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Adaptive Output Feedback Control of General MIMO Systems Using Multirate Sampling

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Abstract—In adaptive output feedback control based on almost strictly positive real conditions, a technical difficulty arises when the multi-input multi-output (MIMO) system under consideration is non-square, and in particular, has less inputs than outputs. To overcome this, we propose an idea of multirate sampled-data control. That is, a lifted discrete-time system, which has the same number of inputs and outputs and does not give rise to the causality constraint, is made by carefully choosing faster input sampling rates. The output feedback based adaptive control strategy can then be applied to the lifted system under certain conditions. The results reported here are validated on numerical simulations through a cart-crane model.

I. INTRODUCTION

Multirate systems arise often in industry due to hardware limitations on sensing and actuating devices. In such inherently multirate sampled systems, one is forced to consider multirate control strategies such as multirate robust control [5], and multirate adaptive control [13], [17], [6], which are more relevant to the topic in this paper.

However, the multirate systems in this paper are motivated from a different viewpoint. In adaptive output feedback control design based on almost strictly positive real (ASPR) conditions, under certain assumptions, closed-loop stability can be proven for single-input, single-output plants and robustness of the adaptive scheme has been shown [9], [7], [12]. Such an adaptive control scheme has several practical advantages and has been applied successfully to many industrial applications. Extending this scheme to multi-input, multi-output systems, the design procedure requires a technical assumption that the systems to be controlled must be square, i.e., the number of inputs must be equal to the number of outputs [3], [1], [8], [15]. Such an assumption can be quite restrictive – in many practical systems, the number of inputs is less than that of the outputs, especially in cases where several control objectives are to be achieved. In such cases, how to apply the adaptive control design methodology is the thrust of this research. Our idea is to consider digital control with a multirate sampling scheme. Thus multirate systems in this paper are obtained by control designers *on purpose* in order to accommodate existing adaptive control strategies.

In selecting the different sampling rates involved, there are two points that we need to keep in mind:

- First, we need to ensure that the resultant systems after lifting [11], [10] are square (with the same number of inputs and outputs).
- Second, we need to avoid further complication by introducing another difficult design limitation – the so called *causality constraint*, see, e.g., [5].

We will show later that these two goals can be achieved simultaneously by carefully choosing faster input updating rates only.

II. MULTIRATE SAMPLING AND LIFTING

For a general discussion, consider a continuous-time, linear, time-invariant plant G_c with m inputs and p outputs, and $p > m$; assume a state-space representation for G_c with input u_c, output y_c, and state x_c:

$$\dot{x}_c(t) = A_c x_c(t) + B_c u_c(t), \qquad (1)$$
$$y_c(t) = C x_c(t) + D_c u_c(t). \qquad (2)$$

Partitioning B_c and D_c according to the input u_c, we get

$$\left[\begin{array}{c} B_c \\ D_c \end{array} \right] = \left[\begin{array}{cccc} B_{c1} & B_{c2} & \cdots & B_{cm} \\ D_{c1} & D_{c2} & \cdots & D_{cm} \end{array} \right], \quad u_c = \left[\begin{array}{c} u_{c1} \\ u_{c2} \\ \vdots \\ u_{cm} \end{array} \right],$$

in which $B_{c1}, B_{c2}, \cdots, B_{cm}$ are n-dimensional column vectors, with n being the dimension of the state space, and $D_{c1}, D_{c2}, \cdots, D_{cm}$ are m-dimensional column vectors.

The system G_c is a typical non-square system; now we would like to derive a *square* lifted discrete-time system. There are many different ways to arrive at a square lifted system by selecting multiple sampling rates, but we target simple ones which do not introduce the causality constraint later in the design of the controller.

We choose to sample all outputs uniformly with a single period, say, T, and update the inputs $u_{c1}, u_{c2}, \cdots, u_{cm}$ through zero-order holds with fast periods T/q_1, T/q_2, $\cdots$, T/q_m, respectively. Here $q_1, q_2, \cdots, q_m$ are all positive integers and are chosen to satisfy
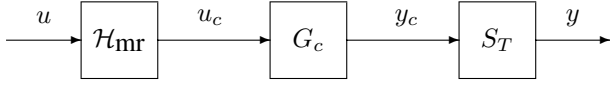
$$q_1 + q_2 + \cdots + q_m = p. \qquad (3)$$

Fig. 1. The multirate sampled-data system

We remark that such q_i's always exist (if $m < p$) and are non-unique; e.g., if $m = 2$ and $p = 5$, there are four possible (q_1, q_2) pairs satisfying (3); they are $(1, 4)$, $(2, 3)$, $(3, 2)$, and $(4, 1)$.

Denote the correspondingly discretized system by G; this is a multirate discrete-time system depicted in Figure 1, where S_T is an ideal sampler with period T (vector-valued), and $\mathcal{H}_{\mathrm{mr}}$ is the *multirate* zero-order hold operator defined as

$$\mathcal{H}_{\mathrm{mr}} = \begin{bmatrix} H_{T/q_1} & & & \\ & H_{T/q_2} & & \\ & & \ddots & \\ & & & H_{T/q_m} \end{bmatrix}$$

with H_{T/q_i} being the synchronized zero-order hold with period T/q_i. Note that y is single-rate with period T ($y = S_T y_c$), but u is multirate with each component having a different period; we can write

$$u = \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix}, \quad u_{ci} = H_{T/q_i} u_i, \quad i = 1, 2, \cdots, m.$$

We get that the multirate G takes multirate u into single-rate y, and can be denoted by

$$G = S_T G_c \mathcal{H}_{\mathrm{mr}}.$$

Next, we need to lift this multirate system to arrive at a time-invariant one with the single period T.

Let v be a discrete-time signal defined on the time set $\{0, 1, 2, \cdots\}$:

$$v = \{v(0), v(1), v(2), \cdots\}.$$

The q-fold lifting operator L_q maps v into $\underline{v}$ as follows [11], [10]:

$$\underline{v} = \left\{ \begin{bmatrix} v(0) \\ v(1) \\ \vdots \\ v(q-1) \end{bmatrix}, \begin{bmatrix} v(q) \\ v(q+1) \\ \vdots \\ v(2q-1) \end{bmatrix}, \cdots, \right\}.$$

Notice that the lifting operation increases the dimension of the signal by a factor of q, but decreases the underlying sampling rate also by a factor of q. The inverse lifting operation, L_q^{-1} is defined obviously.

In order to get a lifted system which is single-rate with period T, we lift the input u_i by L_{q_i} to get $\underline{u}_i$ so that the lifted system $\underline{G}$ maps $\underline{u}(k)$ into $y(k)$ defined as follows:

$$\underline{u}(k) = \begin{bmatrix} \underline{u}_1(k) \\ \vdots \\ \underline{u}_m(k) \end{bmatrix}.$$

Thus we can write

$$\underline{G} = G \begin{bmatrix} L_{q_1}^{-1} & & \\ & \ddots & \\ & & L_{q_m}^{-1} \end{bmatrix}$$

This $\underline{G}$ is square and $m \times m$, based on (3). It is a fact that $\underline{G}$ is time-invariant and admits a state-space model, which can be derived based on the results in [5], [4]. We summarize the results into a few steps.

Step 1 Compute the step-invariant transformation matrices as follows:

$$A = e^{A_c T},$$

$$A_i = e^{A_c T/q_i}, \; B_i = \int_0^{T/q_i} e^{A_c t} B_{ci} \, dt, \; i = 1, 2, \cdots, m.$$

Step 2 Define the matrices

$$\underline{B_i} = \begin{bmatrix} A_i^{q_i-1} B_i & \cdots & A_i B_i & B_i \end{bmatrix},$$
$$\underline{D_i} = \begin{bmatrix} D_{ci} & 0 & \cdots & 0 \end{bmatrix}, \quad i = 1, 2, \cdots, m.$$

Step 3 Then a state-space model for $\underline{G}$ is given by

$$x(k+1) = Ax(k) + B\underline{u}(k), \qquad (4)$$
$$y(k) = Cx(k) + D\underline{u}(k), \qquad (5)$$

where $x(k) = x_c(kT)$, and

$$B = \begin{bmatrix} \underline{B_1} & \underline{B_2} & \cdots & \underline{B_m} \end{bmatrix},$$
$$D = \begin{bmatrix} \underline{D_1} & \underline{D_2} & \cdots & \underline{D_m} \end{bmatrix}.$$

Because of the choice of sampling rates, the causality constraint in the lifted controller (mapping y into $\underline{u}$) will not arise [5].

III. ADAPTIVE CONTROL DESIGN

The adaptive controller design is based on the lifted system $\underline{G}$ in (4) and (5); such controllers map y into $\underline{u}$. Hence in implementation to the original sampled-data system, the controller output $\underline{u}$ needs to be inverse-lifted properly.

A. Problem Statement

Consider the lifted, but square system $\underline{G}$ in (4) and (5); we shall need a few conditions in order to proceed.

Definition 1: (Almost Strictly Positive Realness [1], [9]) The square plant $\underline{G}$ in (4) and (5) is called *almost strictly positive real* (ASPR) if there exists a static output feedback such that the resulting closed loop system is strictly positive real (SPR). Explicitly, $\underline{G}$ is ASPR if there exists a control input with a feedback gain Θ^*,

$$\underline{u}(k) = -\Theta^* y(k) + \underline{v}(k), \qquad (6)$$

(v is any external input) such that the resulting closed loop system from $\underline{v}(k)$ to $y(k)$,

$$x(k+1) = A_c x(k) + B_c \underline{v}(k), \qquad (7)$$
$$y(k) = C_c x(k) + D_c \underline{v}(k), \qquad (8)$$

with

$$A_c = A - B\Theta^*(I + D\Theta^*)^{-1}C , \quad B_c = B(I + \Theta^*D)^{-1},$$
$$C_c = (I + D\Theta^*)^{-1}C , \quad D_c = D(I + \Theta^*D)^{-1},$$
(9)

is SPR.

Definition 2: The square plant $\underline{G}$ is called *strongly ASPR* if there exists a static output feedback,

$$\underline{u}(k) = -\Theta^*y(k) + \underline{v}(k),$$

such that the closed loop system with state matrices (A_c, B_c, C_c, D_c) as given in (9) is SPR and, in addition, a transformed closed loop system with $\underline{\tilde{v}} = (I + \Theta^*D)^{-1}\underline{v}$ as input,

$$x(k + 1) = A_cx(k) + B\underline{\tilde{v}}(k) \quad (10)$$
$$y(k) = C_cx(k) + D\underline{\tilde{v}}(k) \quad (11)$$

is also SPR.

The sufficient conditions for $\underline{G}$ to be ASPR are easily obtained by translating the conditions given for continuous-time systems [2] as follows.

ASPR Conditions:

(1) The relative MacMillan degree of the system is n/n, where n is the dimension of the A-matrix.
(2) The plant is minimum-phase.

Furthermore, for *strong* almost strictly positive realness, an additional condition is required:

(3) $D + D^T > 0$.

We shall impose the following assumptions on the lifted model $\underline{G}$ in (4) and (5).

Assumption 1: The plant $\underline{G}$ given in (4) and (5) is controllable and observable.

This assumption can be related to controllability and observability of the original continuous-time system and a non-pathological sampling condition [4], [5].

Assumption 2: The direct feedthrough matrix D in (5) is known.

Assumption 3: For the plant $\underline{G}$ given in (4) and (5), there exists a known stable parallel feedforward compensator (PFC), denoted by G_f, with a state-space model of an appropriate order,

$$x_f(k + 1) = A_fx_f(k) + B_f\underline{u}(k), \quad (12)$$
$$y_f(k) = C_fx_f(k) + D_f\underline{u}(k), \quad (13)$$

such that the resulting augmented system, $\underline{G} + G_f$, with a state-space model,

$$x_a(k + 1) = A_ax_a(k) + B_a\underline{u}(k), \quad (14)$$
$$y_a(k) = C_ax_a(k) + D_a\underline{u}(k), \quad (15)$$

is strongly ASPR. Here, $x_a(k) = [x(k)^T \ x_f(k)^T]^T$ and

$$A_a = \begin{bmatrix} A & 0 \\ 0 & A_f \end{bmatrix}, \ B_a = \begin{bmatrix} B \\ B_f \end{bmatrix}, \ C_a = [C \ C_f], \ D_a = D + D_f.$$

It should be noted that under Assumption 3, there exists a static output feedback with a feedback gain matrix $\tilde{\Theta}_e^* > 0$,

$$\underline{u}(k) = -\tilde{\Theta}_e^*y_a(k) + \underline{v}(k)$$

such that the resulting closed loop system, after an input transformation $\underline{\tilde{v}} = (I + \tilde{\Theta}_e^*D_a)^{-1}\underline{v}$,

$$x_a(k + 1) = A_{ac}x_a(k) + B_a\underline{\tilde{v}}(k),$$
$$y_a(k) = C_{ac}x_a(k) + D_a\underline{\tilde{v}}(k),$$

is SPR. In this case,

$$A_{ac} = A_a - B_a\tilde{\Theta}_e^*(I + D_a\tilde{\Theta}_e^*)^{-1}C_a,$$
$$C_{ac} = (I + D_a\tilde{\Theta}_e^*)^{-1}C_a.$$

Further, since the system $(A_{ac}, B_a, C_{ac}, D_a)$ is SPR, there exist positive symmetric matrices $P = P^T > 0$, $Q = Q^T > 0$ and appropriate matrices L and W such that based on the Kalman-Yakubovich Lemma, the following hold:

$$A_{ac}^TPA_{ac} - P = -LL^T - Q,$$
$$A_{ac}^TPB_a = C_{ac}^T - LW^T, \quad (16)$$
$$B_a^TPB_a = D_a + D_a^T - WW^T.$$

Our control objective in this paper is to design an adaptive controller that achieves

$$\lim_{k \to \infty} y(k) = \mathbf{0}, \quad (17)$$

for $\underline{G}$ satisfying Assumptions 1 to 3.

B. Controller Design Procedure

An adaptive controller for attaining the control objective in (17) can be designed by the following equations[12], [15]:

$$\underline{u}(k) = -\Theta_e(k)y_p(k), \quad (18)$$
$$y_p(k) = C_ax_a(k). \quad (19)$$

The feedback gain matrix $\Theta_e(k)$ in (18) is adaptively adjusted by the following parameter adjusting law:

$$\Theta_e(k) = \Theta_{Pe}(k) + \Theta_{Ie}(k), \quad (20)$$
$$\Theta_{Pe}(k) = y_a(k)y_p^T(k)\Gamma_{Pe}, \quad (21)$$
$$\Theta_{Ie}(k) = \Theta_{Ie}(k - 1) + y_a(k)y_p^T(k)\Gamma_{Ie}. \quad (22)$$

Here $\Gamma_{Pe} = \Gamma_{Pe}^T > 0$, $\Gamma_{Ie} = \Gamma_{Ie}^T > 0$.

It is noted that $y_a(k)$ in (21) and (22) cannot be directly obtained from measured signals because of a causality problem which arises from the direct feedthrough term D_a in (15). However, $y_a(k)$ can be generated by using available signals. From (15) and (18), we have

$$y_a(k) = y_p(k) + D_a\underline{u}(k)$$
$$= y_p(k) - D_a\{\Theta_{Ie}(k-1) + y_a(k)y_p^T(k)\Gamma_e\}y_p(k)$$
(23)

where $\Gamma_e = \Gamma_{Pe} + \Gamma_{Ie}$. Therefore $y_a(k)$ can be obtained by

$$y_a(k) = \{I + D_ay_p^T(k)\Gamma_ey_p(k)\}^{-1}$$
$$\times \{y_p(k) - D_a\Theta_{Ie}(k - 1)y_p(k)\},$$

by using available signals.

C. Stability Analysis

We have the following stability result for the resulting closed loop control system.

Theorem 1: Under Assumptions 1 to 3, all the signals in the resulting closed loop control system with control input in (18) are uniformly bounded and the output converges to zero, i.e., the objective $\lim_{k\to\infty} y(k) = \mathbf{0}$ is achieved.
Proof. Consider an ideal control input given by

$$\underline{u}^*(k) = -\Theta_e^* y_p(k), \tag{24}$$
$$\Theta_e^* = (I + \tilde{\Theta}_e^* D_a)^{-1}\tilde{\Theta}_e^* = \tilde{\Theta}_e^*(I + D_a\tilde{\Theta}_e^*)^{-1}. \tag{25}$$

The closed loop system with control input in (18) can be represented by

$$x_a(k+1) = \tilde{A}_a x_a(j) + B_a \Delta\underline{u}(k), \tag{26}$$
$$y_a(k) = \tilde{C}_a \boldsymbol{x}_p(j) + D_a \Delta\underline{u}(k), \tag{27}$$

where

$$\tilde{A}_a = A_a - B_a\Theta_e^* C_a \tag{28}$$
$$\tilde{C}_a = (I - D_a\Theta_e^*)C_a \tag{29}$$
$$\Delta\underline{u}(k) = \underline{u}(k) - \underline{u}^*(k)$$

Now, consider the following positive definite function $V(k)$,

$$V(k) = V_1(k) + V_2(k), \tag{30}$$
$$V_1(k) = x_a(k)^T P x_a(k), \tag{31}$$
$$V_2(k) = \text{tr}\{\Delta\Theta_{Ie}(k-1)\Gamma_{Ie}^{-1}\Delta\Theta_{Ie}^T(k-1)\}, \tag{32}$$

where

$$\Delta\Theta_{Ie}(k) = \Theta_{Ie}(k) - \Theta_e^*. \tag{33}$$

Defining $\Delta V(k)$ by

$$\Delta V(k) = V(k+1) - V(k), \tag{34}$$

we have from (30) that

$$\Delta V(k) = \Delta V_1(k) + \Delta V_2(k) \tag{35}$$
$$\Delta V_1(k) = V_1(k+1) - V_1(k)$$
$$\Delta V_2(k) = V_2(k+1) - V_2(k)$$

First, consider the difference ΔV_1. We have from (26) and (31) that

$$\begin{aligned}\Delta V_1(k) &= V_1(k+1) - V_1(k)\\ &= x_a(k)^T\{\tilde{A}_a^T P \tilde{A}_a - P\}x_a(k)\\ &\quad +2x_a(k)^T\tilde{A}_a^T P B_a\Delta\underline{u}(k)\\ &\quad +\Delta\underline{u}(k)^T B_a^T P B_a\Delta\underline{u}(k).\end{aligned} \tag{36}$$

Since it follows from (25) that

$$\begin{aligned}I - D_a\Theta_e^* &= I - D_a\tilde{\Theta}_e^*(I + D_a\tilde{\Theta}_e^*)^{-1}\\ &= (I + D_a\tilde{\Theta}_e^*)^{-1} \neq 0\end{aligned} \tag{37}$$

we have from (28), (29) and (25) that

$$\tilde{A}_a = A_a - B_a\tilde{\Theta}_e^*(I + D_a\tilde{\Theta}_e^*)^{-1}C_a = A_{ac}, \tag{38}$$
$$\tilde{C}_a = (I + D_a\tilde{\Theta}_e^*)^{-1}C_a = C_{ac}, \tag{39}$$

Thus, since the system $(A_{ac}, B_a, C_{ac}, D_a)$ is SPR, $\Delta V_1(k)$ can be expressed from (16) and (36) as

$$\begin{aligned}\Delta V_1(k) &= -x_a(k)^T Q x_a(k) + 2y_a(k)^T\Delta\underline{u}(k)\\ &\quad -\|x_a(k)^T L - \Delta\underline{u}(k)^T W\|^2\end{aligned} \tag{40}$$

Next, consider the difference ΔV_2. From (22) and (33), we have

$$\Delta\Theta_{Ie}(k) = \Delta\Theta_{Ie}(k-1) - y_a(k)y_p(k)^T\Gamma_{Ie} \tag{41}$$

Thus $\Delta V_2(k)$ can be expressed as

$$\begin{aligned}\Delta V_2(k) &= 2\text{tr}\{\Delta\Theta_{Ie}(k)y_p(k)y_a(k)^T\}\\ &\quad -\text{tr}\{y_a(k)y_p(k)^T\Gamma_{Ie}y_p(k)y_a(k)^T\}.\end{aligned}$$

Since

$$\Delta\Theta_{Ie}(k) = \Delta\Theta_e(k) - y_a(k)y_p(k)^T\Gamma_{Pe}, \tag{42}$$
$$\Delta\Theta_e(k) = \Theta_e(k) - \Theta_e^*,$$

it follows that

$$\begin{aligned}\Delta V_2(k) &= -2y_a(k)^T\Delta\underline{u}(k)\\ &\quad -\text{tr}\{y_a(k)y_p(k)^T(\Gamma_{Ie} + 2\Gamma_{Pe})y_p(k)y_a(k)^T\}.\end{aligned} \tag{43}$$

Finally, from (40) and (43), $\Delta V(k)$ can be evaluated as follows:

$$\begin{aligned}\Delta V(k) &= -x_a(k)^T Q x_a(k) - \|x_a(k)^T L - \Delta\underline{u}(k)^T W\|^2\\ &\quad -\text{tr}\{y_a(k)y_p(k)^T(\Gamma_{Ie} + 2\Gamma_{Pe})y_a(k)y_a(k)^T\}\\ &\leq 0.\end{aligned} \tag{44}$$

Consequently we have that $x_a(k)$ and $\Theta_{Ie}(k)$ are uniformly bounded and then it is easy to conclude that all the signals in the control system are uniformly bounded. Further, since we have from (44) that

$$\lim_{N\to\infty}\sum_{k=0}^{N}\|x_a(k)\|^2 < \infty,$$

we can obtain $\lim_{k\to\infty} x_a(k) = 0$ and thus we have $\lim_{k\to\infty} x(k) = 0$. Moreover, from (18) and (19) we have $\lim_{k\to\infty} y_p(k) = 0$ and $\lim_{k\to\infty} \underline{u}(k) = 0$. Therefore, we can conclude from (5) that

$$\lim_{k\to\infty} y(k) = 0.$$

Remark: In practical applications in which disturbances and measurement noises exist, one cannot guarantee the stability of the resulting control system with the ideal adaptive parameter adjusting law (22). However, in such cases, one can adopt robust adaptive parameter adjusting laws such as the one with σ-modification [1], [7] in order to guarantee boundedness of all signals in the control system.
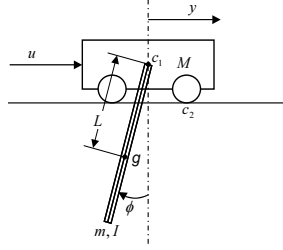
Fig. 2. Cart-crane system

Table 1 Parameters of cart-crane system

Parameter	Value
g	9.81 [m/s^2]
M	$1.168 + 3$ [kg]
m	0.071 [kg]
L	0.358 [m]
I	3.025×10^{-3} [kg·m^2]
c_1	0.01 [N·s/rad]
c_2	10 [N·s/m]

IV. NUMERICAL SIMULATION

In this section, we validate the effectiveness of the proposed method through numerical simulations for a cart-crane model. A simple configuration of the cart-crane system is illustrated in Fig. 2, in which $\phi(t)$ is the angular position of the crane, $y(t)$ is the displacement of the cart, and $u(t)$ is the control force applied to the cart; the system parameters involved are M (mass of the cart), m (mass of the crane), L (length from the axis of rotation to the center of gravity of the crane), I (moment of inertia of the crane), g (gravitational acceleration), c_1 (damping constant at the rotational axis), and c_2 (damping constant between the cart and the horizontal surface). These parameters are given in Table 1 above.

Define $x_c(t) = [\phi(t), \ \dot{\phi}(t), \ y(t), \ \dot{y}(t)]^T$, $y_c(t) = [\phi(t), \ y(t)]$, and $u_c = u$. A linearized state-space model around $\phi = 0$ for the cart-crane system is given as follows:

$$\dot{x}_c(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ a & b & 0 & c \\ 0 & 0 & 0 & 1 \\ d & e & 0 & f \end{bmatrix} x_c(t) + \begin{bmatrix} 0 \\ g_1 \\ 0 \\ h \end{bmatrix} u_c(t) \quad (45)$$

$$y_c(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} x_c(t) \quad (46)$$

with

$$a = \frac{(m-M)mgL}{\Delta}, \ b = \frac{(m-M)c_1L}{\Delta}, \ c = \frac{-mLc_2}{\Delta}$$

$$d = \frac{m^2gL^2}{\Delta}, \ e = \frac{mL^2c_1}{\Delta}, \ f = \frac{-mL^2c_2 - Ic_2}{\Delta}$$

$$g_1 = \frac{mL}{\Delta}, \ h = \frac{mL^2+I}{\Delta}, \ \Delta = mML^2 + I(M-m).$$

In this simulation, we assume that the output $y_c(t)$ is sampled with a period of $T = 0.2$ [s], but the input signal $u_c(t)$ can be updated through a zero-order hold with a fast period $T/2$. Furthermore, to improve the control performance
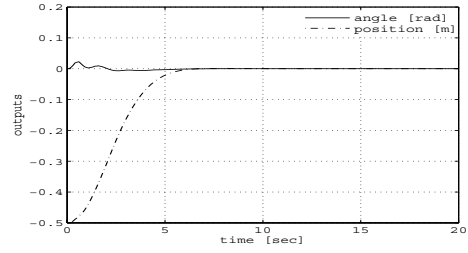

Fig. 3. Simulation result with multirate adaptive controller: cart position and crane angle
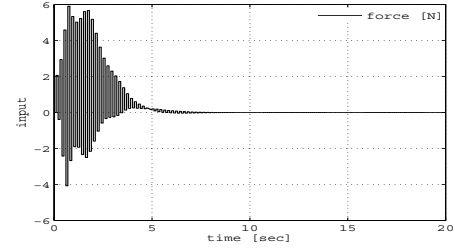

Fig. 4. Simulation result with multirate adaptive controller: control input

for the crane angle, we consider a weighted angle, that is, we generate

$$y_{cw}(t) = \begin{bmatrix} w_1 & 0 \\ 0 & 1 \end{bmatrix} y_c(t) \quad (47)$$

as a new output for the controller design. In this simulation, we set $w_1 = 20$. A PFC, which renders the resulting augmented system ASPR, is designed as follows:

$$D_p = \begin{bmatrix} 0.02 & 0 \\ 0 & 0.02 \end{bmatrix}$$

for the system with y_{cw} as the output. This is the case where $A_f = 0$, $B_f = 0$, $C_f = 0$, and $D_f = D_p$ is the PFC in (12) and (13). Parameters in the adaptive adjusting laws in (20) to (22) are set to

$$\Gamma_{Pe} = \text{diag}[10, 10], \quad \Gamma_{Ie} = \text{diag}[8, 8].$$

Figs. 3 and 4 show simulation results of the proposed multirate control strategy – vibration of the pendulum is effectively suppressed and the cart moves smoothly to the desired position.

By way of comparison, we apply a PI controller to this cart-crane system under the assumption that all the outputs can be sampled with as fast as the control updating period $T = 0.1$ [s]. The (digital) PI controller is then designed as follows:

$$u_{PI}(k) = -\left(K_P\hat{y}_{PI}(k) + TK_I \sum_{m=0}^{k} \hat{y}_{PI}(m) \right) \quad (48)$$

$$\hat{y}_{PI}(k) = w_{p1}\phi(k) + w_{p2}y(k). \quad (49)$$

Here K_p and K_I are the proportional and integral gains of the PI controller; w_{p1} and w_{p2} are weighting factors for the outputs $\phi(t)$ and $y(t)$.

The PI controller parameters are designed to obtain a good cart positional control, and are given as follows:

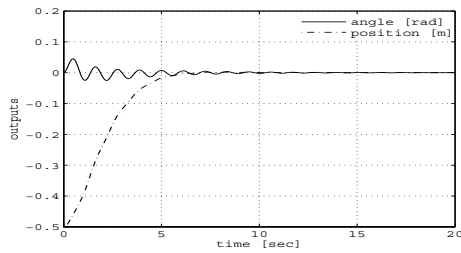$$K_P = 23, \ K_I = 0.01, \ w_{p1} = 1, \ w_{p2} = 0.2.$$

Fig. 5. Simulation result with PI controller for case-1: cart position and crane angle
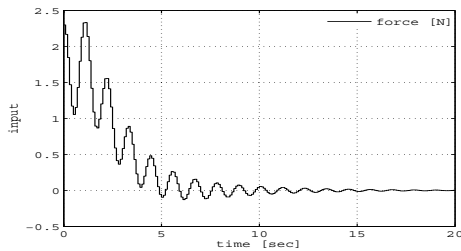

Fig. 6. Simulation result with PI controller for case-1: control input

Figs. 5 and 6 show simulation results with this PI controller, which show a good control performance for the cart position, however, a crane angle vibration exists to some extent.

Comparing the results of the two control strategies, we conclude that the proposed multirate adaptive control is better in that good control performance for both cart position and crane angle is achieved simultaneously.

Next, in order to test the adaptive strategy, we consider a situation in which a load is attached to the tip of the crane, and thus both the weight and center of gravity of the crane are changed. In the simulation we take the load to be $50m$. Fig. 7 shows the result with the PI controller; the control performance is poor and the control system is even unstable. On the other hand, the control result (Fig. 8) with the proposed multirate adaptive controller shows approximately the same performance as in Fig. 3 with no load. Thus the proposed multiratre adaptive controller has a better robustness properties against parameter variations than the conventional PI controller.

V. Conclusions

In this paper, we proposed an adaptive output feedback design scheme for general MIMO systems using the idea of multirate sampled data control. The scheme was based on the ASPR-ness of a controlled system with a condition that the controlled system must be square (namely, with the same number of inputs and outputs). To satisfy this condition, a square lifted discrete-time system was created by using a multirate sampling technique. An ASPR based adaptive control scheme was then applied to this lifted system. The obtained adaptive controller is then multirate and can be implemented to non-square MIMO systems. The effectiveness of the proposed method was validated through numerical simulations for a cart-crane model.
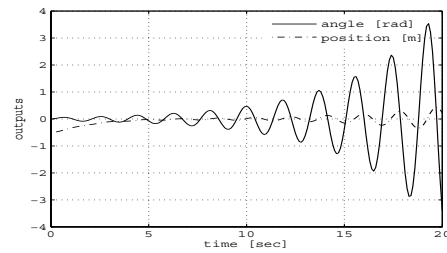

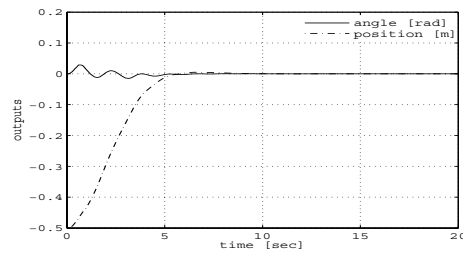Fig. 7. Simulation result with PI controller: additional load

Fig. 8. Simulation result with multirate adaptive controller: additional load

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