Proceedings of the
44th IEEE Conference on Decision and Control, and
the European Control Conference 2005
Seville, Spain, December 12-15, 2005

Controllability of the Dubins problem on surfaces

Yacine Chitour and Mario Sigalotti

***Abstract*— Let M be a complete oriented 2-dim Riemannian manifold. We ask the following question. Given any (p_1, v_1) and (p_2, v_2), v_i velocity at $p_i \in M$, $i = 1, 2$, is it possible to connect p_1 to p_2 by a curve γ with arbitrary small geodesic curvature such that, for $i = 1, 2$, $\dot{\gamma}$ is equal to v_i at p_i? In this paper, we prove that the answer to the question is positive if M verifies one of the following three conditions: (a) M is compact, (b) M is asymptotically flat, (c) M has bounded non negative curvature outside a compact subset.**

I. INTRODUCTION

Let (M, m) be a connected, oriented, complete Riemannian manifold and $N = UM$ its unit tangent bundle. Points of N are pairs (p, v), where $p \in M$ and $v \in T_pM$, $m(v, v) = 1$. Given $\varepsilon > 0$, Dubins' problem consists of finding, for every $(p_1, v_1), (p_2, v_2) \in N$, a curve $\gamma : [0, T] \to M$, $T \geq 0$, parameterized by arc-length such that $\gamma(0) = p_1$, $\dot{\gamma}(0) = v_1$, $\gamma(T) = p_2$, $\dot{\gamma}(T) = v_2$, with geodesic curvature bounded by ε and T as small as possible (depending on $(p_1, v_1), (p_2, v_2)$). When the dimension of M is equal to two, Dubins' problem can be formulated as the time optimal control problem for the following control system,

$$(D_\varepsilon): \qquad \dot{q} = f(q) + ug(q)\,,\ q \in N,\ u \in [-\varepsilon, \varepsilon]\,,$$

where f is the geodesic spray on N (i.e., f is the infinitesimal generator of the geodesic flow on M), g is the smooth vector field generating the fiberwise rotation with angular velocity equal to one and the admissible controls are measurable functions $u : J \to [-\varepsilon, \varepsilon]$, where J is an interval of $\mathbf{R}$. The trajectories of (D_ε) are absolutely continuous curves $\gamma = \gamma_{u,q}(\cdot)$, with γ the solution of (D_ε) starting at q and associated with the admissible control u. A trajectory $\gamma : [0, T] \to N$ of (D_ε) is said to be *time optimal* if, for every trajectory $\gamma' : [0, T'] \to N$ of (D_ε) such that $\gamma'(0) = \gamma(0)$ and $\gamma'(T') = \gamma(T)$, we have $T \leq T'$.

Note that, in the statement of Dubins' problem, the existence of a curve γ of minimal length is not guaranteed. In the language of control theory, a controllability issue should be solved in order to focus on the time optimal problem. Recall that (D_ε) is *completely controllable* (CC) if, for every $q_1, q_2 \in N$, q_2 is *reachable* from q_1, i.e., there exists a trajectory of (D_ε) steering q_1 to q_2. For $q \in N$, let $A_q \subset N$ be the set of points of N reachable from q.

If M is the Euclidean plane, the dynamics defined by (D_ε) represents, in the robotics literature (cf. [1]), the motion of a unicycle (or a rolling penny) and the projections of the trajectories of (D_ε) on the plane are planar curves parameterized by arc-length with curvature bounded by ε. It is easy to see that (D_ε) is completely controllable for every $\varepsilon > 0$ and, given any pair $(p_1, v_1), (p_2, v_2) \in N$, there exists a time optimal trajectory of (D_ε) connecting (p_1, v_1) and (p_2, v_2). In 1957, Dubins ([2]) determined the global structure of time optimal trajectories of (D_ε) in the case where M is the Euclidean plane: he showed that such trajectories are concatenations of at most three pieces made of circles of radius $\frac{1}{\varepsilon}$ or straight lines. Further restrictions on the length of the arcs of an optimal concatenation have been proved by Sussmann and Tang [3].

Dubins'-like problems have been proposed by considering more general manifolds M. For instance, the case where M is a two-dimensional manifold of constant Gaussian curvature was investigated in [4], [5], [6], [7], [8] and the case where $M = \mathbf{R}^n, S^n$, $n \geq 3$ was studied in [9], [10], [11]. Another line of generalization consists of considering the distributional version of (D_ε) (cf. [12]). For simplicity, M is supposed to be two dimensional. The distributional dynamics can be represented by the two-input control system (DD_ε) : $\dot{q} = uf(q) + vg(q)$ with $|u|, |v| \leq \varepsilon$. The controllability issue is trivial since it can be solved infinitesimally: let $h = [f, g]$, where $[.,.]$ denotes the Lie bracket; then the distribution (f, g) is strongly bracket generating, i.e., for every $q \in N$, the triple $(f(q), g(q), h(q))$ spans T_qN.

In this paper, we follow the first path of generalization, i.e., we assume that M is a two-dimensional connected Riemannian manifold, oriented and complete (with possibly non-constant curvature). Our aim is to find geometric or topological conditions on M, such that, for every $\varepsilon > 0$, (D_ε) is completely controllable. We refer to that property as the *unrestricted complete controllability* (UCC) for the Dubins problem (we still use the word "Dubins" although we will not consider any optimal control problem). The (UCC) property can be stated geometrically as follows: for every $(p_1, v_1), (p_2, v_2) \in N$, there exists a curve γ connecting p_1 to p_2 with prescribed initial and final directions v_1 and v_2 and with arbitrary small geodesic curvature.

To establish (CC) of (D_ε), $\varepsilon > 0$, we use a standard reduction (cf., for instance, [5]): we will show that (D_ε) is completely controllable if and only if (D_ε) is *weakly symmetric* i.e., for every $q = (p, v) \in N$, $q^- = (p, -v) \in A_q$. For instance if $M = \mathbf{R}^2$, then a control strategy which shows that (D_ε) is weakly symmetric can be given by u so that the resulting trajectory is a *teardrop* of size $\frac{1}{\varepsilon}$. (See Figure 1).

Let $\Phi : \widetilde{M} \to M$ be a Riemannian covering and $(\widetilde{D}_\varepsilon)$ be

Y. Chitour is with Laboratoire des Signaux et Systèmes, Université Paris-Sud/CNRS/Supelec, 91192 Gif-sur-Yvette, France chitour@lss.supelec.fr

M. Sigalotti is with Projet APICS, INRIA Sophia Antipolis, 06902 Sophia Antipolis, France Mario.Sigalotti@inria.fr

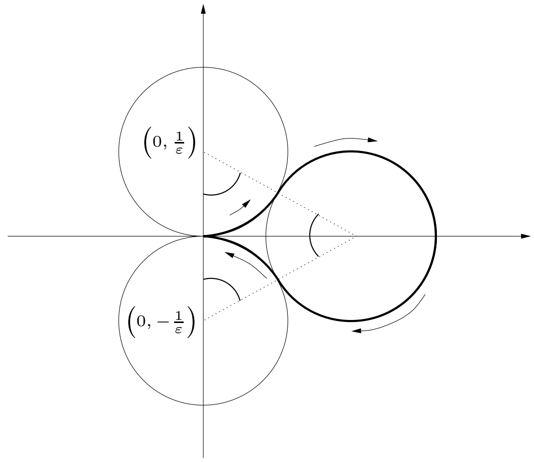

Figure 1: The teardrop trajectory of size $\frac{1}{\varepsilon}$.

the Dubins problem on $\widetilde{M}$; then Φ_* maps trajectories of $(\widetilde{D}_\varepsilon)$ onto trajectories of (D_ε). Therefore, if the (CC) property holds for $(\widetilde{D}_\varepsilon)$, then it also holds for (D_ε). Equivalently, if $(\widetilde{D}_\varepsilon)$ is weakly symmetric, then also (D_ε) is. For instance, if M is flat, then a controllability strategy for M is obtained by projecting the one of the Euclidean plane, seen as the universal covering of M. This simple idea of applying strategies which are valid on a Riemannian covering of the manifold (not necessarily a universal covering) will be repeatedly exploited in the paper.

The first condition ensuring (UCC) which we obtain is purely topological: if M is compact, then, by means of a Poisson stability argument, (UCC) turns out to hold for Dubins' problem. Thus, let us assume that M is non-compact.

The geometric quantity which plays a crucial role in the characterization of controllable Dubins' problems is the Gaussian curvature of M, denoted by K. The curvature appears quite soon in the study of the Lie bracket configuration of the problem and, therefore, for local controllability issues: indeed, for every $q \in N$,

$$[f,[f,g]](q) = -K(\pi(q))g(q)\,,$$

where $\pi : N \to M$ denotes the bundle projection. The importance of K in characterizing (UCC) is again suggested by the following example: if M is the Poincaré half-plane ($K \equiv -1$), then (D_ε) is completely controllable if and only if $\varepsilon > 1$ (cf. [4], [6]). Roughly speaking, this happens because the negativeness of K not only prevents the geodesics to have conjugate points but actually is an obstacle for the controlled turning action to overcome the spreading of the geodesics. When $|u| \leq \varepsilon \leq 1$, the effect of u on g is not strong enough and (CC) fails to hold.

It is therefore natural to formulate necessary conditions for (UCC) in terms of the Gaussian curvature K. For instance, an extension of the case $K \equiv 0$ is given by the situation in which M is *asymptotically flat*, i.e., K tends to zero at infinity. Under this hypothesis, we are able to prove the (UCC) property: the control strategy is based on the possibility of tracking a teardrop loop in a covering domain over a piece of M at infinity. We will see that a suitable covering manifold can be globally described by a single appropriate geodesic chart.

Bearing in mind the previous example of non controllability, it is reasonable to study the situation where the curvature is non negative. In that case, there is no more local spreading effect due to the drift term to compensate. A result by Cohn–Vossen (cf. [13]) implies that, if $K \geq 0$ and K is not identically equal to 0, then M is homeomorphic to a plane and, more importantly for the controllability issue, $\int_M K dA \leq 4\pi$, where dA is the surface element in M. In particular, for any fixed radius $R > 0$, the total curvature on the disk centered at p with radius R tends to zero as p tends to infinity. The same is true under the relaxed hypothesis that K is non-negative outside a compact subset of M. The integral decay of K to zero can be interpreted as a kind of asymptotic flatness condition and it suggests that (D_ε) should be completely controllable for every $\varepsilon > 0$. We are able to confirm that intuition under the additional assumption that K is bounded over M, i.e., $K_\infty = \sup_M K$ is finite. The existence of a control strategy which allows to track a teardrop loop at infinity is much more delicate to prove than in the asymptotically flat case. The key step is the identification of a simply connected covering domain on which the teardrop strategy can be applied. The covering domain $\mathcal{D}$ is not anymore described by one single chart, as in the asymptotic flat case, but by gluing rectangular strips, each of them obtained by one regular geodesic chart. There are $O(\frac{1}{\varepsilon})$ such strips and each of them has width proportional to $\frac{1}{\sqrt{K_\infty}}$. Then, using these strips, one is able to mesh $\mathcal{D}$ by geodesic quadrilaterals $P_{j,k}$ with edge of length proportional to $\frac{1}{\sqrt{K_\infty}}$. The tracking operation is now decomposed in $O(\frac{1}{\varepsilon})$ steps: we design a discrete approximation of the teardrop, by fixing a sequence of $O(\frac{1}{\varepsilon})$ points on the edges of the polygons $P_{j,k}$ and by associating with each of them a corresponding direction. After that, we solve the problem of connecting pairs of subsequent points of the approximating sequence by an admissible trajectory, being tangent to the associated directions. Each elementary problem of this kind can be formulated in a single coordinate strip. Intuitively, its solution is based on the topological description of small time attainable sets for non-degenerate single-input control-affine three dimensional systems, due to Lobry ([14]): the set of points which are reachable from $q_0 \in N$ in small time, is given by the region enclosed by two surfaces, obtained as union of all small-time bang-bang trajectories from q_0 with one switch. What we do in practice, is to estimate the coordinate expression of such surfaces and to check whether they enclose the final state of the elementary problem.

The paper is organized as follows: in section II, we gather the notations used in the paper, we describe the general construction of local covering domains, we establish basic properties for the Dubins problem and, finally, we study the case where M is compact. Section III simply contains an argument in the case M were asymptotically flat. As for the case where M has non negative sectional curvature outside a compact subset, the proof is deferred in the complete version of the paper ([15]).

II. BASIC NOTATIONS AND FIRST RESULTS

A. Differential geometric notions

Let (M, m) be a complete, connected, oriented, two-dimensional Riemannian manifold. Denote by K its Gaussian curvature and by N the unit tangent bundle UM. Let $\pi : N \to M$ be the canonical bundle projection of N onto M. We will usually denote by p a point in M and by $q = (p, v)$ one in N, where $p = \pi(q)$ and $v \in T_pM$, $m(v, v) = 1$. Given $v \in T_pM$, we write $v^\perp$ for its counterclockwise rotation in T_pM of angle $\pi/2$. For every $q = (p, v) \in N$, we set $q^\perp = (p, v^\perp)$ and $q^- = (p, -v)$.

Given $p_1, p_2 \in M$, $d(p_1, p_2)$ denotes the geodesic distance between p_1 and p_2. When no confusion is possible, we simply write $\|p\|$ (respectively, $\|q\|$) to denote the distance $d(p, p_0)$ (respectively, $d(\pi(q), p_0)$) from a fixed point $p_0 \in M$.

Let $f : TM \to T(TM)$ be the *geodesic spray* on TM (i.e., the vector field on TM which generates the geodesic flow). The restriction of f to N (still denoted by f) is a well defined vector field on N. Recall that f is characterized by the following property: $p(\cdot)$ is a geodesic on M if and only if $(p(\cdot), \dot{p}(\cdot))$ is an integral curve of f. In particular, f satisfies the relation $\pi_\star(f(q)) = q$.

Denote by g the smooth vector field on N, whose corresponding flow at time t is the fiberwise rotation of angle t. We write e^{tf} (respectively, e^{tg}) to denote the flow of f (respectively, g) at time t.

For $x_0 \in (X, d_0)$, metric space and $\rho > 0$, $B_\rho(x_0)$ denotes the open ball of center x_0 and radius ρ. Given a subset Y of X, $\mathrm{Clos}(Y)$ and $\mathrm{Int}(Y)$ are, respectively, the closure and the interior of Y.

In the sequel of the paper, we will systematically use as local coordinates the geodesic ones, whose definition is recalled below. Its construction has a crucial role in the present exposition, since it allows to define a wide class of local covering domains of M.

Given $q \in N$, consider the map $\phi_q : \mathbf{R}^2 \longrightarrow M$, $(x, y) \longmapsto \pi(e^{yf} e^{\frac{\pi}{2} g} e^{xf}(q))$. Fix $R = [x_1, x_2] \times [y_1, y_2] \subset \mathbf{R}^2$ and assume that the origin $(0, 0)$ belongs to R. If ϕ_q is a local diffeomorphism at every point of R, then R can be endowed with the Riemannian structure lifted from M, in such a way that ϕ_q becomes a local isometry. If this happens, we denote by $R(q)$ the manifold with boundary which is obtained. The segment $[x_1, x_2] \times \{0\}$, which is the support of a geodesic in $R(q)$, is called the *base curve* of $R(q)$. The Gaussian curvature of $R(q)$ at a point (x, y) is given by $K(\phi_q(x, y))$, and, where no confusion can arise, will be denoted by $K(x, y)$. If R is a neighborhood of $(0, 0)$ and $\phi_q|_R$ is injective, then $\phi_q|_R$ is a *geodesic chart on* M. In the coordinates (x, y), m has the form $m(x, y) = B^2(x, y)dx^2 + dy^2$, where $B : R \to \mathbf{R}$ is the solution of

$$(S_B) : \quad \begin{aligned} B_{yy} + KB &= 0\,, \\ B(x, 0) &\equiv 1\,, \\ B_y(x, 0) &\equiv 0\,, \end{aligned}$$

where the index y appearing in B_y, B_{yy} stands for the partial differentiation with respect to y. Notice that, for every point $q \in N$ and every small enough rectangular neighborhood R of $(0, 0)$, $\phi_q|_R$ is a geodesic chart on M. In general, if B satisfies (S_B) on R with $K = K \circ \phi_q$, then $R(q)$ is well defined if and only if B is everywhere positive on R. We find it useful to define on $R(q)$ a real-valued function F as follows

$$F(x, y) = \frac{B_y(x, y)}{B(x, y)}. \qquad (1)$$

The unit bundle $UR(q)$ is identified with

$$\left\{ (x, y, v_x, v_y) \in \mathbf{R}^4 \middle| \ (x, y) \in R,\ B^2(x, y)v_x^2 + v_y^2 = 1 \right\}.$$

Equivalently, a set of coordinates in $UR(q)$ is given by $(x, y, \theta) \in R \times \mathcal{S}^1$, with the identification

$$Bv_x = \cos\theta, \qquad v_y = \sin\theta\,. \qquad (2)$$

In geodesic coordinates, f and g turn out to be given by

$$\begin{aligned} f(x, y, \theta) &= \left(\frac{\cos\theta}{B(x, y)}, \sin\theta, F(x, y)\cos\theta \right)^T, \\ g(x, y, \theta) &= (0, 0, 1)^T. \end{aligned}$$

Notice that, in the Euclidean case, geodesic and Euclidean coordinates coincide. From (S_B) and (1) one gets $B \equiv 1$ and $F \equiv 0$. The coordinate expression for the flat Dubins problem is, therefore,

$$\dot{x} = \cos\theta\,, \quad \dot{y} = \sin\theta\,, \quad \dot{\theta} = u\,,$$

that is, the standard one.

The pair of vector fields (f, g) define a contact distribution on N, i.e., the vectors $(f(q)$, $g(q)$ and $[f, g](q))$ span T_qN for every $q \in N$, where $[\cdot, \cdot]$ stands for the Lie bracket. The Lie-algebraic structure of the contact distribution $\{f, g\}$ is characterized by the relations

$$\textbf{(i)} \ \ [f, g] = h\,, \qquad \textbf{(ii)} \ \ [g, h] = f\,, \qquad \textbf{(iii)} \ \ [h, f] = Kg\,, \qquad (3)$$

where h, defined by (i), can be represented in geodesic coordinates as

$$h(x, y, \theta) = \left(\frac{\sin\theta}{B(x, y)}, -\cos\theta, F(x, y)\sin\theta \right)^T. \qquad (4)$$

A proof of (3) can be obtained, for instance, by using the expressions of f and g in geodesic coordinates. Equivalently, (3) could have been derived from the structure equations arising from the moving frame approach (see [4]).

A metric $\widetilde{m}$ on N can be introduced by requiring that $(f(q), g(q), h(q))$ is an $\widetilde{m}$-orthonormal basis of T_qN, for every $q \in N$. Such $\widetilde{m}$ is called the *Sasaki metric inherited from* m and endows N with a complete Riemannian structure. (See [16] for instance.)

B. The control system

Recall that, for every $\varepsilon > 0$, (D_ε) denotes the control system $\dot q = f(q) + ug(q)$, $q \in N$ and $u \in [-\varepsilon, \varepsilon]$. By definition, an *admissible control* is a measurable function $u(\cdot)$, defined on some interval of $\mathbf{R}$, with values in $[-\varepsilon, \varepsilon]$. The solutions of (D_ε) corresponding to admissible controls are called *admissible trajectories*. It follows from the definition of g that, for every admissible trajectory $q : [0, T] \to N$ of (D_ε), $d(\pi(q(0)), \pi(q(T))) \leq T$. Therefore, M being complete, for every control function $u : \mathbf{R} \to [-\varepsilon, \varepsilon]$, the non-autonomous vector field $f + u(t)g$ is complete, that is, with any initial condition $q_0 \in N$ we can associate a solution $q(\cdot)$ of (D_ε), defined on the whole real line, such that $q(0) = q_0$. In other words, the control system (D_ε) is *complete*. For every $q \in N$ and $T > 0$, the *attainable set from q within time T* is the set A_q^T consisting of the endpoints of all admissible trajectories starting from q and defined on a time interval of length T. Similarly, the *attainable set from q* is the set A_q consisting of the endpoints of all admissible trajectories starting from q. The control system (D_ε) is called *completely controllable* if $A_q = N$ for every $q \in N$.

Definition 1: We say that the Dubins problem on M has the *unrestricted complete controllability (UCC) property* if, for every $\varepsilon > 0$, (D_ε) is completely controllable.

In local geodesic coordinates, (D_ε) can be written as follows,

$$\dot x = \frac{\cos\theta}{B}, \tag{5}$$

$$\dot y = \sin\theta, \tag{6}$$

$$\dot\theta = u + F\cos\theta. \tag{7}$$

More intrinsically, we can rewrite system (5 – 7) in the form

$$\begin{cases} \dot p = v\,, \\ \nabla_v v = u v^\perp, \end{cases} \tag{8}$$

which accounts for a clear geometric interpretation of the unrestricted controllability property: The Dubins problem on M is unrestrictedly completely controllable if and only if, for every $(p_1, v_1), (p_2, v_2) \in N$, for every $\varepsilon > 0$, there exists a curve $p : [T_1, T_2] \to M$ with geodesic curvature smaller than ε such that $p(T_i) = p_i$, $\dot p(T_i) = v_i$, $i = 1, 2$.

The fact that f and g define a contact distribution on N has the important consequence that, for every $0 < t < T$ and $q \in N$, $e^{tf}(q)$ belongs to $\mathrm{Int}(A_q^T)$. This follows, for instance, from the description of small-time attainable sets for single-input non degenerate three dimensional control systems given by Lobry in [14].

From the viewpoint of control theory, the property that the distribution defining (D_ε) has a contact structure implies that (D_ε) is *bracket generating*, i.e., such that the iterated Lie brackets of f and g span the tangent space to N at every point.

Remark 1: If $q : [0, T] \to N$ is a trajectory of (D_ε) corresponding to some admissible control $u : [0, T] \to [-\varepsilon, \varepsilon]$, then the trajectory $q(T - \cdot)^-$ obtained from $q(\cdot)$ by reflection and time-reversion is itself an admissible trajectory of (D_ε) and steers $q(T)^-$ to $q(0)^-$. Its corresponding control function is given by $-u(T - \cdot)$, which is admissible. Therefore, for every $q, q' \in N$, q' belongs to A_q if and only if q^- belongs to $A_{(q')^-}$.

Remark 2: Assume that, for every q in N, $q^- \in A_q$, i.e., that (D_ε) is *weakly symmetric*. Then, due to Remark 1, $q' \in A_q$ if and only if $A_q = A_{q'}$. It follows that, for every $q \in N$ and every $q' \in A_q$, $q' \in \mathrm{Int}\left(A_{e^{-tf}(q')}\right) = \mathrm{Int}\,(A_{q'}) = \mathrm{Int}\,(A_q)$, where $t > 0$ and the first inclusion follows from the previously quoted Lobry's result. Therefore, $\{A_q\}_{q \in N}$ is an open partition of N. Since N is connected, then (D_ε) is completely controllable.

Thanks again to Remark 1, we obtain the following equivalence: (D_ε) is completely controllable if and only if, for every $q \in N$, there exists $q' \in A_q$ such that $(q')^- \in A_{q'}$.

A sufficient condition for unrestricted complete controllability is the compactness of M. This fact is a consequence of a more general result on controllability on compact manifolds of bracket generating systems made of conservative vector fields due to Lobry ([17]). Lemma 1 gives a stronger formulation of Lobry's result, adapted to the specific control system (D_ε), which implies also that every attainable set is unbounded when M is open. The proof is a variation on the classical one of Poincaré's theorem on volume preserving flows.

Lemma 1: If $q \in N$ exists such that A_q is relatively compact in N, then M is compact and $A_q = N$.

Proof of Lemma 1: Fix $q \in N$ and assume that $G = \mathrm{Clos}(A_q)$ is compact in N. As already remarked, for every $t > 0$ and every $q' \in N$, $e^{tf}(q') \in \mathrm{Int}(A_{q'}^{2t})$. The compactness of G and the continuous dependence of $A_{q'}^2$ on q' imply that there exists $\rho > 0$ such that, for every $q' \in G$,

$$B_\rho(e^f(q')) \subset A_{q'}\,. \tag{9}$$

We want to prove that ∂A_q is empty (and so $A_q = G = M$). Let, by contradiction, $r \in \partial A_q$. A well-known theorem by Krener ([18]) states that any attainable set of a bracket generating system is contained in the closure of its interior. Therefore, $V = A_q \cap B_\rho(r)$ has nonempty interior and, in particular, its volume is strictly positive. Since e^f is a volume preserving diffeomorphism of N (see, for instance, [16]) and A_q has finite volume (it is bounded), then $\{e^{nf}(V)\}_{n \in \mathbf{N}}$ cannot be a disjoint family, being $e^{nf}(V) \subset A_q$ for every $n \in \mathbf{N}$. Therefore, there exist $n_1 < n_2$ such that $e^{n_1 f}(V) \cap e^{n_2 f}(V)$ is not empty. Equivalently, there exists a point in $e^{(n_2 - n_1 - 1)f}(V)$ whose image by e^f lies in V. Due to (9), it follows that $r \in \mathrm{Int}(A_q)$ and the contradiction is reached. ■

Proposition 1: Let M be a complete, connected, oriented, two-dimensional Riemannian manifold. Assume, in addition, that M is compact. Then Dubins' problem is unrestrictedly completely controllable.

For the rest of the paper, we deal with the case M non-compact.

III. ASYMPTOTICALLY FLAT MANIFOLDS

Throughout this section, we assume that M is *asymptotically flat*, that is,

$$\lim_{\|p\|\to\infty} K(p) = 0\,. \tag{10}$$

For every $L > 0$, let $Q_L = [0, 2L] \times [-L, L]$. According to the notation introduced in section II-A, if the map ϕ_{q_0}, $q_0 \in N$, is a local diffeomorphism at every point of Q_L, then $Q_L(q_0)$ denotes the Riemannian manifold (with boundary) obtained endowing Q_L with the Riemannian structure lifted from M.

Let us characterize values of L for which the construction of $Q_L(q_0)$ can be carried out. Let B be the solution of (S_B) on Q_L, with $K = K \circ \phi_{q_0}$. Set $\delta = \max_{Q_L} |K \circ \phi_{q_0}|$.

By Sturm–Liouville theory, we can compare B with the solution of (S_B) corresponding to K constantly equal to δ. We obtain that, if $\sqrt{\delta}|y| \le \frac{\pi}{2}$, then $B(x, y) \ge \cos(\sqrt{\delta} y) \ge 0$ for every $x \in [0, 2L]$. Thus, if $L < \frac{\pi}{2\sqrt{\delta}}$, then $Q_L(q_0)$ is well defined.

In particular, since M is asymptotically flat, then, for every $L > 0$ and every q_0 outside a compact subset of N (depending, in general, on L), $Q_L(q_0)$ is well defined.

Together with m, also the control problem (D_ε) is lifted from N to $UQ_L(q_0)$. Let us stress the trivial, but crucial, property that every admissible trajectory of the lifted control system is projected by ϕ_{q_0} to an admissible trajectory of (D_ε). In the coordinates (x, y, θ) of $UQ_L(q_0)$, the dynamics of the lifted system is described by (5 – 7). Due to Remark 2, the proof of the complete controllability of (D_ε) reduces to show that $q_0^- \in A_{q_0}$ if δ is small enough. This will be done by designing an admissible trajectory for the lifted control problem on $UQ_L(q_0)$, steering $(0, 0, 0)$ to $(0, 0, \pi)$.

Fix $q_0 \in N$, $L > 0$ and assume that $\sqrt{\delta} \le \frac{\pi}{3L}$. Sturm–Liouville theory, together with the well definedness of $Q_L(q_0)$, implies that

$$\cos(\sqrt{\delta} y) \le B(x, y) \le \cosh(\sqrt{\delta} y)\,, \tag{11}$$

$$|F(x, y)| = \left|\frac{B_y(x, y)}{B(x, y)}\right| \le \sqrt{\delta}\frac{\sinh(\sqrt{\delta}|y|)}{\cos(\sqrt{\delta} y)}\,, \tag{12}$$

for every $(x, y) \in Q_L(q_0)$. An upper bound for $|F|$ in $Q_L(q_0)$ is given by $\sqrt{\delta}\frac{\sinh(\sqrt{\delta}L)}{\cos(\sqrt{\delta}L)}$. Then, we can assume that $\max_{Q_L(q_0)} |F| \le \frac{\varepsilon}{2}$, by taking $\sqrt{\delta} \le \frac{\varepsilon}{4\sinh(\frac{\pi}{3})}$. Consider now the control system $(D_{\varepsilon/2})$ on the unit bundle of the Euclidean plane. Let $\overline{u}(\cdot)$ be the control function corresponding to the trajectory whose projection on $\mathbf{R}^2$ is a teardrop of size $2/\varepsilon$ which leaves the origin horizontally and arrives at the origin with the opposite direction. Thus, $\overline{u}(\cdot)$, is piecewise constant, taking alternately the values $-\varepsilon/2$ and $\varepsilon/2$. Denote the coordinates of the teardrop trajectory in $\mathbf{R}^2 \times \mathcal{S}^1$ by $\overline{x}(\cdot)$, $\overline{y}(\cdot)$ and $\overline{\theta}(\cdot)$. It follows from straightforward computations that $(\overline{x}(\cdot), \overline{y}(\cdot))$ takes values in the rectangle $[0, 2(\sqrt{3} + 1)/\varepsilon] \times [-2/\varepsilon, 2/\varepsilon]$ and that the teardrop has length $\frac{14\pi}{3\varepsilon}$. Fix $L = \frac{3}{\varepsilon}$. The idea is to apply to the lifted system the time-variant feedback control

$$u(t) = \overline{u}(t) - F(x, y)\cos\theta\,, \tag{13}$$

which is admissible, as long as the corresponding trajectory stays in $UQ_L(q_0)$.

Consider the solution $\gamma(\cdot) = (x(\cdot), y(\cdot), \theta(\cdot))$ of (5 – 7) corresponding to $u(\cdot)$, with initial condition $\gamma(0) = (x_0, 0, 0)$. As long as $(x(t), y(t))$ stays in Q_L, we have $y(t) = \overline{y}(t)$ and $\theta(t) = \overline{\theta}(t)$. Therefore,

$$\begin{aligned} |x(t) - \overline{x}(t) - x_0| &\le \int_0^t |\cos(\theta(s))| \left|\frac{1}{B(x(s), y(s))} - 1\right| \\ &\le \frac{14\pi}{3\varepsilon} \max_{Q_L(q_0)} \left|\frac{1}{B} - 1\right|. \end{aligned}$$

It follows from (11) that, for every $\alpha \in \left(0, \frac{\pi}{2}\right)$, if $\sqrt{\delta}L \le \alpha$, then

$$\max_{Q_L(q_0)} \left|\frac{1}{B} - 1\right| \le \frac{\cosh(\alpha) - 1}{\cos(\alpha)}.$$

Therefore, it is possible to fix α, independent of ε, such that, whenever δ satisfies $\sqrt{\delta}L \le \alpha$.

$$\frac{14\pi}{3\varepsilon} \max_{Q_L(q_0)} \left|\frac{1}{B} - 1\right| \le \frac{1}{4\varepsilon}\,.$$

Fix $x_0 = \frac{1}{4\varepsilon}$. Then $\gamma(\cdot)$ is defined for the entire time duration of $\overline{u}(\cdot)$. At its final point its coordinates are of the type $(x_1, 0, \pi)$. Concatenating γ with two trajectories corresponding to control equal to zero, we obtain an admissible trajectory for the Dubins problem lifted to $UQ_L(q_0)$, steering $(0, 0, 0)$ to $(0, 0, \pi)$. We proved the following theorem.

Theorem 1: Let M be a complete, connected, oriented, two-dimensional Riemannian manifold. Assume, in addition, that M is asymptotically flat. Then, Dubins' problem is unrestrictedly completely controllable.

Actually, from the nature of the above argument, a stronger result follows.

Proposition 2: There exists a universal constant $\mu > 0$ such that, if $\limsup_{\|p\|\to\infty} |K(p)| \le \mu\varepsilon^2$, then (D_ε) is completely controllable.

REFERENCES

[1] R.N. Murray, Z.X. Li, and S.S. Sastry, *A mathematical introduction to robotic manipulation*, CRC Press, Boca Raton, FL; 1994.

[2] L.E. Dubins, On curves of minimal length with a constraint on average curvature and with prescribed initial and terminal positions and tangents, *Am. J. Math.*, vol. 79, 1957, pp 497-516.

[3] H.J. Sussmann and G. Tang, Shortest paths for the Reeds–Shepp car: a worked out example of the use of geometric techniques in nonlinear optimal control, Rutgers Center for Systems and Control Technical Report 91-10, 1991.

[4] Y. Chitour, Applied and theoretical aspects of the controllability of nonholonomic control system, PhD Thesis, Rutgers University, 1996.

[5] V. Jurdjevic, *Geometric control theory*, Cambridge Studies in Adv. Math., vol. 52, Cam. Univ. Press, Cambrdige; 1997.

[6] D. Mittenhuber, "Dubins' problem in the hyperbolic plane using the open disc model", in *Geometric control and non-holonomic mechanics (Mexico City, 1996)*, CMS Conf. Proc., vol. 25, Amer. Math. Soc., Providence, RI, 1998, pp 115-152.

[7] U. Boscain and Y. Chitour, Time optimal synthesis for left-invariant control systems on $SO(3)$, preprint, 2003.

[8] F. Monroy-Perez, Non-Euclidean Dubins' problem: A control theoretic approach, PhD thesis, University of Toronto, 1995.

[9] H.J. Sussmann, "Shortest 3-dimensional paths with a prescribed curvature bound", in *Proc. IEEE CDC*, New Orleans, LA, 1995, pp 3306-3312.

[10] D. Mittenhuber, Dubins' problem is intrinsically three-dimensional, *ESAIM Control Optim. Calc. Var.*, vol. 3, 1998, pp 1-22.

[11] F. Monroy-Perez, "Three-dimensional non-Euclidean Dubins' problem", in *Geometric control and non-holonomic mechanics (Mexico City, 1996)*, CMS Conf. Proc., vol. 25, Amer. Math. Soc., Providence, RI, 1998, pp 153-181.

[12] U. Boscain and Y. Chitour, On the minimum time problem for driftless left-invariant control systems on $SO(3)$, *Commun. Pure Appl. Anal.*, vol. 1, 2002, pp 285-312.

[13] S. Cohn–Vossen, Kürzeste Wege und Totalkrümmung auf Flächen, *Compositio Math.*, vol. 2, 1935, pp 69-133.

[14] C. Lobry, Contrôlabilité des systèmes non linéaires, *SIAM J. Cont.*, vol. 8, 1970, pp 573-605.

[15] Y. Chitour and M. Sigalotti Dubins' problem on surfaces. I. Nonnegative curvature, *J. Geom. Anal.*, to appear.

[16] T. Sakai, *Riemannian Geometry*, Trans. Math. Mono., vol. 149, Amer. Math. Soc., Providence, RI; 1996.

[17] C. Lobry, Controllability of nonlinear systems on compact manifolds, *SIAM J. Control*, vol. 12, 1974, pp 1-4.

[18] A. Krener, A generalization of Chow's theorem and the Bang-Bang theorem to non-linear control problems, *SIAM J. Control*, vol. 12, 1974, pp 43-51.