Proceedings of the
44th IEEE Conference on Decision and Control, and
the European Control Conference 2005
Seville, Spain, December 12-15, 2005

Extension of a result by Moreau on stability of leaderless multi-agent systems.

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Abstract— **The paper presents a result which relates connectedness of the interaction graphs in a multi-agent systems with the capability for global convergence to a common equilibrium of the system. In particular we extend a previously known result by Moreau by including the possibility of arbitrary bounded time-delays in the communication channels and relaxing the convexity of the allowed regions for the state transition map of each agent.**

I. Introduction

Recent years have witnessed a growing interest in the study of the dynamical behaviour of the so called multi-agent systems. Roughly speaking these can be thought of as complex dynamical systems composed by a high number of simpler units, the agents. Each of them updates its state according to some rule, whose Input-Output dynamics are typically much simpler and much better understood, and on the basis of the available information coming from the other agents. All of them, though not necessarily identical, share in fact some common feature of interest (say for instance a given output variable) and are coupled together by communication channels. The focus of the current research is precisely on how the global behaviour of the system, (for instance questions concerning the global stability or the overall synchronization) is influenced by the topology of the coupling on one hand (this is an analysis problem in many respects) or the dual question of how to induce a certain desired property of the ensemble based on some form of local coupling for the agents. Problems of this nature arise in many different fields, such as in the theory of coupled oscillators [7], [13], in neural networks [5], in economics or in the manouvering of groups of vehicles [8]. For instance in [9] the so called *rendezvous* problem is considered, namely how to design a local updating rule, based on nearest neighbor interactions, which would ensure convergence of all of the agents to an unspecified common meeting point. Emergence of a global behaviour is therefore a consequence of the local updating rule, without the need for a leader nor of centralized directions being broadcasted.

Despite the common traits, the most powerful results are obtained when specializing to systems of a given simple form. Hereby we take a slightly different approach. The emphasis is on how the topology of interconnections between agents (possibly time-varying) affects the convergence of all agents to a common equilibrium. This analysis will be carried out in the presence of limited transmission speed of the information between the agents. In particular, we propose an extension of the contributions by Moreau [10], [11], mainly in two directions:

- The new setting allows the presence of arbitrary bounded communication delays.
- A central assumption in the results [10], [11], namely that the future evolution of the studied system is constrained to occur in the convex hull of the agents states, is removed.

The first aspect comes as a very natural question both from a practical and a theoretical point of view. Communication delays are in fact ubiquitous in the "real" world and it is well-known their potential destabilizing effect in conjunction with feedback loops, here induced by the graph topology of the communication channels which need not be of a hierarchical type. It is therefore remarkable to see how, at least in the specific set-up we are considering, this destabilizing effect does not take place and the same global behaviour of the multi-agent system in terms of convergence to a common equilibrium follows also in the extended set-up.

The second extension deals with convexity issues; one of the technical tools used in order to enforce a common behaviour in systems whose state takes value in Euclidean space, is to have local evolutions point always inside the convex hull of all variables. This makes life easier in a certain respect but it is an unnatural assumption in more general contexts, for instance when oscillators networks are considered (these are typically modeled as systems evolving on a torus) or systems evolving in partially obstructed Euclidean spaces (for instance on a plane minus a circle). Relaxing convexity is meant as a first step in the quest for stability conditions which can work in more general spaces.

Before going on further, we present the main elements of this construction, developed below. The multi-agent system under study will be described by a *time-dependent graph* $\mathscr{A}(t)$, describing the transfer of information between the agents at time t, and a *set of rules* according to which each agent updates its state at time $t+1$. The definition of the latter is done by the introduction of two types of objects, which we present now (complete definitions are to be found in Section II below).

- A *set-valued map* σ, is defined, which associates to the set of present and past states of the agents a compact set in the state space common to all the agents. This map will play the central role of a *set-valued Lyapunov*

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function for the system.

- It is then necessary to define the rules according to which the agents update their state, given the (possibly delayed) information on the position of the other agents they received. For this, each agent k is attributed a set-valued map e_k which, *given the communication graph* $\mathscr{A}(t)$, defines the set of allowed positions $e_k(\mathscr{A}(t))$. An important point here is that, whatever the information received by each agent, the new positions cannot induce an increase of the set-valued Lyapunov function along the trajectories.

The definition of the new class of multi-agent systems studied here is done and commented in Section II. The stability is studied and the main results are given in Section III. Complete demonstration of the results presented in this note may be found in [1]. Last, it is fair to underline the deep connections of the present work with the results on *partially asynchronous iterative methods* presented in [3, Chapter 7].

Notations: As often as possible, we use the notations introduced by Moreau [10], [11]. Following him, we distinguish between the inclusion, denoted $\subseteq$, and the strict inclusion, denoted $\subset$. The topological interior of a set is denoted $\mathtt{int}$.

We study systems with n agents whose position at time t are written as $x_1(t),\ldots,x_n(t)$ in the finite-dimensional space X. In the setting introduced in Moreau's contributions, the corresponding overall state variable is $x(t) = (x_1(t),\ldots,x_n(t)) \in X^n$. Here, we consider systems with delay smaller than a given integer $h > 0$. In consequence, the complete state variable of the system is $(x_1(t), x_1(t-1),\ldots,x_1(t-h+1),\ldots,x_n(t),\ldots,x_n(t-h+1)) \in X^{hn}$.

We denote $\tilde{x} = (x_1,\ldots,x_{hn})$ an arbitrary element of X^{hn} and, when considering the dynamical system, we write $\tilde{x}_k(t) = (x_k(t), x_k(t-1),\ldots,x_k(t-h+1))$ for all $k \in \mathscr{N} \doteq \{1,\ldots,n\}$ and $\tilde{x}(t) = (\tilde{x}_1(t),\ldots,\tilde{x}_n(t))$. We also use the corresponding decomposition of any element $\tilde{x}$ of X^{hn} as $\tilde{x} = (\tilde{x}_1,\ldots,\tilde{x}_n)$ (which amounts to identify X^{hn} to $(X^h)^n$). When needed, any $\tilde{x}_k \in X^h$ is decomposed according to $\tilde{x}_k = (x_{k,0},\ldots,x_{k,h-1})$, in such a way that for the variables of the dynamical systems under study $x_{k,j}(t) = x_k(t-j)$, $k \in \mathscr{N}$, $j \in \mathscr{H} \doteq \{0,\ldots,h-1\}$. Similarly we denote by $\mathscr{H}\mathscr{N} \doteq \{1,2,\ldots,hn\}$. The previous notation is necessary, in order to distinguish between the delayed and the actual values of the position of the agents. Coherently with the notations introduced above, we sometimes abbreviate $x_{k,0}$ and write simply x_k.

Last, given any $\tilde{x} \in X^{hn}$ we often need to embed it on 2^X, according to the following rule: $\pi(\tilde{x}) \doteq \{x_1, x_2, \ldots x_{nh}\}$. In this way the state of the system is mapped to a finite collection of points in the X space.

II. A CLASS OF MULTI-AGENT DYNAMICAL SYSTEMS

This section is devoted to the presentation of the dynamical system under study. We study here a special class of *nonlinear difference inclusions with delay*, that we write:

$$x_k(t+1) \in e_k(\mathscr{A}(t))(\tilde{x}(t)) \ . \tag{1}$$

Recall that $x_k(t)$ represents the "position" at time t of the agent k. The evolution of the latter depends upon the complete system state $\tilde{x}(t)$ (including delayed components), through the time-varying map $e_k(\mathscr{A}(t))$. For a trajectory of (1), we call *decision set of agent* k at time t the value taken by $e_k(\mathscr{A}(t))(\tilde{x}(t))$. The specificity of the problem lies in these maps: they depend upon the topology of the inter-agent communications, modeled by the *graph* $\mathscr{A}(t)$.

The modeling of the communication network is presented below in Section II-A. The construction of the decision sets inside which, given the communication network, each agent may update the value of its state, is made in Section II-B. Last, we provide some examples in Section II-C.

A. Inter-agent communications modeling

The first ingredient of the construction is the family of *continuous set-valued maps* $e_k(\mathscr{A}) : X^{hn} \rightrightarrows X$ taking on *compact values*, and defined for $k \in \mathscr{N}$ and any directed graph $\mathscr{A}$. The latter will define, according to the position of the other agents, in which subset of X agent k is allowed to choose its future state.

Here, we are concerned by information transfer from the past to the present. In other words, we need to consider graphs in X^{hn} linking some past and/or present values $x_k(t-j)$ of the states of an agent k to another agent l. Consequently, at each time, the communication graph $\mathscr{A}$ is a *weighted, directed multigraph* defined on the set $\mathscr{N}$ of the *nodes,* that is a set of ordered couples of nodes (with possible repetitions), called *arcs*[1]. To each of these arcs is associated a *weight,* chosen in $\mathscr{H}$, to be interpreted as the corresponding information delay[2]. All the considered graphs will contain all the loops of zero weight, corresponding to the ability for each agent to use without delay the knowledge on its own state. The graphs fulfilling all these conditions will be called in the sequel *admissible graphs*.

We shall write $i \overset{j}{\rightsquigarrow}_{\mathscr{A}} k$ when an arc of weight j links in $\mathscr{A}$ the node i to the node k (with $i,k \in \mathscr{N}$, $j \in \mathscr{H}$). A node $k \in \mathscr{N}$ is said to be *connected* to a node $l \in \mathscr{N}$ if there exists a *path* from k to l in the admissible graph $\mathscr{A}$ which respects the orientation of the arcs. Last, given a sequence of admissible graphs $\mathscr{A}(t)$, $t \in \mathbb{N}$, a node $k \in \mathscr{N}$ is said *connected* to a node $l \in \mathscr{N}$ *on an interval* $I \subseteq \mathbb{N}$ if k is connected to l for the graph $\bigcup_{t \in I} \mathscr{A}(t)$.

Figure 1 provides an example of admissible graph. For the graph represented therein, agents 1 and 2 are mutually connected and agent 3 is connected to 1 and 2, but neither 1 nor 2 is connected to 3. Notice that generally speaking there may exist more than one arc between two distinct nodes, and that a node may be connected to itself (via delayed values).

Definition 1: Consider an admissible graph $\mathscr{A}$ and a nonempty subset $\mathscr{L} \subseteq \mathscr{N}$. The set Neighbors$(\mathscr{L},\mathscr{A})$ is the set of those nodes $k \in \mathscr{N} \setminus \mathscr{L}$ for which there is $l \in \mathscr{L}$ such that (at least) one arc from k to l exists. When $\mathscr{L}$ is a

[1] For details on the basic graph-theoretic notions needed here, the reader is referred e.g. to [].

[2] Recall that $\mathscr{N} = \{1,\ldots,n\}$, $\mathscr{H} = \{0,\ldots,h-1\}$, where n is the number of agents and $h-1$ the larger transmission delay.

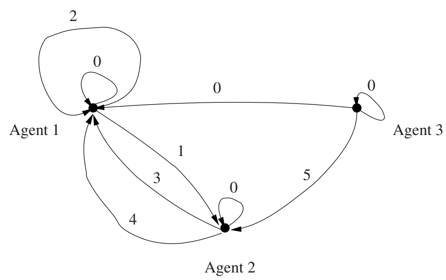

Fig. 1. An example of admissible graph for a system with three agents.

singleton $\{l\}$, the notation Neighbors$(l, \mathscr{A})$ is used instead of Neighbors$(\{l\}, \mathscr{A})$. ■

We impose to the maps e_k the following assumption.

Assumption A: For all $k \in \mathscr{N}$ and all admissible graph $\mathscr{A}$, the set-valued map e_k is continuous and takes on compact values. Moreover,

- $e_k(\mathscr{A})(\tilde{x}) = \{x_k\}$ if $\{x_{i,j} \ : \ i \overset{j}{\sim}_{\mathscr{A}} k\} = \{x_k\}$;
- $e_k(\mathscr{A})(\tilde{x}) \subset \mathtt{ri}\ \sigma\left(\{x_k\} \cup \{x_{i,j} \ : \ i \overset{j}{\sim}_{\mathscr{A}} k\}\right)$ otherwise. ■

The exact meaning and the properties of the set-valued map $\mathtt{ri}\ \sigma$ are the subject of Section II-B. However, we may already make some remarks on the form of the right-hand side of the problem. Clearly, Assumption A implies that the evolution of each agent depends only upon the possibly delayed information received from its neighbors. The case where $\{x_{i,j} \ : \ i \overset{j}{\sim}_{\mathscr{A}} k\} = \{x_k\}$ is realized when either the agent k has no neighbor and the set involved in the formula is empty, or all the (possibly delayed) positions received from the neighboring agents are also equal to the present position x_k of agent k; in this case, no motion is allowed. We shall see below that in the present framework the use by each agent of the present value of its own position is mandatory for stability, see counterexample in Example 6.

B. Construction of the decision sets

The second ingredient necessary for the construction of the dynamical system under study is a *set-valued map* $\sigma : 2^X \rightrightarrows X$, taking on *compact values*. It has a central role in the definition of the dynamics, and it will be shown afterwards (cf. in particular the proof of Theorem 2) that it plays the role of a "set-valued Lyapunov function" for the studied system.

In order to state the properties that σ should fulfil, we have to introduce beforehand some notions. First of all, define $\mathscr{S}$, a set of subsets of X in which σ will be compelled to take on its values, as:

$$\mathscr{S} \doteq \{S \subset X \ : \ S \text{ compact and } \exists \varphi : X \to X, \varphi \text{ bijective}, \varphi, \varphi^{-1} \text{ Lipschitz and } \varphi(S) \text{ convex}\} \ . \quad (2)$$

Important consequences will proceed from the fact that σ takes on values in $\mathscr{S}$, inherited from properties summarized in the following result.

Lemma 1: Let $\mathscr{S}$ be defined by (2).

1) for any $S \in \mathscr{S}$, the function $d_S(x^0, x^1) : S \times S \to [0, +\infty)$ defined as

$$d_S(x^0, x^1) \doteq \inf\Big\{\mathtt{length}(\psi) \ : \ \psi : [0,1] \overset{\text{Lipschitz}}{\longrightarrow} S, \ \psi(0) = x^0, \psi(1) = x^1\Big\}$$

is well-defined and continuous. Define $\mu : \mathscr{S} \to \mathbb{R}^+$:

$$\mu(S) \doteq \max_{x^0, x^1 \in S} d_S(x^0, x^1). \quad (3)$$

Then, for all $S \in \mathscr{S}$,

- $\mu(S) < +\infty$.
- $\mu(S) = 0$ if and only if S is a singleton.
- $\mu(S)$ is at least equal to the (euclidian) diameter of S, and equal to this value if S is convex.
- μ is lower semicontinuous in S, but nowhere continuous.

2) for any $S \in \mathscr{S}$, let φ be as in (2) and

$$\mathtt{ri}(S) \doteq \varphi^{-1}\left(\mathtt{ri}(\varphi(S))\right) \ ,$$

where $\mathtt{ri}(\varphi(S))$ designates the relative interior[3] of the convex set $\varphi(S)$. Then, for all $S \in \mathscr{S}$,

- $\mathtt{ri}(S)$ is independent of the choice of φ.
- $\mathtt{ri}(S) = \emptyset$ if and only if S is a singleton.
- $\mathtt{int}\ S \subseteq \mathtt{ri}\ S \subset S$.
- $\mathtt{ri}(S)$ is the relative interior of S if S is convex. ■

Lemma 1 permits to measure the distance between points of a set $S \in \mathscr{S}$ "along the arcs". It permits to define extended notions of diameter and of relative interior, which coincide with the usual ones for convex subsets of X. By definition, we call "relative boundary" of sets S in $\mathscr{S}$ the following set:

$$\mathtt{r}\partial(S) \doteq S \setminus \mathtt{ri}(S) \ .$$

Also, according to the definition of d_S in Lemma 1, we define, for any subsets S', S'' of a set S in $\mathscr{S}$ the *S-distance from S' to S''* as:

$$d_S(S', S'') \doteq \inf_{x^0 \in S', x^1 \in S''} d_S(x^0, x^1) \ . \quad (4)$$

We now gather the properties that σ must fulfil, and afterwards comment on their meaning and consequences.

Assumption B: The set-valued map $\sigma : 2^X \rightrightarrows X$ is continuous with respect to the topology induced by Hausdorff metric and maps the bounded subsets of X to $\mathscr{S}$. Moreover, the following should hold:

1) $S \subseteq \sigma(S)$ with equality if S is a singleton.
2) $\sigma(S) = \sigma \circ \sigma(S)$ for all $S \in 2^X$.
3) $S' \subseteq S \Rightarrow \sigma(S') \subseteq \sigma(S)$ for all $S, S' \in 2^X$.
4) If S is bounded and not a singleton, for all $x \in S$, there exists $\Sigma_x \subseteq \mathtt{r}\partial\sigma(S)$ such that $\Sigma_x \cap S \neq \emptyset$ and $x \notin \Sigma_x$. Moreover, if $S' \subseteq \sigma(S)$:
   a) if $\mathtt{ri}\ \sigma(S') \cap \Sigma_x \neq \emptyset$, then $S' \subseteq \Sigma_x$ (and in particular, $x \notin S'$).
   b) if $d_{\sigma(S)}(S', \Sigma_x) > 0$, then $\mu(\sigma(S')) < \mu(\sigma(S))$.

[3] i.e. its interior when regarded as a topological subspace of its affine hull.

5) $\mu \circ \sigma$ is continuous. ■

Remark that at this point, the problem under study is fully understandable: our goal is to find stability conditions for systems defined by (1), where the maps e_k verify Assumption A for a given map σ fulfilling Assumption B, and where the meaning of the relative interior ri has been defined previously by Lemma 1.

Important consequences of Assumptions B.1 to B.5 are now discussed. We shall see further in Theorem 1, that Assumptions B.1–B.3 are indeed sufficient to forbid increase along time of the natural set-valued Lyapunov function of the system. The additional Assumptions B.4–B.5 induce the *strict* decrease of the set-valued Lyapunov function (see Theorem 2) . We provide in the following lemma a direct consequence of Assumption B.1.

Lemma 2: Assume Assumption B.1 is fulfilled. Then, for any bounded $S \subset X$, card $S > 1 \Rightarrow \mathtt{ri}\ \sigma(S) \neq \emptyset$ and $\mu(\sigma(S)) > 0$. ■

We now come to the central hypothesis, stated in Assumption B.4. This Assumption applies to arbitrary (but non trivial) groups of agents S, which may comprise indifferently true agents or "virtual" agents, viz. informations relative to the position of a true agent at previous sampling times. More closely, for each agent x, there exists a portion of the boundary of $\sigma(S)$, denoted by Σ_x, whose elements are *irreversibly* attracted outside of it when using information received from any agent not in Σ_x (such as x itself) according to the rule edicted in Assumption A. The second part of Assumption B.4 imposes that such an irreversible escape from Σ_x comes with a *strict decrease* of the diameter of the set-valued Lyapunov function of the system (for *convex* sets $S, S' \subseteq X$, $S' \subseteq S$ implies $\mu(S') \leq \mu(S)$, but this is not true for general sets in $\mathscr{S}$ defined by (2)).

Generally speaking, the set Σ_x, defining a critical part of the relative boundary of $\sigma(S)$ relative to x, looks like an union of "faces" of $\mathtt{r}\partial\sigma(\tilde{x})$ containing an extremity of each geodesic in $\sigma(S)$ originating in x and which are maximal (for the inclusion) among the set of these geodesics. Remark that sets Σ_x, Σ_y associated to different points x, y in S may be equal.

Similarly to what happens within Moreau's setting, one has the following result.

Lemma 3: Assume Assumptions B.1-B.4 be fulfilled. Then, card $S > 1 \;\Rightarrow\;$ card$(S \cap \mathtt{r}\partial\sigma(S)) > 1$. ■

Last, notice that Lemma 1 and the continuity assumption on σ implies that the map $\mu \circ \tilde{V}$ is already lower semicontinuous on X^{hn}. Assumption B.5 thus represents a slightly stronger regularity assumption.

C. Examples

We present here different examples and counter-examples of maps σ fulfilling the properties previously defined.

Example 1 (convex hull): In Moreau's work, $\sigma(S)$ is taken to be the convex hull of S, see Figure 2. One may check easily that Assumptions B.1 to B.5 are all fulfilled. Here, the sets Σ_x involved in Assumption B.4 can be defined as follows:

$$\Sigma_x \doteq \bigcup_{c \in TC_{\sigma(S)}(x), |c|=1} x + \max\{t : x + ct \in \sigma(S)\}c \ ,$$

where $TC_{\sigma(S)}(x)$ denotes the Bouligand contingent cone to the set $\sigma(S)$ at x (otherwise called tangent cone, as $\sigma(S)$ is convex here; see [2, pp. 176–177 and 219] for details). ■

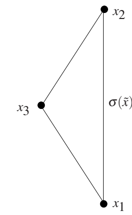

Fig. 2. The convex-hull, Moreau's set-valued Lyapunov function.

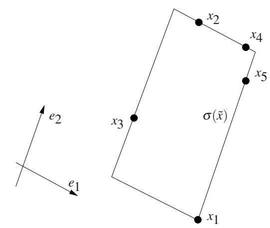

Fig. 3. Illustration of Example 2.

Example 2 (a different convex example): For a given basis e_j, $j = 1, \dots, p$ of X, take

$$\sigma(S) \doteq \left[\min_{x \in S} e_1^T x, \max_{x \in S} e_1^T x\right] \times \cdots \times \left[\min_{x \in S} e_p^T x, \max_{x \in S} e_p^T x_i\right] \ .$$

In this example, the convex hull is applied "componentwise", see Figure 3. Remark that conv$(S) \subseteq \sigma(S)$ for this case, but this relation is not mandatory, see Example 4 below.

In the example depicted on Figure 3, one may check that the choice consisting in taking for $\Sigma_x \doteq \bigcup_{c \in TC_{\sigma(S)}(x), |c|=1} x + \max\{t : x + ct \in \sigma(S)\}c$, fulfills the Assumptions. ■

Example 3 (other convex examples): One may also define $\sigma(S)$ as the smaller set containing S and with boundary parallel to given $p+1$ non-parallel hyperplans (where $X = \mathbb{R}^p$), see Figure 4. More precisely, let $\Sigma = \mathrm{conv}(S)$ and $e_1, \dots, e_{p+1}$ be $(p+1)$ vectors in X such that for some positive $\lambda \in \mathbb{R}^{p+1}$ we have $\sum_j \lambda_j e_j = 0$. The set $\sigma(S)$ is a polytope defined as: $\left\{x \in X \ : \ e_j^T x \leq \max_{x' \in \Sigma} e_j^T x', \ j = 1, \dots, p+1\right\}$, containing the points $x_1, \dots, x_{hn}$. Symmetrically we may define $\sigma(x) = \left\{x \in X \ : \ e_j^T x \geq \min_{x' \in \Sigma} e_j^T x', \ j = 1, \dots, p+1\right\}$. Similarly to what occurs in Example 2, one may take for Σ_x the portion of the boundary obtained by following the vectors coming out from the tangent cone at x all the way to their extreme intersection point with the boundary of $\sigma(S)$, and the Assumptions B.1-B.5 are fulfilled. ■

Remark that the smallest ball or the smallest hypercube containing S does *not* fulfil the requested properties. For instance the smallest circle containing a triangle never contains

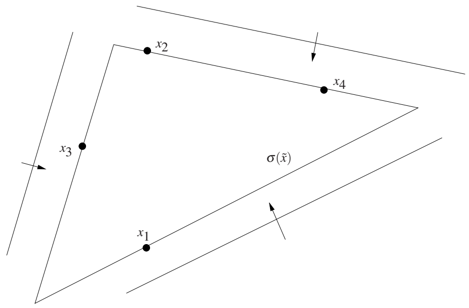

Fig. 4. Other convex examples of set-valued Lyapunov function, see Example 3.

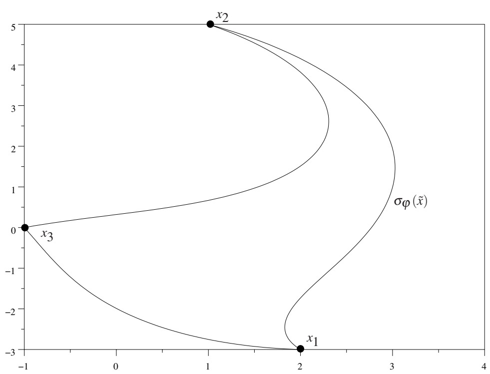

Fig. 5. An example of map σ giving rise to nonconvex sets. Notice that $\mathrm{conv}(S) \not\subseteq \sigma(S)$, and that $\mu(\sigma(S))$ is larger than $\mu(\mathrm{conv}(S))$, the diameter of $\mathrm{conv}(S)$.

the smallest circle containing the shortest of its edges, which violates monotonicity of the map σ.

Example 4 (nonconvex examples): For any bijective transformation $\varphi : X \to X$ which is Lipschitz together with its inverse, one may take

$$\sigma_\varphi(S) \doteq \varphi^{-1}\left(\sigma(\varphi(S))\right) \ ,$$

where σ fulfils all the Assumptions. In general $\sigma_\varphi(S) \not\subseteq \mathrm{conv}(S)$ and is not convex: indeed, this latter property is not essential. Such an example of nonconvex sets is given in Figure 5, obtained for $X = \mathbb{R}^2$, $x_1 = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$, $x_2 = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$, $x_3 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$, $\varphi(x) = \begin{pmatrix} \cos\alpha\|x\|^2 & \sin\alpha\|x\|^2 \\ -\sin\alpha\|x\|^2 & \cos\alpha\|x\|^2 \end{pmatrix} x$, $\alpha = 0.04$, and $\sigma(S) = \mathrm{conv}(S)$.

Notice that, generally speaking, the systems generated along this principle are such that the map φ in (2) is identical for *all* the sets $\sigma(S)$. The sets Σ_x may be obtained as for Example 1, up to transformation by φ. ∎

Example 5 (intersection of decision sets): When σ and σ' fulfil the properties stated above, an interesting issue is to see whether $\sigma \cap \sigma'$ do. One sees easily that Assumptions B.1–B.3 are fulfilled. The validity of B.4 and B.5 depends upon the configuration of the sets Σ_x, Σ'_x corresponding to σ and σ'. In Figure 6 is presented an example where the

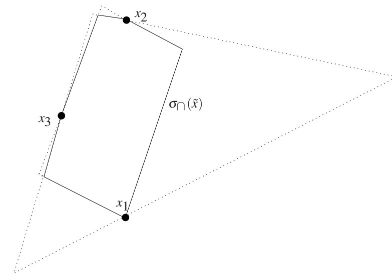

Fig. 6. Map obtained by intersection of the maps from Figures 3 and 4.

resulting map fulfills all the properties. ∎

III. Results

Before stating the results of this paper, we recall the notions under discussion below, see [10], [12]. As in Moreau's papers, we call *equilibrium point* any element of the state space which is the constant value of an *equilibrium solution*.

Definition 2: Let $\mathscr{X}$ be a finite-dimensional Euclidean space and consider a continuous set-valued map $e : \mathbb{N} \times \mathscr{X} \rightrightarrows \mathscr{X}$ taking on closed values, giving rise to the difference inclusion

$$x(t+1) \in e(t, x(t)) \ . \tag{5}$$

Consider a collection of equilibrium solutions of this equation and denote the corresponding set of equilibrium points by Φ: $\varphi \in \Phi$ if and only if $\varphi \in e(t, \varphi)$ for all $t \in \mathbb{N}$.

With respect to the considered collection of equilibrium solutions, the dynamical system is called

1) *stable* if for each $\varphi \in \Phi$, for all $c_2 > 0$ and $t_0 \in \mathbb{N}$, there is $c_1 > 0$ such that every solution ζ of (5) satisfies: if $|\zeta(t_0) - \varphi| < c_1$ then $|\zeta(t) - \varphi| < c_2$, $t \geq t_0$.
2) *bounded* if for each $\varphi \in \Phi$, for all $c_1 > 0$ and $t_0 \in \mathbb{N}$, there is $c_2 > 0$ such that every solution ζ of (5) satisfies: if $|\zeta(t_0) - \varphi| < c_1$ then $|\zeta(t) - \varphi| < c_2$, $t \geq t_0$.
3) *globally attractive* if for each $\varphi_1 \in \Phi$, for all $c_1, c_2 > 0$ and $t_0 \in \mathbb{N}$, there is $T \geq 0$ such that every solution ζ of (5) satisfies: if $|\zeta(t_0) - \varphi_1| < c_1$ then there is $\varphi_2 \in \Phi$ such that $|\zeta(t) - \varphi_2| < c_2$, $t \geq t_0 + T$.
4) *globally asymptotically stable* if it is stable, bounded and globally attractive.

If c_1 (respectively c_2 and T) may be chosen independently of t_0 in Item 1 (respectively Items 2 and 3) then the dynamical system is called uniformly stable (respectively uniformly bounded and uniformly globally attractive) with respect to the considered collection of equilibrium solutions. ∎

Notice that the above notions are uniform with respect to all trajectories of (5).

We now state a first result on boundedness and (simple) stability, analogous to [10, Theorem 2].

Theorem 1: Assume that Assumptions A and B.1–B.3 are fulfilled. Then the discrete-time system (1) is uniformly globally bounded and uniformly globally stable with respect to the collection of equilibrium solutions $x_1(t) \equiv \cdots \equiv x_n(t) \equiv$ constant. ∎

The proof of Theorem 1 [1] is based on the evolution of the following set-valued function $\tilde{V} : X^{hn} \rightrightarrows X$,

$$\tilde{V}(\tilde{x}) \doteq \sigma(\pi(\tilde{x})) \quad (6)$$

along the solutions of (1). The fact that $t \mapsto \tilde{V}(\tilde{x}(t))$ is non-increasing is stated in the following result.

Lemma 4: Let x be a solution of equation (1). Then, for all $t \in \mathbb{N}$,

$$\tilde{V}(\tilde{x}(t+1)) \subseteq \tilde{V}(\tilde{x}(t)) \ . \qquad \blacksquare$$

In view of Lemma 4, one may now have a clearer understanding of the fact that the map σ has a double role: it is necessary to define the flow, but also serves as a set-valued Lyapunov function of the systems. Indeed, Assumption A states that each agent has to remain in the set $\tilde{V}(\tilde{x}(t))$, of which it has only an imperfect knowledge, and does its best to come closer from the other agents it has detected (this is the meaning of the use of the relative interior). In particular, when no new information is received, the only possible choice is to stay at the same place.

As detailed in Section II-B, contrary to σ, the map $\mathtt{ri}\ \sigma$ is not monotone: violation of this rule may occur when $S' \subset S$ and the σ-hulls $\sigma(S), \sigma(S')$ have different topological dimensions as spheres. Up to this subtlety, a consequence of Assumption A is that, in general, *the larger the quantity of information received by agent k from its neighborhood, the largest the set of possible updates it may choose* (see the monotony property in Assumption B.3). Although this may sound paradoxical at first glance, this increase of the decision possibilities is quite natural: it means that supplementary information either leads to make a choice which could have been done otherwise (it is ignored or makes more valuable the decision) or allows to adopt choices which would not have been done otherwise. The "subtlety" comes from the fact that, when the information available to an agent is poor, some decisions are taken which would not have been possible with richer data. For example, the possibility of staying in the same place, which occurs when an agent, say agent 1, is isolated from the other world, disappears when the position of another agent located elsewhere, agent 2, is received. However, the unique choice $\sigma(\{x_1\}) = \{x_1\}$ is then located "on the boundary" of the decision set $\mathtt{ri}\ \sigma(\{x_1, x_2\})$, see Lemma 3.

The key result of the paper is now stated. It provides a *necessary and sufficient stability condition* for system (1), which extends [10, Theorem 3].

Theorem 2: Assume that Assumptions A and B are fulfilled. Then the discrete-time system (1) is uniformly globally attractive with respect to the collection of equilibrium solutions $x_1(t) \equiv \cdots \equiv x_n(t) \equiv$ constant *if and only if* there exists $T \geq 0$ such that for all $t_0 \in \mathbb{N}$ there is a node connected to all other nodes across $[t_0, t_0 + T]$. $\blacksquare$

The argument of the proof of Theorem 2 is based on an abstract stability result on difference inclusions with delay — not reproduced here for sake of space, see [1] —, and on the following estimate [1]: for any $t, t' \in \mathbb{N}$,

$$t' > t + (n-1)^2(h+T) \Rightarrow \mu(\tilde{V}(\tilde{x}(t'))) < \mu(\tilde{V}(\tilde{x}(t))) \ .$$

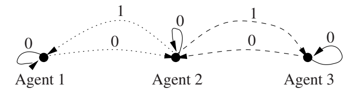

Fig. 7. Graph representing the information flow for Example 6: even (dots) and odd (dash) times.

The uniformity which is meant in the statement of Theorems 1 and 2 is with respect to *time*. One may check from the proofs in [1] that it is also valid with respect to the different trajectories of (1).

Theorem 2 states asymptotic stability for any (finite) values of the delay. Of course, as may be checked elementarily, the values of the latter has a determining impact on the convergence speed of the solution. Quantitative analysis of this issue is scheduled as a next step.

Example 6: The necessity for each agent to use the *undelayed* value of its own position may be seen by the following counter-example, see Figure 7. Here, $n = 3$ and $h = 2$. Agent 2 sends alternativaly to agent 1 and 3 the value of its position at the previous instant, and receives the present value of their position. Assume the agents use at time t the value of their position at time $t-1$ to elaborate the update applied at time $t+1$. Clearly, for the corresponding graph, the agent 2 is connected to all other agents across any interval $[t, t+1]$. However, provided that agents 1 and 3 are initially located at different points, the positions of agent 2 at even and odd times tend in general toward two different values. As indicated by the existence of periodic motion, the strict decrease of the map $t \mapsto \mu(\tilde{V}(\tilde{x}(t)))$ may fail. $\blacksquare$

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