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Local Convergence of the Feedback Product via the Asymptotics of the Catalan Numbers

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***Abstract*— Given two analytic nonlinear input-output systems represented as Fliess operators, F_c and F_d, their feedback connection $y = F_c[u + F_d[y]]$ can be described in terms of a feedback product of their corresponding generating series c and d, namely $y = F_{c@d}[u]$. In this paper, sufficient conditions are given under which $F_{c@d}$ is always well-defined on a closed ball in a suitable input signal space and over a nonzero interval of time. In the process of establishing this result, a connection is derived between the radius of convergence and the asymptotic behavior of the sequence of Catalan numbers or, more specifically, the binomial transform of the sequence of Catalan numbers. This suggests a deeper connection between feedback structures involving analytic systems and algebraic combinatorics on words.**

I. INTRODUCTION

Let $X = \{x_0, x_1, \ldots, x_m\}$ be an alphabet and X^* the free monoid comprised of all words over X (including the empty word $\emptyset$) under the catenation product. A formal power series in X is any mapping of the form $X^* \to \mathbb{R}^\ell$, and the set of all such mappings will be denoted by $\mathbb{R}^\ell\langle\langle X\rangle\rangle$. For each $c \in \mathbb{R}^\ell\langle\langle X\rangle\rangle$, one can formally associate an m-input, ℓ-output operator F_c in the following manner. Let $p \geq 1$ and $a < b$ be given. For a measurable function $u : [a, b] \to \mathbb{R}^m$, define $\|u\|_p = \max\{\|u_i\|_p : 1 \leq i \leq m\}$, where $\|u_i\|_p$ is the usual L_p-norm for a measurable real-valued function, u_i, defined on $[a, b]$. Let $L_p^m[a, b]$ denote the set of all measurable functions defined on $[a, b]$ having a finite $\|\cdot\|_p$-norm and $B_p^m(R)[a, b] := \{u \in L_p^m[a, b] : \|u\|_p \leq R\}$. With $t_0, T \in \mathbb{R}$ fixed and $T > 0$, define recursively for each $\eta \in X^*$ the mapping $E_\eta : L_1^m[t_0, t_0 + T] \to \mathcal{C}[t_0, t_0 + T]$ by $E_\emptyset = 1$, and

$$E_{x_i\bar{\eta}}[u](t, t_0) = \int_{t_0}^{t} u_i(\tau)E_{\bar{\eta}}[u](\tau, t_0)\, d\tau,$$

where $x_i \in X$, $\bar{\eta} \in X^*$ and $u_0(t) \equiv 1$. The input-output operator corresponding to c is then

$$F_c[u](t) = \sum_{\eta\in X^*} (c, \eta)\, E_\eta[u](t, t_0),$$

which is referred to as a *Fliess operator*. All Volterra operators with analytic kernels, for example, are Fliess operators. In the classical literature, where these operators first appeared [3], [5], [6], [14], it is normally assumed that there exist real numbers $K, M > 0$ such that $|(c, \eta)| \leq KM^{|\eta|}|\eta|!$, $\forall\, \eta \in X^*$, where $|z| = \max\{|z_1|, |z_2|, \ldots, |z_\ell|\}$ when $z \in \mathbb{R}^\ell$, and $|\eta|$ denotes the number of symbols in η. This growth condition on the coefficients of c ensures that there exist positive real numbers R and T_0 such that for all piecewise continuous u with $\|u\|_\infty \leq R$ and $T \leq T_0$, the series defining F_c converges uniformly and absolutely on $[t_0, t_0 + T]$. Therefore, a power series c is said to be *locally convergent* when its coefficients satisfy such a growth condition. The set of all locally convergent series in $\mathbb{R}^\ell\langle\langle X\rangle\rangle$ will be denoted by $\mathbb{R}^\ell_{LC}\langle\langle X\rangle\rangle$. More recently, it was shown in [8] that local convergence also implies that F_c constitutes a well-defined operator from $B_p^m(R)[t_0,\, t_0 + T]$ into $B_q^\ell(S)[t_0,\, t_0 + T]$ for sufficiently small $R, S, T > 0$, where the numbers $p, q \in [1, \infty]$ are conjugate exponents, i.e., $1/p + 1/q = 1$ with $(1, \infty)$ being a conjugate pair by convention.

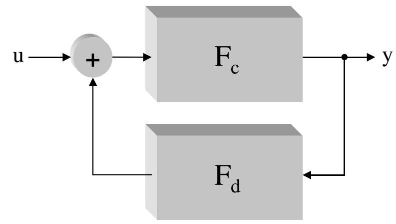

Fig. 1. Feedback connection of F_c and F_d

In many applications input-output systems are interconnected in a variety of ways. Given two Fliess operators F_c and F_d, where $c, d \in \mathbb{R}^m_{LC}\langle\langle X\rangle\rangle$, Fig. 1 shows the feedback connection. Ferfera was the first to describe the generating series for such a connection via the composition product $c\circ d$ of two formal power series [2]. However, the local convergence of this generating series was not explicitly addressed. The feedback connection is a fundamentally more complex case to analyze than other elementary interconnections. For example, when F_c is a linear operator, the formal solution to the feedback equation

$$y = F_c[u + F_d[y]] \tag{1}$$

is

$$y = F_c[u] + F_c \circ F_d \circ F_c[u] + \cdots .$$

It is not immediately clear that this series converges in any manner and, in particular, converges to another Fliess operator, say $F_{c@d}$ for some $c@d \in \mathbb{R}^m_{LC}\langle\langle X\rangle\rangle$. When F_c is nonlinear, the problem is further complicated by the fact that operators of the form $I + F_d$, where I denotes the identity

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map, *never* have a Fliess operator representation. In this paper, the problem is circumvented by introducing a simple variation of the composition product so that an appropriate *feedback product*, $c@d$, is well-defined, and $y = F_{c@d}[u]$ satisfies the feedback equation (1) in the sense that every analytic input u produces an analytic output y with (u, y) satisfying (1). In this case, $c@d$ is referred to as being *input-output locally convergent*, and explicit expressions are derived for one set of growth constants, K_{c_y} and M_{c_y}, for the series representation of the output function, c_y. In the process of establishing this result, an interesting connection is derived between the radius of convergence and the asymptotic behavior of the sequence of Catalan numbers, C_n, or more specifically, the binomial transform of the sequence of Catalan numbers, s_n. The positive integer sequences C_n and s_n each have a variety of combinatoric interpretations in graph theory and the theory of formal languages. Of particular interest to system theorists is the fact that C_n is equivalent to the number of ways to binary bracket the letters in a word of length $n + 1$ [16]. Such bracketing is typically encountered while characterizing the controllability of any state space realization an input-output system might possess [14]. This suggests a deeper connection between feedback structures of analytic systems and classical topics in algebraic combinatorics on words. This observation is further strengthened by the fact that the binomial transformation is known to preserve the Hankel transform of a sequence, in which case, C_n and s_n have identical Hankel transformations [11].

II. PRELIMINARIES

In this section, a brief overview of the shuffle product and composition product is provided. Given an alphabet $X = \{x_0, x_1, \ldots, x_m\}$, the shuffle product of two words $\eta, \xi \in X^*$ is defined recursively by

$$\eta \sqcup\!\sqcup \xi = (x_j\eta') \sqcup\!\sqcup (x_k\xi') := x_j[\eta' \sqcup\!\sqcup \xi] + x_k[\eta \sqcup\!\sqcup \xi']$$

with $\emptyset \sqcup\!\sqcup \emptyset = \emptyset$ and $\xi \sqcup\!\sqcup \emptyset = \emptyset \sqcup\!\sqcup \xi = \xi$. It is easily verified that $\eta \sqcup\!\sqcup \xi$ is always a polynomial consisting of words each having length $|\eta| + |\xi|$. The definition is extended to any two series $c, d \in \mathbb{R}\langle\langle X\rangle\rangle$ by

$$c \sqcup\!\sqcup d = \sum_{\eta,\xi \in X^*} [(c, \eta)(d, \xi)]\, \eta \sqcup\!\sqcup \xi.$$

In general, the shuffle product is commutative. It is also associative and distributes over addition. Thus, the vector space $\mathbb{R}\langle\langle X\rangle\rangle$ with the shuffle product forms a commutative $\mathbb{R}$-algebra, the so-called *shuffle algebra*, with multiplicative identity element $\emptyset$.

For any $\eta \in X^*$ and $d \in \mathbb{R}^m\langle\langle X\rangle\rangle$ the composition product is defined recursively in term of the shuffle product by

$$\eta \circ d = \begin{cases} \eta & : |\eta|_{x_i} = 0, \ \ \forall i \neq 0 \\ x_0^{n+1}[d_i \sqcup\!\sqcup (\eta' \circ d)] & : \eta = x_0^n x_i \eta', \ \ n \geq 0, \ \ i \neq 0, \end{cases}$$

where $|\eta|_{x_i}$ denotes the number of symbols in η equivalent to x_i and $d_i : \xi \mapsto (d, \xi)_i$, the i-th component of the coefficient $(d, \xi) \in \mathbb{R}^m$. Consequently, if

$$\eta = x_0^{n_k} x_{i_k} x_0^{n_{k-1}} x_{i_{k-1}} \cdots x_0^{n_1} x_{i_1} x_0^{n_0},$$

where $i_j \neq 0$ for $j = 1, \ldots, k$, then it follows that

$$\eta \circ d = x_0^{n_k+1}[d_{i_k} \sqcup\!\sqcup x_0^{n_{k-1}+1}[d_{i_{k-1}} \sqcup\!\sqcup \cdots \sqcup\!\sqcup x_0^{n_1+1}[d_{i_1} \sqcup\!\sqcup x_0^{n_0}] \cdots]].$$

Alternatively, for any $\eta \in X^*$, one can uniquely associate a set of right factors $\{\eta_0, \eta_1, \ldots, \eta_k\}$ by the iteration

$$\eta_{j+1} = x_0^{n_{j+1}} x_{i_{j+1}} \eta_j, \ \ \eta_0 = x_0^{n_0}, \ \ i_{j+1} \neq 0, \qquad (2)$$

so that $\eta = \eta_k$ with $k = |\eta| - |\eta|_{x_0}$. In which case, $\eta \circ d = \eta_k \circ d$, where $\eta_{j+1} \circ d = x_0^{n_{j+1}+1}[d_{i_{j+1}} \sqcup\!\sqcup (\eta_j \circ d)]$ and $\eta_0 \circ d = x_0^{n_0}$. For any $c \in \mathbb{R}^\ell\langle\langle X\rangle\rangle$ and $d \in \mathbb{R}^m\langle\langle X\rangle\rangle$, the composition product is defined as

$$c \circ d = \sum_{\eta \in X^*} (c, \eta)\, \eta \circ d.$$

The summation can also be written using the set of all right factors as described by equation (2). Let X^i be the set of all words in X^* of length i. For each word $\eta \in X^i$, the j-th right factor, η_j, has exactly j letters not equal to x_0. Therefore, given any $\nu \in X^*$:

$$(c \circ d, \nu) = \sum_{i=0}^{|\nu|} \sum_{j=0}^{i} \sum_{\eta_j \in X^i} (c, \eta_j)(\eta_j \circ d, \nu).$$

The rightmost summation is understood to be the sum over the set of all possible j-th right factors of words of length i.

It is easily verified that the composition product is linear in its first argument, but not its second. It was shown in [2] that the composition product is associative and distributive from the right over the shuffle product. But in general it is neither commutative nor has an identity element. The lack of an identity element is precisely the reason the identity map I in (1) is not realizable as a Fliess operator. One can characterize the continuity of the composition product with respect to a metric induced topology. The set $\mathbb{R}^m\langle\langle X\rangle\rangle$ forms a complete metric space under the ultrametric

$$\begin{aligned} \text{dist} \ &: \ \mathbb{R}^m\langle\langle X\rangle\rangle \times \mathbb{R}^m\langle\langle X\rangle\rangle \to \mathbb{R}^+ \cup \{0\} \\ &: \ (c, d) \mapsto \sigma^{\text{ord}(c-d)}, \end{aligned}$$

where $\sigma \in (0, 1)$ is arbitrary [1]. The following theorem states that the composition product on $\mathbb{R}^m\langle\langle X\rangle\rangle \times \mathbb{R}^m\langle\langle X\rangle\rangle$ is continuous in both arguments.

Theorem 2.1: [7] Let $\{c_i\}_{i\geq 1}$ be a sequence in $\mathbb{R}^m\langle\langle X\rangle\rangle$ with $\lim_{i\to\infty} c_i = c$. Then $\lim_{i\to\infty}(c_i \circ d) = c \circ d$ for any $d \in \mathbb{R}^m\langle\langle X\rangle\rangle$. Likewise, let $\{d_i\}_{i\geq 1}$ be a sequence in $\mathbb{R}^m\langle\langle X\rangle\rangle$ with $\lim_{i\to\infty} d_i = d$. Then $\lim_{i\to\infty}(c \circ d_i) = c \circ d$ for all $c \in \mathbb{R}^m\langle\langle X\rangle\rangle$.

Other useful facts concerning the composition product are that it produces a contractive mapping on $\mathbb{R}^m\langle\langle X\rangle\rangle$, and it preserves local convergence.

Theorem 2.2: [7] For any $c \in \mathbb{R}^m\langle\langle X\rangle\rangle$, the mapping $d \mapsto c \circ d$ is a contraction on $\mathbb{R}^m\langle\langle X\rangle\rangle$, that is,

$$\text{dist}(c \circ d, c \circ e) \leq \sigma\, dist(d, e), \ \ \forall d, e \in \mathbb{R}^m\langle\langle X\rangle\rangle.$$

Theorem 2.3: [7] Suppose $c \in \mathbb{R}^{\ell}_{LC}\langle\langle X\rangle\rangle$ and $d \in \mathbb{R}^{m}_{LC}\langle\langle X\rangle\rangle$ with growth constants K_c, M_c and K_d, M_d, respectively. Then $c \circ d \in \mathbb{R}^{\ell}_{LC}\langle\langle X\rangle\rangle$ with

$$|(c\circ d,\nu)| \leq K_c((\phi(mK_d)+1)M)^{|\nu|}(|\nu|+1)!, \quad \forall \nu \in X^*,$$

where $\phi(x) := x/2 + \sqrt{x^2/4 + x}$ and $M = \max\{M_c, M_d\}$. This local convergence property makes the following result possible concerning the cascade connection of two Fliess operators.

Theorem 2.4: [7] If $c, d \in \mathbb{R}^{m}_{LC}\langle\langle X\rangle\rangle$ then the composition of F_c and F_d produces another well-defined Fliess operator, namely $F_c \circ F_d = F_{c\circ d}$.

Example 2.1: The composition product provides an alternative interpretation of the symbolic calculus of Fliess [4], [6], [10]. Specifically, consider an input-output system represented by F_c with $c \in \mathbb{R}^{\ell}_{LC}\langle\langle X\rangle\rangle$. Any input u, which is analytic at $t = t_0$, can be represented near t_0 by a series $c_u \in \mathbb{R}^{m}_{LC}\langle\langle X_0\rangle\rangle$, where $X_0 = \{x_0\}$, i.e., $u = F_{c_u}[v]$ for some locally convergent series $c_u = \sum_{k\geq 0}(c_u, x_0^k)\, x_0^k$ and arbitrary $v \in B_p^m(R)[t_0, t_0+T]$. In effect, c_u is the formal Laplace-Borel transform of the input u (see [12]). The analyticity of $y = F_c[u]$ follows from [15, Lemma 2.3.8], and therefore the formal Laplace-Borel transform of y, namely c_y, can be related to c and c_u via

$$F_{c_y}[v] = y = F_c[F_{c_u}[v]] = F_{c\circ c_u}[v].$$

From [15, Corollary 2.2.4], it follows directly that $c_y = c \circ c_u$. □

This last example motivates the following definition.

Definition 2.1: A series $c \in \mathbb{R}^{\ell}\langle\langle X\rangle\rangle$ is ***input-output locally convergent*** if for every $c_u \in \mathbb{R}^{m}_{LC}\langle\langle X_0\rangle\rangle$ it follows that $c \circ c_u \in \mathbb{R}^{\ell}_{LC}\langle\langle X_0\rangle\rangle$.

It is immediate that every locally convergent series is input-output locally convergent, but the converse claim is only known to hold at present in certain special cases.

Lemma 2.1: [7] Let $c \in \mathbb{R}^{\ell}\langle\langle X\rangle\rangle$ be an input-output locally convergent series with non-negative coefficients. Then c is locally convergent.

Lemma 2.2: [7] Let $c \in \mathbb{R}^{\ell}\langle\langle X\rangle\rangle$ be an input-output locally convergent linear series of the form $c = \sum_{j\geq 0}(c, x_0^j x_{i_j})\, x_0^j x_{i_j}$, where $i_j \in \{1, 2, \ldots, m\}$ for all $j \geq 0$. Then c is locally convergent.

III. THE FEEDBACK CONNECTION

Given any $c, d \in \mathbb{R}^{m}_{LC}\langle\langle X\rangle\rangle$ and any input admissible $u \in B_p^m(R)[t_0, t_0+T]$, the general goal of this section is to determine when there exists a $y \in B_q^m(S)[t_0, t_0+T]$ which satisfies the feedback equation (1) and, in particular, when there exists a generating series e so that $y = F_e[u]$. In the latter case, the feedback equation becomes equivalent to

$$F_e[u] = F_c[u + F_{d\circ e}[u]], \tag{3}$$

and the *feedback product* of c and d is defined by $c@d = e$. An initial obstacle in this analysis is that F_e is required to be the composition of two operators, F_c and $I + F_{d\circ e}$, where the second operator is *never* a Fliess operator. This does not prevent the composition from being a Fliess operator, but to compensate for the presence of the direct feed term, I, a *modified* composition product is needed. Specifically, for any $\eta \in X^*$ and $d \in \mathbb{R}^m\langle\langle X\rangle\rangle$, define the modified composition product as

$$\eta\,\tilde{\circ}\,d = \begin{cases} \eta & : \ |\eta|_{x_i} = 0, \ \forall i \neq 0 \\ x_0^n x_i(\eta'\,\tilde{\circ}\,d) + & : \ \eta = x_0^n x_i \eta', \\ \quad x_0^{n+1}[d_i \sqcup\!\sqcup (\eta'\,\tilde{\circ}\,d)] & \quad\ n \geq 0, \ i \neq 0. \end{cases}$$

For $c \in \mathbb{R}^{\ell}\langle\langle X\rangle\rangle$ and $d \in \mathbb{R}^m\langle\langle X\rangle\rangle$, the definition is extended as

$$c\,\tilde{\circ}\,d = \sum_{\eta\in X^*}(c,\eta)\,\eta\,\tilde{\circ}\,d.$$

It can be verified in a manner completely analogous to the original composition product that the modified composition product is always well-defined (summable), continuous in both arguments, and locally convergent when both c and d are. In particular, the following theorems are central to the analysis in this section.

Theorem 3.1: [7] For any $c \in \mathbb{R}^{\ell}_{LC}\langle\langle X\rangle\rangle$ and $d \in \mathbb{R}^{m}_{LC}\langle\langle X\rangle\rangle$, it follows that

$$F_{c\,\tilde{\circ}\,d}[u] = F_c[u + F_d[u]]$$

for all admissible $u \in B_p^m(R)[t_0, t_0+T]$.

Theorem 3.2: [7] For any $c \in \mathbb{R}^m\langle\langle X\rangle\rangle$, the mapping $d \mapsto c\,\tilde{\circ}\,d$ is a contraction on $\mathbb{R}^m\langle\langle X\rangle\rangle$ with contraction coefficient σ.

The first main result of this section is given next.

Theorem 3.3: Let c, d be fixed series in $\mathbb{R}^m\langle\langle X\rangle\rangle$. Then:

1) The mapping

$$\begin{aligned} S \ &: \ \mathbb{R}^m\langle\langle X\rangle\rangle \to \mathbb{R}^m\langle\langle X\rangle\rangle \\ &: \ e_i \mapsto e_{i+1} = c\,\tilde{\circ}\,(d\circ e_i) \end{aligned}$$

has a unique fixed point in $\mathbb{R}^m\langle\langle X\rangle\rangle$, $c@d = \lim_{i\to\infty} e_i$, which is independent of e_0.

2) If c, d and $c@d$ are locally convergent, then $F_{c@d}$ satisfies the feedback equation (3).

Proof:

1. The mapping S is a contraction since, by Theorems 2.2 and 3.2,

$$\mathrm{dist}(S(e_i), S(e_j)) \leq \sigma\,\mathrm{dist}(d\circ e_i, d\circ e_j) \leq \sigma^2\,\mathrm{dist}(e_i, e_j).$$

Therefore, the mapping S has a unique fixed point, $c@d$, that is independent of e_0, i.e.,

$$c@d = c\,\tilde{\circ}\,(d\circ(c@d)). \tag{4}$$

2. From the stated assumptions concerning c, d and $c@d$, it follows that

$$F_{c@d}[u] = F_{c\,\tilde{\circ}\,(d\circ(c@d))}[u] = F_c[u + F_d[F_{c@d}[u]]]$$

for any admissible u. ■

The obvious question is whether $c@d$ is always locally convergent, or at least input-output locally convergent, when both c and d are locally convergent. The first step in the analysis is to show that when $e = c \circ e$, where c is locally convergent with growth constants K_c and M_c, then

$$|(e, x_0^n)| \leq K_c\, \tilde{\psi}_n(K_c)\, M_c^n\, n!, \;\; \forall n \geq 0,$$

where each $\tilde{\psi}_n(K_c)$ is a polynomial in K_c of degree n. The next lemma establishes the claim using a family of polynomials of the form

$$\tilde{\psi}_n(K_c) = \sum_{i=0}^{n} \sum_{j=0}^{i} \sum_{\eta_j \in X^i} K_c^j \tilde{S}_{\eta_j}(K_c, n) |\eta_j|!, \;\; n \geq 0.$$

Given a fixed n, every word η_j in the innermost summation satisfies $j \leq |\eta_j| \leq n$ and has a corresponding set of right factors $\{\eta_0, \eta_1, \ldots, \eta_j\}$. When $j > 0$, each polynomial $\tilde{S}_{\eta_j}(K_c, n)$ is computed iteratively using its right factors and the previously computed polynomials $\{\tilde{\psi}_0(K_c), \tilde{\psi}_1(K_c), \ldots, \tilde{\psi}_{n-1}(K_c)\}$:

$$\begin{aligned}
\tilde{S}_{\eta_0}(K_c, n) &= \frac{1}{|\eta_0|!}, \;\; 0 \leq |\eta_0| \leq n \\
\tilde{S}_{\eta_1}(K_c, n) &= \frac{1}{(n)_{n_1+1}}\, \tilde{\psi}_{n-|\eta_1|}(K_c)\, \tilde{S}_{\eta_0}(K_c, n), \\
&\qquad\qquad 1 \leq |\eta_1| \leq n \\
&\;\;\vdots \\
\tilde{S}_{\eta_j}(K_c, n) &= \frac{1}{(n)_{n_j+1}} \sum_{i=0}^{n-|\eta_j|} \tilde{\psi}_i(K_c) \cdot \\
&\qquad \tilde{S}_{\eta_{j-1}}(K_c, n-(n_j+1)-i), \;\; 2 \leq j \leq |\eta_j| \leq n.
\end{aligned}$$

Here $(n)_i = n!/(n-i)!$ is the falling factorial.

Lemma 3.1: Let $c \in \mathbb{R}^m_{LC}\langle\langle X \rangle\rangle$ with growth constants K_c, M_c, and $e \in \mathbb{R}^m\langle\langle X \rangle\rangle$ such that $e = c \circ e$. Then

$$|(e, x_0^n)| \leq K_c\, \tilde{\psi}_n(K_c)\, M_c^n\, n!, \;\; \forall n \geq 0. \tag{5}$$

Proof: The proof involves *nested inductions*. The outer induction is on n. The claim is trivial when $n = 0$ and $n = 1$. Suppose (5) holds up to some fixed $n - 1 \geq 1$. Given any η_j, where $j \leq |\eta_j| \leq n$, it will first be shown by induction on j (the inner induction) that

$$|(\eta_j \circ e, x_0^n)| \leq K_c^j M_c^{-|\eta_j|} M_c^n\, n!\, \tilde{S}_{\eta_j}(K_c, n), \;\; 0 \leq j \leq n. \tag{6}$$

The $j = 0$ case is trivial. Suppose $j = 1$. Then $0 \leq n - |\eta_1| \leq n - 1$ and

$$\begin{aligned}
&|(\eta_1 \circ e, x_0^n)| \\
&\quad= \left|\left(x_0^{n_1+1}(e_{i_1} ⧢ x_0^{n_0}), x_0^n\right)\right| \\
&\quad= \left|\left(e_{i_1} ⧢ x_0^{n_0}, x_0^{n-(n_1+1)}\right)\right| \\
&\quad= \left|\left(e_{i_1}, x_0^{n-|\eta_1|}\right)\left(x_0^{n-|\eta_1|} ⧢ x_0^{n_0}, x_0^{n-(n_1+1)}\right)\right| \\
&\quad\leq \left(K_c\, \tilde{\psi}_{n-|\eta_1|}(K_c)\, M_c^{n-|\eta_1|}(n-|\eta_1|)!\right) \cdot \\
&\qquad\qquad \binom{n-(n_1+1)}{n-|\eta_1|} \\
&\quad= K_c M_c^{-|\eta_1|} M_c^n\, n!\, \tilde{S}_{\eta_1}(K_c, n).
\end{aligned}$$

Now assume that inequality (6) holds up to some fixed j, where $1 \leq j \leq n-1$. Then $0 \leq n - |\eta_{j+1}| \leq n - (j+1)$ and

$$\begin{aligned}
&|(\eta_{j+1} \circ e, x_0^n)| \\
&\quad= \left|\left(e_{i_{j+1}} ⧢ (\eta_j \circ e), x_0^{n-(n_{j+1}+1)}\right)\right| \\
&\quad= \left|\sum_{i=0}^{n-(n_{j+1}+1)} \left(e_{i_{j+1}}, x_0^i\right)\left(\eta_j \circ e, x_0^{n-(n_{j+1}+1)-i}\right) \cdot \right. \\
&\qquad\qquad \left. \binom{n-(n_{j+1}+1)}{n-(n_{j+1}+1)-i}\right|.
\end{aligned}$$

Since $\left(\eta_j \circ e, x_0^{n-(n_{j+1}+1)-i}\right) = 0$ when $n-(n_{j+1}+1)-i < |\eta_j|$ or, equivalently, $i > n - |\eta_{j+1}|$, it follows that using the coefficient bound (5) for e (because $0 \leq i \leq n-1$) and the bound (6) for $\eta_j \circ e$,

$$\begin{aligned}
&|(\eta_{j+1} \circ e, x_0^n)| \\
&\quad\leq \sum_{i=0}^{n-|\eta_{j+1}|} \left(K_c \tilde{\psi}_i(K_c) M_c^i\, i!\right)\left(K_c^j M_c^{-|\eta_j|} M_c^{n-(n_{j+1}+1)-i} \cdot \right. \\
&\qquad \left. (n-(n_{j+1}+1)-i)!\, \tilde{S}_{\eta_j}(K_c, n-(n_{j+1}+1)-i)\right) \cdot \\
&\qquad \binom{n-(n_{j+1}+1)}{n-(n_{j+1}+1)-i} \\
&\quad= K_c^{j+1} M_c^{-|\eta_{j+1}|} M_c^n\, n!\, \frac{1}{(n)_{n_{j+1}+1}} \cdot \\
&\qquad \sum_{i=0}^{n-|\eta_{j+1}|} \tilde{\psi}_i(K_c)\, \tilde{S}_{\eta_j}(K_c, n-(n_{j+1}+1)-i) \\
&\quad= K_c^{j+1} M_c^{-|\eta_{j+1}|} M_c^n\, n!\, \tilde{S}_{\eta_{j+1}}(K_c, n).
\end{aligned}$$

Hence, the claim is true for all $0 \leq j \leq n$.

To complete the outer induction with respect to n, observe that

$$\begin{aligned}
|(e, x_0^n)| &= |(c \circ e, x_0^n)| \\
&= \left|\sum_{i=0}^{n} \sum_{j=0}^{i} \sum_{\eta_j \in X^i} (c, \eta_j)(\eta_j \circ e, x_0^n)\right| \\
&\leq \sum_{i=0}^{n} \sum_{j=0}^{i} \sum_{\eta_j \in X^i} \left(K_c M_c^{|\eta_j|} |\eta_j|!\right) \cdot \\
&\qquad \left(K_c^j M_c^{-|\eta_j|} M_c^n\, n!\, \tilde{S}_{\eta_j}(K_c, n)\right) \\
&= K_c\, \tilde{\psi}_n(K_c)\, M_c^n\, n!.
\end{aligned}$$

Therefore, inequality (5) holds for all $n \geq 0$. ■

The next lemma provides an upper bound on the growth of the sequence $\tilde{\psi}_n(K_c)$, $n \geq 0$, when K_c is fixed.

Lemma 3.2: For any $K_c \geq 1$, it follows that

$$\tilde{\psi}_n(K_c) \leq \phi_g(mK_c(2+\phi_g)+1)^n s_n, \;\; \forall n \geq 0, \tag{7}$$

where $\phi_g = \phi(1)$ (the golden ratio), $s_0 = 1/\phi_g$, and s_n, $n \geq 1$, is an integer sequence equivalent to the binomial transform of the sequence of Catalan numbers, C_n, $n \geq 1$ (specifically, sequence A007317 in [13]).

Proof: The proof has two main parts. First, it is shown by a nested induction that for any $\epsilon > 0$, there exists a sequence of positive real numbers, $\xi_n(\epsilon)$, such that

$$\tilde{\psi}_n(K_c) \leq (mK_c(2+\epsilon)+1)^n \xi_n(\epsilon), \quad n \geq 0, \quad K_c \geq 1. \quad (8)$$

Then inequality (7) is produced for $n \geq 1$ by setting $\epsilon = \phi_g$ and showing that $\xi_n(\phi_g) = \phi_g s_n$ when $n \geq 1$. ($n = 0$ is a trivial special case.)

Let $\epsilon > 0$ and define two sequences of positive real numbers, $\xi_n(\epsilon)$ and $\Gamma_n(\epsilon)$, via the recurrence equations

$$\xi_{n+1}(\epsilon) = \xi_n(\epsilon) + \Gamma_{n+1}(\epsilon), \ n \geq 0, \ \xi_0 = 1, \ \Gamma_1 = 1/\epsilon \quad (9)$$

$$\Gamma_{n+1}(\epsilon) = \frac{1}{\epsilon}\left[\xi_n(\epsilon) + \sum_{i=1}^{n} \xi_i(\epsilon)\Gamma_{n-i+1}(\epsilon)\right], \ n \geq 1. \quad (10)$$

By definition, $\Gamma_0 = 1$. The claim is straightforward when $n = 0$ and $n = 1$. Suppose the inequality holds up to some fixed $n - 1 \geq 1$. Given any word η_j, where $j \leq |\eta_j| \leq n$, an inner induction with respect to j will now show that

$$\tilde{S}_{\eta_j}(K_c, n) \leq \frac{(mK_c(2+\epsilon)+1)^{n-|\eta_j|}\,(2+\epsilon)^j\,\Gamma_{n-|\eta_j|}(\epsilon)}{|\eta_j|!}, \quad (11)$$

where $0 \leq j \leq |\eta_j|$. The $j = 0$ case is trivial. Suppose $j = 1$. Since $n - |\eta_1| < n$, it follows that

$$\begin{aligned}\tilde{S}_{\eta_1}(K_c, n) &= \frac{1}{(n)_{n_1+1}} \frac{\tilde{\psi}_{n-|\eta_1|}(K_c)}{|\eta_0|!} \\ &\leq \frac{(mK_c(2+\epsilon)+1)^{n-|\eta_1|}\,\xi_{n-|\eta_1|}(\epsilon)}{|\eta_1|!} \\ &\leq \frac{(mK_c(2+\epsilon)+1)^{n-|\eta_1|}\,(2+\epsilon)\,\Gamma_{n-|\eta_1|}(\epsilon)}{|\eta_1|!},\end{aligned}$$

when $n \geq |\eta_1|$. This last inequality employs the general properties for any $j \geq 0$ that $\xi_{n-|\eta_j|}(\epsilon) = \Gamma_{n-|\eta_j|}(\epsilon)$ when $n = |\eta_j|$ and

$$\sum_{i=0}^{n-|\eta_j|} \xi_i(\epsilon)\Gamma_{n-|\eta_j|-i}(\epsilon) = (2+\epsilon)\,\Gamma_{n-|\eta_j|}(\epsilon) \quad (12)$$

when $n > |\eta_j|$. Now suppose inequality (11) holds up to some fixed $j \geq 1$. Then

$$\begin{aligned}&\tilde{S}_{\eta_{j+1}}(K_c, n) \\ &= \frac{1}{(n)_{n_{j+1}+1}} \sum_{i=0}^{n-|\eta_{j+1}|} \tilde{\psi}_i(K_c)\,\tilde{S}_{\eta_j}(K_c, n-(n_{j+1}+1)-i) \\ &\leq \frac{1}{|\eta_{j+1}|!} \sum_{i=0}^{n-|\eta_{j+1}|} (mK_c(2+\epsilon)+1)^i \xi_i(\epsilon)\,\cdot \\ &\quad \left[(mK_c(2+\epsilon)+1)^{n-|\eta_{j+1}|-i}(2+\epsilon)^j\,\Gamma_{n-|\eta_{j+1}|-i}(\epsilon)\right] \\ &= \frac{(mK_c(2+\epsilon)+1)^{n-|\eta_{j+1}|}(2+\epsilon)^j}{|\eta_{j+1}|!}\,\cdot \\ &\quad \sum_{i=0}^{n-|\eta_{j+1}|} \xi_i(\epsilon)\Gamma_{n-|\eta_{j+1}|-i}(\epsilon) \\ &= \frac{(mK_c(2+\epsilon)+1)^{n-|\eta_{j+1}|}\,(2+\epsilon)^{j+1}\,\Gamma_{n-|\eta_{j+1}|}(\epsilon)}{|\eta_{j+1}|!},\end{aligned}$$

where $|\eta_j| < |\eta_{j+1}| \leq n$ and again identity (12) was used to derive the final equality above. Hence, inequality (11) holds for all $0 \leq j \leq |\eta_j|$. To complete the outer induction with respect to n, observe that

$$\begin{aligned}&\tilde{\psi}_{n+1}(K_c) \\ &= \sum_{i=0}^{n+1}\sum_{j=0}^{i}\sum_{\eta_j \in X^i} K_c^j \tilde{S}_{\eta_j}(K_c, n+1)|\eta_j|! \\ &\leq \sum_{i=0}^{n+1}\sum_{j=0}^{i}\binom{i}{j}\,\cdot \\ &\quad \left[\frac{(mK_c(2+\epsilon)+1)^{n+1-i}\,(mK_c(2+\epsilon))^j\,\Gamma_{n+1-i}(\epsilon)}{i!}\right] i! \\ &= (mK_c(2+\epsilon)+1)^{n+1}\sum_{i=0}^{n+1}\Gamma_{n+1-i}(\epsilon) \\ &= (mK_c(2+\epsilon)+1)^{n+1}\xi_{n+1}(\epsilon).\end{aligned}$$

Thus, inequality (8) must hold for all $n \geq 0$.

Now consider setting $\epsilon = \phi_g$ in the system of equations (9)–(10). Eliminating by substitution the sequence $\Gamma_n(\phi_g)$ gives the recurrence relation

$$\xi_{n+1}(\phi_g) = \phi_g + \frac{1}{\phi_g}\sum_{i=1}^{n}\xi_i(\phi_g)\xi_{n-i+1}(\phi_g), \quad n \geq 1$$

with $\xi_1(\phi_g) = \phi_g$, or, equivalently,

$$\left(\frac{\xi_{n+1}(\phi_g)}{\phi_g}\right) = 1 + \sum_{i=1}^{n}\left(\frac{\xi_i(\phi_g)}{\phi_g}\right)\left(\frac{\xi_{n-i+1}(\phi_g)}{\phi_g}\right), \quad n \geq 1,$$

with $\xi_1(\phi_g)/\phi_g = 1$. It is known that s_n satisfies the recurrence equation

$$s_{n+1} = 1 + \sum_{i=1}^{n} s_i s_{n-i+1}, \quad n \geq 1, \quad s_1 = 1 \quad (13)$$

(see [13] and the references therein). Hence, the conclusion that $\xi_n(\phi_g) = \phi_g s_n$, $n \geq 1$, is immediate. ■

The recurrence equation (13) can be derived from the well-known recurrence relation for the Catalan numbers: $C_{n+1} = \sum_{i=0}^{n} C_i C_{n-i}$ with $C_0 = 1$, which in turn is equivalent to Segner's recurrence formula given in the year 1758 as a solution to Euler's polygon division problem [16]. It is also worth noting that the sequence $t_n := \Gamma_n(\phi_g)/\phi_g$, $n \geq 1$, the increments of s_n, is sequence A002212 in [13]. The positive integer sequences C_n, s_n and t_n each have a variety of combinatoric interpretations in graph theory and the theory of formal languages. The asymptotic behavior of s_n,

$$s_n \sim \frac{1}{8}\sqrt{\frac{5}{\pi}}\,\frac{5^n}{n^{3/2}}$$

(see [9, sequence 124]), motivates the following central result concerning local convergence.

Theorem 3.4: If $c \in \mathbb{R}^m_{LC}\langle\langle X\rangle\rangle$ with growth constants K_c, M_c and $e = c \circ e$, then $e \in \mathbb{R}^m_{LC}\langle\langle X_0\rangle\rangle$. Specifically, for any $K_c \geq 1$,

$$|(e, x_0^n)| \leq K_c((mK_c(2+\phi_g)+1)5M_c)^n\, n!, \;\; \forall n \geq 0.$$

Proof: The result is trivial when $n = 0$. When $n \geq 1$, it is first necessary to show by induction that $s_{n+1} < 5s_n$. The claim is clearly true when $n = 1$ or $n = 2$. Suppose it is known to hold up to some fixed integer $n+1 \geq 2$. Sequence s_n is known to satisfy another recurrence equation [9], [13]:

$$(n+2)s_{n+2} = (6n+4)s_{n+1} - 5ns_n.$$

Therefore,

$$s_{n+2} < [(6n+4)s_{n+1} - ns_{n+1}]/(n+2) < 5s_{n+1},$$

which proves the claim for all $n \geq 1$. Next, substituting the upper bound $\phi_g s_n \leq 5^n$, $n \geq 0$, into (7) gives

$$\tilde{\psi}_n(K_c) \leq ((mK_c(2+\phi_g)+1)5)^n, \;\; \forall n \geq 0.$$

The theorem is finally proved by simply applying Lemma 3.1. ■

The final step of the analysis is to use Theorem 3.4 to prove the input-output local convergence of the feedback product.

Theorem 3.5: If $c, d \in \mathbb{R}^m_{LC}\langle\langle X\rangle\rangle$, then $c@d$ is input-output locally convergent. Specifically, when $K_c \geq 1$, then

$$((c@d)\circ b, x_0^n) \;\; \leq \;\; K_c([mK_c(2+\phi_g)+1] \cdot [\phi(m(K_b+K_d))+1]\, 10M)^n\, n!$$

for any $b \in \mathbb{R}^m_{LC}\langle\langle X_0\rangle\rangle$ and where $M = \max\{M_b, M_c, M_d\}$.

Proof: Select any series $b \in \mathbb{R}^m_{LC}\langle\langle X_0\rangle\rangle$. It follows from (4) that

$$(c@d)\circ b = (c\,\tilde{\circ}\,(d\circ(c@d)))\circ b = c\circ(b+d)\circ((c@d)\circ b).$$

Since b, c and d are all locally convergent, so is the series $c\circ(b+d)$. Now apply Theorem 3.4, replacing c with $c\circ(b+d)$ and e with $(c@d)\circ b$. This implies that $(c@d)\circ b$ is always locally convergent, and therefore $c@d$ must be input-output locally convergent. To produce the given growth condition for the output series, note that

$$K_{c\circ(b+d)} = K_c \qquad M_{c\circ(b+d)} = 2(\phi(m(K_b+K_d))+1)M,$$

using Theorem 2.3 and the fact that $n+1 \leq 2^n$ for all $n \geq 0$. Substituting these growth constants for K_c and M_c, respectively, in Theorem 3.4 produces the desired result. ■

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