^{k_{\bar m}-1}\left|h(\bar n+\hat t_{k_{\bar m}},\hat x_{k_{\bar m}}(\bar n+\hat t_{k_{\bar m}}))\right|^2\le\lim_{\bar m\to\infty}1/k_{\bar m}=0,$$

for each $n\in Z_+$ where the first equality is from the assumption that h is continuous at any x, uniformly in n. According to WZSD, it can be concluded that $\inf_{n\in Z_+}|\bar x(n)|=0$. On the other hand, for each $n\in Z_+$, the inequality $n\le k_{\bar m}$ holds for large enough $\bar m$. This together with (8) shows that

$$|\bar x(n)|=\lim_{\bar m\to\infty}\left|\hat x_{k_{\bar m}}(n+\hat t_{k_{\bar m}})\right|\ge\delta_0,\forall n\in Z_+.$$

We reach a contradiction. Thus, the origin is UGAS and this completes the proof of the proposition. ■

Let us further improve Proposition 1 by considering the following condition related to the output function. Roughly speaking, it says that the output energy is almost bounded.

(H1) There exists a continuous function $\mu:\Re^q\to[0,\infty)$, with $\mu(0)=0$ and $\mu(v)>0$ for all $v\neq 0$ such that, for each $\sigma>0$ and each $r>0$ there exists a constant $\tilde M(\sigma,r)>0$ such that, for all $n_0\in Z_+$ and all solutions $x:Z_+^{n_0}\to B_r$ of (1),

$$\sum_{n=n_0}^{n_0+m}[\mu(h(n,x(n)))-\sigma]\le\tilde M(\sigma,r),\qquad\forall\, m\in\aleph.\qquad(9)$$

Now, (C2) can be replaced by (H1) that is more easily verified in practical applications to guarantee UGAS of the origin. Particularly, the following result can be proposed.

Proposition 3. Consider a system of the form (1)-(2) where h is continuous with respect to x, uniformly in n. Suppose the origin is uniformly globally stable. Assume further that (H1) holds and the system is WZSD. Then, the origin is UGAS.

Proof. Define a new output as $\tilde y=\tilde h=\sqrt{\mu(h(n,x))},\forall n\in Z_+,\forall x\in\Re^p$. Then, it is not difficult to see that $\tilde h$ is also continuous in argument x, uniformly in n. For each $k\in\aleph$, let $\sigma=1/(4k^2)$. By virtue of (H1), for each $r>0$, there exists a positive constant $\tilde M(\sigma,r)$ such that (9) holds. Choose a positive integer $\hat k$ satisfying the inequality $2k\tilde M<\hat k\le 2k\tilde M+1$. Then, the following inequality can be derived:

$$\sum_{m=1}^{\hat k}\sum_{n=n_0+mk}^{n_0+(m+1)k-1}\left|\tilde h(n,x(n))\right|^2\le\sum_{n=n_0}^{n_0+(\hat k+1)k-1}\mu(h(n,x(n)))\le\tilde M(\sigma,r)+\sigma(\hat k+1)k,$$

for any $n_0\in Z_+$ and any solutions $x:Z_+^{n_0}\to B_r$ of (1). By the choices of σ and $\hat k$, we have

$$\min_{m=1,2,\ldots\hat k}\sum_{n=n_0+mk}^{n_0+(m+1)k-1}\left|\tilde h(n,x(n))\right|^2\le\frac{\tilde M}{\hat k}+\frac{\hat k+1}{\hat k}\frac{k}{4k^2}<\frac{1}{k}.$$

Thus, (C2) holds for the new output function $\tilde h$ with $M=\hat k k$. Since h is continuous at x, uniformly in n, with $h(n,0)=0,\forall n\in Z_+$, h is also uniformly bounded according to Lemma 3. Moreover, $\sqrt{\mu}$ is a positive definite continuous function. Thus, (3) holds for the original output function h when it holds for the new output function $\tilde h=\sqrt{\mu\circ h}$. Hence WZSD also holds for the new output function $\tilde h=\sqrt{\mu\circ h}$. The proposition follows thus from Proposition 2. ■

In the following, let us show that the converse of Theorem 1 is also true. First, it is interesting to note that (H1) is a *necessary* condition for UGAS, as stated in the following.

Lemma 6. Consider the system (1)-(2) where h is continuous in argument x, uniformly in n. Then, (H1) holds provided that the origin is UGAS.

Proof. Let $\mu:\Re^q\to[0,\infty)$ be any continuous function with $\mu(0)=0$ and $\mu(v)>0$ for all $v\neq 0$. Since μ is continuous, it is easy to see that $\mu\circ h$ is also continuous in argument x, uniformly in n by the assumption of h. Moreover, we have $\mu(h(n,0))=0,\forall n\in Z_+$. In particular, for any positive constant σ, there exists a positive constant $\varepsilon_0(\sigma)$ such that $\mu(h(n,x))<\sigma$, for all $n\in Z_+$ and all $|x|<\varepsilon_0$. Since the origin is UGAS, it holds that for any $r>0$, there exist two positive constants $\tilde r(r)$ and $T(\sigma,r)\in Z_+$ so that for all solutions $x:Z_+^m\to\Re^p$ of (1) satisfying $|x(m)|\le r$, we have $|x(n)|\le\tilde r,\forall n\ge m$, and $|x(n)|<\varepsilon_0,\forall n\ge m+T$. Since $\mu\circ h$ is continuous in argument x, uniformly in n, with $\mu(h(n,0))=0,\forall n\in Z_+$, it is also uniformly bounded according to Lemma 3. Let $\bar M=\sup\left\{\mu(h(n,x))\middle|n\in Z_+,|x|\le\tilde r\right\}<\infty$. Then, for all $n_0\in Z_+$ and all solutions $x:Z_+^{n_0}\to B_r$ of (1), we have:

$$\sum_{n=n_0}^{n_0+m}[\mu(h(n,x(n)))-\sigma]\le\bar MT,\forall 0\le m<T,\text{ and}$$

$$\sum_{n=n_0+T}^{n_0+m}[\mu(h(n,x(n)))-\sigma]\le 0,\forall m\ge T.$$

This implies that (H1) holds with $\tilde M=\bar M T$. The proof is therefore completed. ■

Notice that (C1) implies (4) in the definition of WZSD. Thus, the WZSD condition naturally holds under (C1). Particularly, the following result is readable from Lemma 2, Proposition 3 and Lemma 6 by showing the implications: (1) ⇒ (2) ⇒ (3) ⇒ (1).

Theorem 1. Consider a system of the form (1)-(2) where h is continuous at x, uniformly in n. Suppose the origin is uniformly globally stable. Then, the following conditions are equivalent.

(1) The origin is UGAS.
(2) (C1) and (H1) holds.
(3) The system is WZSD and (H1) holds. ■

Remark 2. Theorem 1 can be applied to analyze asymptotic stability of cascaded time-varying systems. Previous related work can be seen in [7] for autonomous systems and [9] for time-varying systems. Particularly, it is possible to show that the local Lipschitz assumption (on Lyapunov functions) introduced in the recent work [9] can be relaxed by merely requiring the continuity. To save space, the detailed will be presented in our forthcoming paper. ■

A direct application of Theorem 1 yields the following Lyapunov function based result that generalizes the classical Lyapunov Theorem. Its proof is omitted here.

Proposition 4. Consider a system of the form (1)-(2) where h is continuous in argument x, uniformly in n. Let $\mu : \Re^q \to [0,\infty)$ be a positive-definite continuous function and $V : Z_+ \times \Re^p \to [0,\infty)$ be a function such that

$$W_1(x) \le V(n,x) \le W_2(x), \tag{10}$$

$$V(n+1, f(n,x)) - V(n,x) \le -\mu(h(n,x)) \le 0, \tag{11}$$

for all $n \in Z_+$, all $x \in \Re^p$ where $W_1 : \Re^p \to [0,\infty)$ and $W_2 : \Re^p \to [0,\infty)$ are continuous positive definite functions with $\lim_{|x|\to\infty} W_1(x) = \infty$. Then, the origin is UGAS if and only if the system is WZSD. ■

Remark 3. Consider a special case where $h(n,x) = x, \forall n \in Z_+, \forall x \in \Re^p$. In this case, it is easy to see that the system is WZSD and Proposition 4 is reduced to the classical Lyapunov theorem. ■

B. Second characterization of UGAS in terms of reduced limiting systems

In this subsection, the second characterization of UGAS is proposed in terms of reduced limiting systems, instead of the original system. We use it to obtain a generalization of a discrete-time version of Krasovskii-LaSalle theorem.

First, what is a reduced limiting system is defined as follows. Consider a system of the form (1)-(2) given in Section II. Throughout this subsection, we assume that f and h are both continuous in argument x, uniformly in n. Consider the extended function $g = (f^T, h^T)^T$. Then, g is also continuous in x, uniformly in n. Moreover, $g(n,0) = (f^T(n,0), h^T(n,0))^T = 0, \forall n \in Z_+$. Thus, for any unbounded sequence $\{t_m\}$ in Z_+ there exists a subsequence $\gamma \in \Lambda(g)$ of $\{t_m\}$ in view of Proposition 1. Particularly, the set $\Lambda(f) \cap \Lambda(h) = \Lambda(g)$ is plenty but not empty. Thus, the following definition makes sense.

Definition 4. A *reduced limiting system of system (1)-(2) associated with* $\gamma \in \Lambda(f) \cap \Lambda(h)$ is defined as:

$$\bar{x}(n+1) = f_\gamma(n, \bar{x}(n)) \tag{12}$$

$$\bar{y} = h_\gamma(n, \bar{x}(n)). \tag{13}$$

We impose the following simplified detectability hypothesis that, roughly speaking, describes a "*weak zero-state detectability*" on reduced limiting system (12)-(13) (see [6] for a similar definition in continuous-time systems).

(H2) For *any* admissible sequence $\gamma \in \Lambda(f) \cap \Lambda(h)$ and any bounded solution $\bar{x} : Z_+ \to \Re^p$ of reduced limiting system (12)-(13) satisfying the equation $h_\gamma(n, \bar{x}(n)) = 0, \ \forall n \in Z_+$, it holds that $\inf_{n \in Z_+} |\bar{x}(n)| = 0$.

The main result of this subsection is proposed as follows.

Theorem 2. Consider a system of the form (1)-(2) where f and h are both continuous in argument x, uniformly in n. Suppose that the origin is uniformly globally stable. Then, the origin is UGAS if and only if (H1) and (H2) hold.

Proof. First, let us prove the "if" part. In view of Theorem 1, it remains to show that the system is WZSD. Let $\bar{x}$ be any limiting solution of (1) with respect to an unbounded sequence $\{t_m\}$ in Z_+, satisfying the following equation

$$\lim_{m\to\infty} h(n + t_m, \bar{x}(n)) = 0, \forall n \in Z_+. \tag{14}$$

By Proposition 1 and the discussion before the theorem, there exists a subsequence $\gamma = \{t_{m_k}\} \in \Lambda(f) \cap \Lambda(h)$ of $\{t_m\}$ and a limiting function $(f_\gamma^T, h_\gamma^T)^T$ of $(f^T, h^T)^T$ such that

$$h_\gamma(n, \bar{x}(n)) = \lim_{k\to\infty} h(n + t_{m_k}, \bar{x}(n)) = 0, \forall n \in Z_+. \tag{15}$$

Since $\bar{x}$ is a limiting solution of (1) with respect to $\{t_m\}$, there exist a $r > 0$ and a sequence $\{x_m : Z_+^{t_m} \to B_r\}$ of solutions of (1) such that for each $n \in Z_+$, the associated sequence $\{x_m(n + t_m)\}$ converges to $\bar{x}(n)$. By the output injection form and (14), we have

$$\begin{aligned} &\lim_{m\to\infty} \left|\Delta f(n + t_m, x_m(n + t_m))\right| \le \lim_{m\to\infty} \lambda_r(h(n + t_m, x_m(n + t_m))) \\ &= \lambda_r(\lim_{m\to\infty} h(n + t_m, \bar{x}(n))) = 0, \end{aligned} \tag{16}$$

for all n in Z_+ and some continuous function $\lambda_r : \Re^q \to [0,\infty)$ with $\lambda_r(0) = 0$ where the last equality used the fact that h is continuous in argument x, uniformly in n. Thus, the following equation can be derived:

$$\begin{aligned} &\bar{x}(n+1) = \lim_{k\to\infty} x_{m_k}(n + 1 + t_{m_k}) = \lim_{k\to\infty} f(n + t_{m_k}, x_{m_k}(n + t_{m_k})) \\ &+ \lim_{k\to\infty} \Delta f(n + t_{m_k}, x_{m_k}(n + t_{m_k})) = f_\gamma(n, \bar{x}(n)), \forall n \in Z_+, \end{aligned} \tag{17}$$

where the last equation used the fact that f is continuous in argument x, uniformly in n. That is to say that the limiting solution $\bar{x}$ is also a solution of (12) satisfying $h_\gamma(n, \bar{x}(n)) = 0, \ \forall n \in Z_+$. The WZSD condition follows from (H2). Thus, the origin is UGAS by using Theorem 1.

Now, let us prove the "only if" part. According to Lemma 6, it remains to show that (H2) is a necessary condition for UGAS. In view of Lemma 2, this is done provided that we can show that every solution $\bar{x} : Z_+ \to \Re^p$ of reduced limiting system (12)-(13) satisfying the equation

$h_\gamma(n,\bar{x}(n))=0,\ \ \forall n\in Z_+$, must be a limiting solution of (1), $\forall\gamma=\{t_m\}\in\Lambda(f)\cap\Lambda(h)$. Indeed, let $\{x_m : Z_+^{t_m}\to\Re^p\}$ be a sequence of solutions of (1) starting from $x_m(t_m)=\bar{x}(0),\forall m\in\aleph$. Since the solutions are globally uniformly bounded by the assumption, there exists a $r>0$ such that the range of x_m is contained in B_r for all *m* in $\aleph$. Using Lemma 1, there exist a function $\tilde{x}:Z_+\to\Re^p$ and a subsequence $\{m_k\}$ of $\{m\}$ such that for each $n\in Z_+$, the sequence $\{x_{m_k}(n+t_{m_k})\}$ converges to $\tilde{x}(n)$, i.e., $\tilde{x}$ is a limiting solution of (1) with respect to $\{t_{m_k}\}$. We claim that $\bar{x}\equiv\tilde{x}$. The claim will be proven by induction. First, note that $\tilde{x}(0)=\lim_{k\to\infty}x_{m_k}(t_{m_k})=\bar{x}(0)$. By induction, assume that $\bar{x}(n)=\tilde{x}(n)$. It is shown that the same is true in the case of $n+1$. Indeed, by inductive hypothesis, $h_\gamma(n,\tilde{x}(n))=h_\gamma(n,\bar{x}(n))=0$. Again using the output injection form, we have

$$\lim_{k\to\infty}\left|\Delta f(n+t_{m_k},x_{m_k}(n+t_{m_k}))\right|\le\lim_{k\to\infty}\lambda_r(h(n+t_{m_k},x_{m_k}(n+t_{m_k})))$$
$$=\lim_{k\to\infty}\lambda_r(h(n+t_{m_k},\tilde{x}(n)))=\lambda_r(h_\gamma(n,\tilde{x}(n)))=0.$$

Thus, the following equation can be derived:

$$\tilde{x}(n+1)=\lim_{k\to\infty}x_{m_k}(n+1+t_{m_k})$$
$$=\lim_{k\to\infty}f(n+t_{m_k},x_{m_k}(n+t_{m_k}))+\lim_{k\to\infty}\Delta f(n+t_{m_k},x_{m_k}(n+t_{m_k}))$$
$$=\lim_{k\to\infty}f(n+t_{m_k},\tilde{x}(n))=f_\gamma(n,\tilde{x}(n))=f_\gamma(n,\bar{x}(n))=\bar{x}(n+1).$$

Thus, the claim is true by the inductive principle. Particularly, every solution $\bar{x}:Z_+\to\Re^p$ of reduced limiting system (12)-(13) satisfying the equation $h_\gamma(n,\bar{x}(n))=0,\ \ \forall n\in Z_+$, must be a limiting solution of (1). By Lemma 2, $\lim_{n\to\infty}\bar{x}(n)=0$ and hence (H2) holds. This completes the proof of the theorem. ■

Remark 4. It is natural to ask how to determine limiting systems and check their detectability? In general, a single function may yield many limiting functions as in continuous-time case [6]. The same conclusion can be applied to limiting systems of a system. However, many functions from practical systems are in the form described in Lemma 5 with *g* being a pure time-function. Thus, their limiting functions have the same form as the original functions when we replace the function *g* by its limiting functions. Then, one can impose certain properties like persistent excitation condition in continuous-time systems on *g* to guarantee the required detectability condition. Since the space is limited, some interesting examples and more discussions will be given in our forthcoming paper. Interested readers can contact the authors for such examples. ■

As a special case, consider a system consisting of continuous periodic functions with the same period. In this case, it is not difficult to see that f and h are both continuous in argument *x*, uniformly in *n*. Moreover, every limiting function is just a time-shifting of the original function. Then, (H2) is reduced to the following condition.

(C3) For any $m\in\aleph$ and bounded solution $x:Z_+^m\to\Re^p$ of $x(n+1)=f(n,x(n))$ that satisfies the equation $h(n,x(n))=0,\ \ \forall n\in Z_+^m$, it holds that $\inf_{n\in Z_n^m}|x(n)|=0.$

Now, the following result is a simple consequence of Theorem 2. It can be viewed as a generalized Krasovskii-LaSalle theorem in discrete-time systems [7].

Corollary 2. Consider system (1)-(2) where f and h are continuous periodic functions with the same period. Suppose that the origin is UGS. Then, the origin is UGAS if and if (H1) and (C3) hold. ■

Remark 5. Let us remark that the discrete-time version of the well-known Krasovskii-LaSalle theorem can be proposed and deduced from Corollary 2 as in the continuous-time case [6] by choosing $\mu(\hat{y})=|\hat{y}|,\forall\hat{y}\in\Re^q$, and defining a virtual output as $y=h(n,x)=V(n+1,f(n,x))-V(n,x)$ for all $n\in Z_+$, all $x\in\Re^p$, with *V* being a Lyapunov function. ■

IV. Conclusion

Two new characterizations of UGAS have been proposed for nonlinear and time-varying discrete-time systems, based on detectability and limiting equations. They generalize both the classical Lyapunov theorem and a discrete-time version of Krasovskii-LaSalle theorem. Our future work will be directed at nonlinear sampled-data stabilization and tracking control of nonholonomic systems based on their exact discrete-time models and the achieved new stability results.

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