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Lyapunov functions for time varying systems satisfying generalized conditions of Matrosov theorem

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Abstract— **The classical Matrosov theorem concludes uniform asymptotic stability of time varying systems via a** *weak Lyapunov function* **(positive definite, decrescent, with** *negative semi-definite derivative* **along solutions) and another auxiliary function with derivative that is strictly non-zero where the derivative of the Lyapunov function is zero [10]. Recently, several generalizations of the classical Matrosov theorem that use a finite number of Lyapunov-like functions have been reported in [5]. None of these results provides a construction of a** *strong Lyapunov function* **(positive definite, decrescent, with** *negative definite derivative* **along solutions) that is a very useful analysis and controller design tool for nonlinear systems. Inspired by generalized Matrosov conditions in [5], we provide a construction of a strong Lyapunov function via an appropriate weak Lyapunov function and a set of Lyapunov-like functions whose derivatives along solutions of the system satisfy a particular triangular structure. Our results will be very useful in a range of situations where strong Lyapunov functions are needed, such as robustness analysis and Lyapunov function based controller redesign.**

Key words. Lyapunov functions, Matrosov Theorem, Nonlinear, Stability, Time-Varying.

I. INTRODUCTION

Lyapunov second method is ubiquitous in stability and robustness analysis of nonlinear systems. In recent years, its different versions were used for controller design, e.g. control Lyapunov functions, nonlinear damping, backstepping, forwarding, and so on [7], [11]. While it is often useful to obtain a *strong Lyapunov function* (positive definite, decrescent, with *negative definite derivative* along solutions) to analyze robustness or redesign the given controller, it is often the case that only a *weak Lyapunov function* (positive definite, decrescent, with *semi-negative definite derivative* along solutions) can be constructed for a problem at hand [1], [2], [3], [8], [9], [14]. For example, controller design methods that are based on the passivity property typically require the use of the La Salle invariance principle [8] which exploits weak Lyapunov functions to conclude asymptotic stability.

The La Salle Theorem in its original form applies only to time-invariant systems. On the other hand, the Matrosov Theorem [10] concludes uniform asymptotic stability of time-varying systems via a *weak Lyapunov function* and another auxiliary function with derivative that is strictly non-zero where the derivative of the Lyapunov function is zero [10]. Different generalizations of the Matrosov theorem that use an arbitrary number of auxiliary functions to conclude uniform asymptotic stability have been recently reported in the literature [5]. Moreover, results in [5] make use of the recently proposed notion of uniform δ persistency of excitation (uδ-PE condition) [6] that allows to further relax the original Matrosov conditions.

The proofs presented in [5], [10] do not provide a construction of a strong Lyapunov function and they conclude uniform asymptotic stability by considering directly the behavior of the trajectories of the system. Nevertheless, these results are extremely important in applications since it is often too hard to construct a strong Lyapunov function. The main purpose of this paper is to construct strong Lyapunov functions using appropriate generalized Matrosov conditions that are inspired by main results in [5]. In particular, each of our results assumes existence of an appropriate weak Lyapunov function and a set of Lyapunov-like functions, similar to [5], to provide an explicit formula for constructing a strong Lyapunov function. Moreover, our results parallel main results in [5] and we present constructions that exploit the uδ-PE condition. We apply our results to construct a strong Lyapunov function for time-varying systems in cases when an appropriate weak Lyapunov function is known and certain uniform observability conditions are satisfied. When applied to time-invariant systems, this result provides an alternative construction of a strong Lyapunov function to the one presented in [13]. A special case of our results also generalizes the construction of a strong Lyapunov function given in [12].

Constructions provided in this paper will be useful in a range of situations when the knowledge of a

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strong Lyapunov function is useful, such as robustness analysis and Lyapunov based controller redesign. In particular, our results can be applied to non holonomic systems and model reference adaptive control examples reported in [5]. This analysis is not presented in this paper due to space constraints.

II. Preliminaries

Unless otherwise stated, we assume throughout the paper that the functions encountered are sufficiently smooth. We often omit arguments of functions to simplify the notations. A continuous function $k : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be of class $\mathcal{K}_\infty$ if $k(0) = 0$ and k is increasing and unbounded. A function $V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^p$ is said to be bounded uniformly in t if there exists $\alpha \in \mathcal{K}_\infty$ such that for all t and x we have $|V(t,x)| \leq \alpha(|x|)$. A continuous function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is positive semi-definite if $V(0) = 0$ and $V(x) \geq 0$ for all $x \in \mathbb{R}^n$. It is positive definite if $V(0) = 0$ and $V(x) > 0$ for all $x \neq 0$. It is negative semi-definite (definite) if $-V$ is positive semi-definite (definite).

Consider the time varying system:

$$\dot{x} = f(t,x) \tag{1}$$

with $t \in \mathbb{R}, x \in \mathbb{R}^n$. In order to simplify the notation, we use the following notation:

$$DV := \frac{\partial V}{\partial t}(t,x) + \frac{\partial V}{\partial x}(t,x) f(t,x) \ ,$$

where $V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$. We need the following definition and assumptions:

Definition 2.1: Suppose that there exist functions $V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, $\alpha_1, \alpha_2, \alpha_4 \in \mathcal{K}_\infty$ and $\alpha_3 : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ such that for all x and all t the following holds:

$$\alpha_1(|x|) \leq V(t,x) \leq \alpha_2(|x|) \tag{2}$$

$$DV \leq -\alpha_3(x) \tag{3}$$

$$\left|\frac{\partial V}{\partial t}(t,x)\right| \leq \alpha_4(|x|). \tag{4}$$

If the function α_3 is positive semi-definite, then we say that V is a *weak* Lyapunov function for the system (1). If, on the other hand, α_3 is positive definite, then V is referred to as a *strong* Lyapunov function for the system (1).

Assumption 2.2: A weak Lyapunov function V_1 for the system (1) is known.

Assumption 2.3: The following functions are known: $V_i : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$, $i = 2, 3, \ldots, j$, such that V_i, $\frac{\partial V_i}{\partial t}(t,x)$ and $\frac{\partial V_i}{\partial x}(t,x)$ are bounded uniformly in t; positive semi-definite functions $N_i : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ for $i = 2, \ldots, N$; $\chi_i : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{i-2} \to \mathbb{R}$ for $i = 3, \ldots, N$, with $\chi_i(t,x,0,0,\ldots,0) \equiv 0$ such that χ_i and N_i are bounded uniformly in t. Moreover, for all $t \in \mathbb{R}$ and all $x \in \mathbb{R}^n$ we have:

$$DV_2 \leq -N_2 =: Y_2 \tag{5}$$

$$DV_3 \leq -N_3 + \chi_3(t,x,Y_2) =: Y_3 \tag{6}$$

$$DV_4 \leq -N_4 + \chi_4(t,x,Y_2,Y_3) =: Y_4 \tag{7}$$

$$\vdots \quad \vdots \quad \vdots$$

$$DV_j \leq -N_j + \chi_j(t,x,Y_2,\ldots,Y_{j-1}) =: Y_j \tag{8}$$

III. Main results

In this section, we establish main results of this paper that are summarized in Theorem 3.1 and Corollaries 3.4 and 3.6. Each of these results provides a construction of a strong Lyapunov function using an existing weak Lyapunov function (Assumption 2.2), a set of Lyapunov-like functions (Assumption 2.3) and other appropriate conditions.

To state the first main result, we will suppose that the system (1) admits the decomposition:

$$\dot{x}_1 = f_1(t,x) \ , \ \dot{x}_2 = f_2(t,x) \tag{9}$$

with $x_1 \in \mathbb{R}^{n_1}$, $x_2 \in \mathbb{R}^{n_2}$, $n_1 + n_2 = n$. Note that we allow for the cases when either $n_1 = n$ or $n_2 = n$ that correspond to $x_1 = x$ and $x_2 = x$, respectively.

The main result of our paper is stated next:

Theorem 3.1: Consider the system (9) and suppose that Assumptions 2.2 and 2.3 hold. Suppose also that the following conditions hold:

C1. There exist a positive definite real-valued function ω, and a nonnegative continuously differentiable function $M : \mathbb{R} \times \mathbb{R}^{n_2} \to \mathbb{R}$ such that $M(t,x_2)$ and $\frac{\partial M}{\partial x_2}(t,x_2)$ are bounded uniformly in t and the following holds for all x and t

$$\sum_{i=2}^{j} N_i(t,x) \geq \omega(|x_1|) + M(t,x_2) \tag{10}$$

and

$$|f_2(t,x)| \leq \chi_f(t,x,Y_2,Y_3,\ldots,Y_{j-1}) \ , \tag{11}$$

where $\chi_f(t,x,0,\ldots,0) \equiv 0$ and χ_f is bounded uniformly in t.

C2. There exist a differentiable function $\theta : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ and a positive definite function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that for all (t,x), we have:

$$\int_t^{t+\theta(|x_2|)} M(s,x_2)ds \geq \gamma(|x_2|) \ . \tag{12}$$

Then, one can determine nonnegative functions p_i and a positive definite function δ such that the following function:

$$W(t,x) = \sum_{i=1}^{j} p_i(V_1(t,x))V_i(t,x) + p_{j+1}(V_1(t,x))\delta(|x_2|)A(t,x_2) \tag{13}$$

with

$$A(t,x_2)=\int_{t-\theta(|x_2|)}^{t}\left(\int_s^t M(l,x_2)dl\right)ds \quad (14)$$

is a strong Lyapunov function for system (9).

Remark 3.2: We note that a construction of the functions p_i and δ in (13) is provided in the proofs of our main results.

Remark 3.3: Conditions of Theorem 3.1 can be regarded as generalized Matrosov theorem conditions and they are directly related to conditions used in [5, Theorem 1]. Indeed, our Assumption 2.2 corresponds to [5, Assumption 1]. Our Assumption 2.3 corresponds to [5, Assumptions 2 and 3], and so on. In particular, our condition **C2** corresponds to the so called $u\delta$-persistency of excitation condition introduced in [6]. Note, however, that our conditions are stronger in that we assume that we know all the bounding functions since they are required in construction of the strong Lyapunov function W. For instance, we assume that we know the functions θ and γ in the condition **C2** of Theorem 3.1 whereas this is not needed in main results of [5]. This is the main difference between our conditions and those given in [5]. A consequence of our stronger assumptions is that we construct a strong Lyapunov function W, which was not done in [5].

There are several important consequences of Theorem 3.1 that we state next. The following corollary is a direct consequence of Theorem 3.1 - it is a special case of this theorem when $x_1 = x$ in (16). It is interesting in the sense that it provides with a family of strong Lyapunov function for systems satisfying the conditions of the classical Mastrosov theorem. In this special case V_1 is the weak Lyapunov function, $V_1 = V_2$ and V_3 is the auxiliary function from the classical Matrosov theorem.

Corollary 3.4: Consider the system (1) and suppose that Assumptions 2.2 and 2.3 hold and that $f(t,x)$ is bounded uniformly in t. Suppose also that:

$$\sum_{i=2}^{j} N_i(t,x) \geq \omega(x) \quad (15)$$

where $\omega(\cdot)$ is a positive definite function. Then, one can determine nonnegative functions p_i such that the following function:

$$W(t,x)=\sum_{i=1}^{j} p_i(V_1(t,x))V_i(t,x) \quad (16)$$

is a strong Lyapunov function for system (1).
The following corollary is devoted to the case where $x_2 = x$. In particular, we can state:

Corollary 3.5: Consider the system (9) and suppose that Assumptions 2.2 and 2.3 hold. Suppose also there exists a nonnegative continuously differentiable function $M : \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}$ such that $M(t,x)$ and $\frac{\partial M}{\partial x}(t,x)$ are bounded uniformly in t and the following holds for all x and t

$$\sum_{i=2}^{j} N_i(t,x) \quad \geq \quad M(t,x) \quad (17)$$

and

$$|f(t,x)| \leq \chi_f(t,x,Y_2,Y_3,\ldots,Y_{j-1}) \quad (18)$$

holds, where $\chi_f(t,x,0,\ldots,0)\equiv 0$ and χ_f is bounded uniformly in t. Moreover, there exist a differentiable function $\theta : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ and a positive definite function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that for all (t,x), we have:

$$\int_t^{t+\theta(|x|)} M(s,x)ds \geq \gamma(|x|) \ . \quad (19)$$

Then, one can determine nonnegative functions p_i, $i = 1$ to $j+1$ and a positive definite function δ such that the following function:

$$W(t,x)=\sum_{i=1}^{j} p_i(V_1(t,x))V_i(t,x)+ p_{j+1}(V_1(t,x))\delta(|x|)\int_{t-\theta(|x|)}^{t}\left(\int_s^t M(l,x)dl\right)ds \quad (20)$$

is a strong Lyapunov function for system (9).
It is possible to strengthen the persistency condition (19) and at the same time relax the condition (18) to provide a similar Lyapunov function construction that is presented in the next corollary. Observe that the strong Lyapunov functions we obtain are given by expressions slightly simpler that (13). Besides, this corollary will be instrumental in establishing the result of the next section.

Corollary 3.6: Consider the system (1) and suppose that Assumptions 2.2 and 2.3 hold and that $f(t,x)$ is bounded uniformly in t. Suppose also that

$$\sum_{i=2}^{j} N_i(t,x) \geq M(t,x) = \overline{p}(t)\mu(x) \quad (21)$$

where μ is a positive definite function and $\overline{p}(t)$ is a nonnegative function such that, for all $t\in\mathbb{R}$,

$$\int_{t-\tau}^{t}\overline{p}(l)dl \geq p_m \quad , \quad \overline{p}(t)\leq p_M \quad (22)$$

where $\tau > 0, p_m > 0, p_M > 0$.

Then, one can determine nonnegative functions p_i such that the following function:

$$W(t,x)=\sum_{i=1}^{j} p_i(V_1(t,x))V_i(t,x)+ +p_{j+1}(V_1(t,x))\left(\int_{t-\tau}^{t}\left(\int_s^t \overline{p}(l)dl\right)ds\right) \quad (23)$$

is a strong Lyapunov function for system (1).

IV. Strong Lyapunov functions via uniform observability

The result of this section is an extension of the main result of [13] to time-varying systems. Similar uniform observability conditions were used in [5, Section 3.3] to conclude uniform asymptotic stability of time-varying system. We apply Corollary 3.6 to construct a strong Lyapunov function for this case:

Corollary 4.1: Consider the system (1). Assume that f is a function bounded uniformly in t and that Assumption 2.2 holds. Let $N \geq 2$ be an integer and let us assume that the functions

$$a(t,x) := DV_1(t,x) \leq 0 \tag{24}$$

and

$$b_1(t,x) := Da(t,x)\ ,\ b_{i+1}(t,x) := Db_i(x,t)\ , \tag{25}$$

$i = 2, \ldots, N$ are bounded uniformly in t and such that for all (t,x),

$$a(t,x) + b_1(t,x)^2 + \cdots + b_N(t,x)^2 \geq \overline{p}(t)\mu(x) \tag{26}$$

where $\mu(x)$ is a positive definite function and $\overline{p}(t)$ is nonnegative function such that (21) is satisfied. Then, one can determine nonnegative functions p_i such that the strong Lyapunov function for system (1) has the form (23), where

$$\begin{array}{l} V_2(t,x) := V_1(t,x)\ , \\ V_3(t,x) := -a(t,x)b_1(t,x)\ , \\ V_i(t,x) := -b_{i-3}(t,x)b_{i-2}(t,x)\ ,\ \forall i \geq 4\ . \end{array} \tag{27}$$

Proof. Observe that

$$DV_3 = -b_1(t,x)^2 - a(t,x)Db_1(t,x) \tag{28}$$

and, for all $i \in \{4, ..., N\}$,

$$DV_i = -b_{i-2}(t,x)^2 - b_{i-3}(t,x)Db_{i-2}(t,x)\ . \tag{29}$$

Let $\rho(\cdot)$ be a positive function such that

$$|Db_i(t,x)| \leq \rho(V_1(t,x))\ ,\ \forall i \geq 1. \tag{30}$$

Then the following inequalities, for all $i \in \{4, ..., N\}$,

$$DV_3 \leq -b_1(t,x)^2 + a(t,x)\rho(V_1(t,x)) \tag{31}$$

$$DV_i \leq -b_{i-2}(t,x)^2 + |b_{i-3}(t,x)|\rho(V_1(t,x)) \tag{32}$$

hold. Hence, all conditions of Corollary 3.6 are satisfied and the result follows.

V. Proofs of Main Results

The proof of main results is carried out by first proving Corollary 3.4. Then, the proof of Theorem 3.1 is carried out by showing that under the given conditions it is possible to construct a function V_{j+1} such that the functions $V_2, \ldots, V_{j+1}$ satisfy all conditions of Corollary 3.4. Corollary 3.6 is proved in a similar way. Due to space limitations, we only give the proof of Corollary 3.4.

Proof of Corollary 3.4: We first outline the steps in the proof. Using functions V_i, $i = 1, 2, \ldots, j$, we define

$$S_{j+1}(t,x) := \sum_{i=2}^{j} V_i(t,x) \tag{33}$$

and then recursively construct the functions:

$$S_{j-i} := V_1 + V_{j-i} + l_{j-i}(V_1)S_{j-i+1}$$

for $i = 0, 1, 2, \ldots, j-1$. In particular, we construct these functions to satisfy for any $i = 0, 1, \ldots, j-2$ the following:

$$\begin{aligned} DS_{j-i} &\leq -w_{j-i}(x) \\ &\quad + \phi_{j-i}\big(\sum_{i=2}^{j-i-1} N_k\big)\rho_{j-i}(V_1)\ , \end{aligned} \tag{34}$$

$$|S_{j-i}| \leq \sigma_{j-i}(|x|)\ , \tag{35}$$

for some positive definite w_{j-i}, positive function ρ_{j-i} and $\sigma_{j-i}, \phi_{j-i} \in \mathcal{K}_\infty$ where we let $N_1 \equiv 0$. In particular, this implies for all (t,x) that

$$DS_2 \leq -w_2(x)\ , \tag{36}$$

$$|S_2| \leq \sigma_2(|x|)\ . \tag{37}$$

Note that S_2 is not necessarily a positive definite function. Finally, using Lemma 1.1 we can choose the function l_1 so that $W(t,x) := S_1(t,x)$ is a strong Lyapunov function for the system.

First, using conditions in Assumption 2.3 and (15), one can determine $\phi \in \mathcal{K}_\infty$ and a positive function ρ such that

$$DV_2 \leq -N_2 \tag{38}$$

$$DV_i \leq -N_i + \phi\big(\sum_{2}^{i-1} N_k\big)\rho(V_1)\ , \forall i = 3, \ldots, j \tag{39}$$

$$DS_{j+1} \leq -\omega(x) + \phi\big(\sum_{2}^{j} N_k\big)\rho(V_1) \tag{40}$$

where S_{j+1} is the function defined in (33) and ω comes from (15). From Assumption 2.3 it follows that there exists $\sigma_{j+1} \in \mathcal{K}_\infty$ such that for all (t,x) we have

$$|S_{j+1}(t,x)| \leq \sigma_{j+1}(|x|)\ .$$

Consider now

$$S_j(t,x) \;:=\; V_1(t,x)+V_j(t,x) \quad (41)$$
$$+l_j(V_1(t,x))S_{j+1}(t,x) \quad (42)$$

where l_j is any function that satisfies for all $(t,x)\neq(t,0)$ the following:

$$|l_j'(V_1(t,x))| \leq \min\left\{\psi_1(|x|), \frac{1}{\sigma_{j+1}(|x|)}\right\}\ , \quad (43)$$
$$l_j(V_1(t,x)) \leq \min\left\{\psi_2(|x|), \zeta(x)\right\}\ , \quad (44)$$

ω, ϕ, N_j, ρ come from (40), $\psi_1, \psi_2 \in \mathcal{K}_\infty$ are arbitrary and $\zeta : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is such that for all (t,x) we have[1]:

$$\zeta(x) \leq \frac{\phi^{-1}\left(\frac{\omega(x)}{2\rho(V_1(t,x))}\right)}{2\phi(2N_j(t,x))\rho(V_1(t,x))}\ .$$

Note that Lemma 1.1 in the appendix guarantees that such l_j always exists. From Assumption 2.3 we know that there exists $\bar{\alpha}_2 \in \mathcal{K}_\infty$ such that $|V_2(t,x)| \leq \bar{\alpha}_2(|x|)$ for all (t,x). Hence, for all (t,x) we have:

$$|S_j(t,x)| \leq \alpha_2(|x|)+\bar{\alpha}_2(|x|)+\psi_2(|x|)\sigma_{j+1}(|x|) =: \sigma_j(|x|)\ .$$

Moreover, its derivative along the trajectories of (1) satisfies

$$DS_j \;=\; DV_j + [l_j'(V_1)S_{j+1}+1]DV_1 + l_j(V_1)DS_{j+1}\ . \quad (45)$$

Note that (43) implies that $l_j'(V_1(t,x))S_{j+1}(t,x)+1\geq 0$ and since $DV_1\leq 0$ and $\phi(a+b)\leq\phi(2a)+\phi(2b)$ we can write:

$$\begin{aligned} DS_j &\leq DV_j + l_j(V_1)DS_{j+1} \\ &\leq -N_j - l_j(V_1)\omega + \phi(\sum_2^{j-1} N_k)\rho(V_1) \\ &\quad + l_j(V_1)\phi(\sum_2^{j} N_k)\rho(V_1) \\ &\leq -N_j - l_j(V_1)\omega + \phi(\sum_2^{j-1} N_k)\rho(V_1) \\ &\quad + l_j(V_1)\phi(2\sum_2^{j-1} N_k)\rho(V_1) \\ &\quad + l_j(V_1)\phi(2N_j)\rho(V_1)\ . \end{aligned} \quad (46)$$

We distinguish between two cases. In the first case we assume $2\phi(2N_j)\rho(V_1)\leq\omega(x)$ and we obtain:

$$\begin{aligned} DS_j &\leq -\tfrac{1}{2}l_j(V_1)\omega + \phi(\sum_2^{j-1} N_k)\rho(V_1) \\ &\quad + l_j(V_1)\phi(2\sum_2^{j-1} N_k)\rho(V_1)\ . \end{aligned} \quad (47)$$

[1]Note that such a function always exists since N_j are uniformly bounded and V_1 satisfies conditions (2) of Definition 2.1. Moreover, note that ζ is not necessarily zero at zero.

In the second case we assume that $2\phi(2N_j)\rho(V_1)\geq\omega(x)$ and then we get:

$$N_j \geq \frac{1}{2}\phi^{-1}\left(\frac{\omega(x)}{2\rho(V_1)}\right) \quad (48)$$

and therefore

$$\begin{aligned} DS_j &\leq -\tfrac{1}{2}\phi^{-1}\left(\tfrac{\omega(x)}{2\rho(V_1)}\right) - l_j(V_1)\omega(x) \\ &\quad + l_j(V_1)\phi(2N_j)\rho(V_1) \\ &\quad + \phi(\sum_2^{j-1} N_k)\rho(V_1) \\ &\quad + l_j(V_1)\phi(2\sum_2^{j-1} N_k)\rho(V_1)\ . \end{aligned} \quad (49)$$

With this choice and using (44) the inequality

$$\begin{aligned} DS_j &\leq -l_j(V_1)\omega(x) + \phi(\sum_2^{j-1} N_k)\rho(V_1) \\ &\quad + l_j(V_1)\phi(2\sum_2^{j-1} N_k)\rho(V_1) \end{aligned} \quad (50)$$

is obtained. Thus, according to (47) and (50), we have, for all (t,x) that:

$$\begin{aligned} DS_j &\leq -\tfrac{1}{2}l_j(V_1)\omega(x) \\ &\quad + [1+l_j(V_1)]\phi(2\sum_2^{j-1} N_k)\rho(V_1)\ . \end{aligned} \quad (51)$$

Using properties of functions l_j, V_1, ϕ, ρ, there exist a positive definite w_j, positive function ρ_j and $\phi_j \in \mathcal{K}_\infty$ such that $w_j(x)\leq\omega(x)$ for all x and $\rho_j(s)\geq\rho(s)$ and $\phi(s)\leq\phi_j(s)$ for all $s\geq 0$ such that for all (t,x) we have:

$$DS_j \;\leq\; -w_j(x) + \phi_j(\sum_2^{j-1} N_k)\rho_j(V_1)\ . \quad (52)$$

Therefore, we have

$$DV_2 \;\leq\; -N_2\ , \quad (53)$$
$$DV_k \;\leq\; -N_k + \phi_j(N_{k-1})\rho_j(V_1)\ , \quad (54)$$

for $k = 3,...,j-1$. The inequalities (52) to (54) have the same form as the inequalities (38) to (40). It is clear that the construction of S_{j-i} can be applied repeatedly for $i = 1,2,\ldots,j-2$. That way, one can construct the function S_2, a positive definite function w_2 and $\sigma_2 \in \mathcal{K}_\infty$ such that (36) and (37) hold. Using Lemma 1.1, we can determine a positive definite function l_1 such that (43) and (44) hold with $j = 1$ and, moreover, $l_1(V_1(t,x))|S_2(t,x)| \leq V_1(t,x)$ for all (t,x). Then, the function

$$S_1(t,x) = 2V_1(t,x) + l_1(V_1(t,x))S_2(t,x) \quad (55)$$

is a strong Lyapunov function for the system (1). Indeed, for all (t,x) we have that

$$\begin{aligned} \alpha_1(|x|) &\leq S_1(t,x) \leq 3\alpha_2(|x|) , \\ DS_1 &\leq -w_1(x) , \end{aligned}$$

where w_1 is some positive definite function and $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ come from Assumption 2.2. Finally, since all derivatives of l_j functions are bounded by $\psi(|x|)$ where $\psi \in \mathcal{K}_\infty$ and also all derivatives of the functions V_i are uniformly bounded, we have that there exists $\tilde{\alpha}_4 \in \mathcal{K}_\infty$ such that for all (t,x)

$$\left| \frac{\partial S_1}{\partial x}(t,x) \right| \leq \tilde{\alpha}_4(|x|) .$$

Note, moreover, that S_1 has the form given in (16) where the functions p_i are given by the following formulae:

$$\begin{aligned} p_1(s) &= 2 + \sum_{k=0}^{j-2} \prod_{t=0}^{k} l_{1+t}(s) , \\ p_k(s) &= \prod_{t=0}^{k-2} l_{1+t}(s) + \prod_{t=0}^{j-1} l_t(s) , \end{aligned} \quad (56)$$

for $k = 2, 3, \ldots, j$ and the functions l_t, $t = 1, 2, \ldots, j$ were constructed above.

VI. Conclusion

We provided several constructions of strong Lyapunov functions for time-varying systems that satisfy generalized conditions of the Matrosov theorem. We expect that our results will have significant implications in several areas of nonlinear control, especially in the areas of tracking and adaptive control. We will address these issues in our future work.

References

[1] D. Aeyels, *Asymptotic stability of nonautonomous systems by Lyapunov's direct method*. Systems and Control Letters, Vol. 25, pp. 273-280, 1995.

[2] D. Angeli, *Input-to-State Stability of PD-controlled robotic systems*. Automatica, 35, pp. 1285-1290, 1999.

[3] V. Jurdjević, J.P. Quinn, *Controllability and stability.* J. Differential Equations, 28 (1978) 381-389.

[4] A. Loria, E. Panteley, D. Popović, A. Teel, *An extension of Mastrosov's theorem with application to stabilization of nonholonomic control systems.* 41th Conf. on Decision and Control, pp. 1528-1533, Las Vegas, Nevada, 2002.

[5] A. Loria, E. Panteley, D. Popović, A. Teel, *A nested Matrosov theorem and persistency of excitation for uniform convergence in stable non-autonomous systems*, to appear in IEEE Trans. Automat. Contr., 2005.

[6] A. Loria, E. Panteley, A. Teel, *Relaxed persistency of excitation for uniform asymptotic stability.* IEEE Trans. Aut. Contr., Vol. 46, pp. 1874-1886, 2001.

[7] H.K. Khalil, *Nonlinear systems*. Prentice Hall: New Jersey, 1996.

[8] J.P. LaSalle, *Some extensions of Lyapunov's second method.* IRE Trans. Circuit Theory, CT-7(4): 520-527, December 1960.

[9] T.C. Lee, *On the equivalence relations of detectability and PE conditions with applications to stability analysis of time-varying systems.* 2003 American Control Conference, Denver, Colorado, pp. 1873-1878, 2003.

[10] V. M. Mastrosov, *On the stability of motion,* J. Appl. Math. Mech., Vol. 26, pp. 1337-1353, 1962.

[11] F. Mazenc, *Interconnected Nonlinear Systems, Local and Global Stabilization.* Systems & Control Letters, **35** (5), 317-323, 1998.

[12] F. Mazenc, *Strict Lyapunov Functions for Time-varying Systems.* Automatica 39, pp. 349-353 (2003).

[13] F. Mazenc, D. Nešić, *Strong Lyapunov functions for systems satisfying the conditions of La Salle.* IEEE Trans. Aut. Contr., Vol. 49, no.6, pp. 1026-1030, (2004).

[14] K.S., Narendra, A.M. Annaswamy, *Persistent excitation in adaptive systems.* Int. Journal of Control, **45**, 127-160, 1987.

Appendix

Lemma 1.1: Let $w_i : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ $i = 1, 2$ be two positive definite functions; $V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ and γ_1, γ_2 of class $\mathcal{K}_\infty$ such that for all (t,x) we have:

$$\gamma_1(|x|) \leq V(t,x) \leq \gamma_2(|x|) . \quad (57)$$

Then, one can construct a real-valued function L of class C^N, where $N \geq 1$ is an integer, such that $L(0) = 0$, $L(s) > 0$ for all $s > 0$ and for all (t,x) we have:

$$L(V(t,x)) \leq w_1(x) \quad (58)$$

$$|L'(V(t,x))| \leq w_2(x) \quad (59)$$

Proof. This easy and technical proof is omitted.