Proceedings of the
44th IEEE Conference on Decision and Control, and
the European Control Conference 2005
Seville, Spain, December 12-15, 2005

On Robust Output Feedback Control For Polytopic Systems

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Abstract— **Robust dynamic output feedback design is an open problem, computationally speaking, since its determination asks for the solution of nonlinear matrix inequalities, namely bilinear ones. This is particularly the case, for polytopic uncertainty. Here a new sufficient condition is proposed by the use of bounds and scaling for completion of squares. The usefulness of the provided conditions stands in the fact that its solution can be performed using the Frank-Wolfe algorithm which runs in only one shot. The control design of an inverted pendulum with uncertain friction coefficients illustrates the theory.**

I. INTRODUCTION

Robust control has grown, in the last past years, as one of the most important area in modern control design since the pioneering works by Zames [8], [9] and Doyle [2], [3], [4], among many others. Now the progresses in the domain have, to a large extent, reached the initial purpose of filling the gap between the advantages of the frequency domain approaches in terms of robustness and engineering understanding and the time domain ones in terms of mathematical description and computational efficiency. Specifically, the Linear Matrix Inequalities - LMI, in semi-definite programming is recognized to have a big interest because of its abilities to describe non trivial control design problems integrating various specifications such as robustness, structural and performances constraints, as well its suitability for efficient numerical processing through various solvers available to date (see [1] and the references therein). Although there is still room for efficiency improvement the actual solvers give a satisfactory answer for practical design on real processes. There are, however, some control problems which are still open for the determination of solutions enabling the direct use of LMI for numerical determination. This is the case of the robust dynamic output control problem where the plant is subject to structured uncertainty.

In this paper polytopic uncertainty is considered which is recognized as one of the most difficult structured uncertainty since, in this case, not even any nominal system is formally defined in the system description. In the case of norm bounded uncertainty which is equivalent in the one block case to the H_∞ control, the dynamic output controller is clearly derived as a filter like controller based on the nominal model [3], [5].

This research was supported by grants from "Conselho Nacional de Desenvolvimento Científico e Tecnológico - CNPq" and "Fundação de Amparo à Pesquisa do Estado de São Paulo- Fapesp ", Brazil.

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In the polytopic uncertain case, the problem has been addressed in a preceding work where a bilinear matrix inequality was found as a necessary and sufficient condition for quadratic stability and used for H_2 and H_∞ guaranteed cost control design. An LMI processing was performed in a cross decomposition algorithm working by successive projection and relaxation techniques [7]. In the present paper, a new sufficient condition is given in terms of LMI whose sufficiency is reduced by the use of suitable bounding and scaling determined from an adequate completion of squares, leading to a Frank-Wolfe algorithm for optimization which generates a sequence of feasible LMI problems with a remarkable speed of convergence.

The organization of the paper is as follows. In the next section the problem is stated together with a summary of the preceding results pointing out the difficulty of its numeric resolution. Section 3 is devoted to the main result, that is the LMI sufficient condition for robust dynamic output feedback control design in face of polytopic uncertainty. A discussion is developed on the use and the processing of the proposed bounds and scaling in order to reduce the conservativeness of the approach. The H_2 norm guaranteed cost problem is treated. The paper ends with a numerical example and concluding remarks about the interest for the reported results.

The notation used throughout is standard. Capital letters denote matrices, small letters denote vectors and small Greek letters denote scalars. For matrices or vectors ($'$) indicates transpose. For symmetric matrices, $X > 0$ (≥ 0) indicates that X is positive definite (nonnegative definite). The set of real numbers is denoted by $\mathbb{R}$. For square matrices $\mathrm{trace}(X)$ denotes the trace function of X being equal to the sum of its eigenvalues and, for the sake of easing the notation of partitioned symmetric matrices, the symbol ($\bullet$) denotes generically each of its symmetric blocks.

II. PROBLEM FORMULATION

Let the uncertain time invariant linear system be given by the following state space representation

$$\dot{x}(t) = Ax(t) + B_1 w(t) + B_2 u(t) \quad (1)$$

$$z(t) = C_1 x(t) + D_1 u(t) \quad (2)$$

$$y(t) = C_2 x(t) + D_2 w(t) \quad (3)$$

where as usual, $x(\cdot) \in \mathbb{R}^n$ is the state, $u(\cdot) \in \mathbb{R}^m$ the control, $w(t) \in \mathbb{R}^r$ the exogenous perturbation, $z(t) \in \mathbb{R}^p$ the controlled output and $y(t) \in \mathbb{R}^q$ the measured output. To ease the presentation it is assumed that only the pair of matrices (A, B_2) is uncertain and belongs to the polytopic

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domain

$$(A, B_2) \in \mathbb{D} := \text{co}\{(A_i, B_{2i})\ ,\ \ i = 1, \cdots, N\} \qquad (4)$$

where $\text{co}\{\cdot\}$ denotes the convex hull of the indicated vertices. More general uncertainty models including uncertainty on the output matric C_2 can also be handled with slight modifications. The control is given by a full order strictly proper dynamic output feedback controller

$$\dot{x}_c(t) = A_c x_c(t) + B_c y(t) \qquad (5)$$
$$u(t) = C_c x_c(t) \qquad (6)$$

where accordingly $x_c(\cdot) \in \mathbb{R}^n$ and the real matrices A_c, B_c, C_c are of appropriate dimensions. Here, too, strictly proper controllers have been considered for simplicity. The case $D_c \neq 0$ can also be handled with slight modifications. The conventional H_2 guaranteed cost control is worked out. The control, that is, the design matrices (A_c, B_c, C_c), leading to the controller transfer function $H_c(s) = C_c(sI - A_c)^{-1}B_c$, are to be determined such as to minimize an upper bound of the squared H_2 norm between the output $z(t)$ and the input $w(t)$ over the uncertainty set, namely

$$\min\{\mu\ :\ \|H_{zw}(s)\|_2^2 \leq \mu\ ,\ \ \forall (A, B_2) \in \mathbb{D}\} \qquad (7)$$

Connecting the controller (5-6) to the system (1-3) the H_2 norm of the transfer function under consideration is readily given by[1]

$$\min_{\mathcal{P}>0}\{\text{trace}(\mathcal{B}'\mathcal{P}\mathcal{B})\ :\ \mathcal{A}'\mathcal{P} + \mathcal{P}\mathcal{A} + \mathcal{C}'\mathcal{C} < 0\} \qquad (8)$$

where

$$\mathcal{A} := \begin{bmatrix} A & B_2C_c \\ B_cC_2 & A_c \end{bmatrix} \qquad (9)$$
$$\mathcal{B} := \begin{bmatrix} B_1 \\ B_cD_2 \end{bmatrix} \qquad (10)$$
$$\mathcal{C} := \begin{bmatrix} C_1 & D_1C_c \end{bmatrix} \qquad (11)$$

Now, using the uncertainty domain (4) together with (9) - (11) it is seen that the closed loop matrix $\mathcal{A}$ depends on the unknown parameters in such a way that $\mathcal{A} \in \mathcal{D} := \text{co}\{\mathcal{A}_i\ ,\ i = 1, \cdots, N\}$ where the vertices $\mathcal{A}_i$ are determined from those of $\mathbb{D}$. Hence, defining the following $N+1$ matrix inequalities

$$\begin{bmatrix} W & \mathcal{B}'\mathcal{P} \\ \bullet & \mathcal{P} \end{bmatrix} > 0 \qquad (12)$$

and

$$\begin{bmatrix} \mathcal{A}_i'\mathcal{P} + \mathcal{P}\mathcal{A}_i & \mathcal{C}' \\ \bullet & -I \end{bmatrix} < 0 \qquad (13)$$

for all $i = 1, \cdots, N$. The guaranteed cost control problem (7) can be rewritten in the final form

$$\min\{\text{trace}(W)\ :\ (12) - (13)\} \qquad (14)$$

being apparent that whenever a given stabilizing output feedback controller is fixed, the associated guaranteed H_2 cost is readily calculated from the solution of (14) with respect to matrices W and $\mathcal{P}$. In this case (14) becomes a convex problem expressed in terms of LMIs. The situation is much more complicated when the controller matrices (A_c, B_c, C_c) are included in the set of variables. In this case, problem (14) is non convex and the determination of a solution still remains an open problem. Some attempts have been already done in order to solve this type of optimization problem (see [7] and the references therein). They generally work by successive approximation applying relaxation techniques involving LMIs at each iteration. Although, in many cases, convergence may be observed it is not mathematically proved. The cross decomposition algorithm given in [7] can be cast in this class although it has been adapted to the problem above in such a way that at each iteration a maximum number of variables is used so that there are common overlapping terms, a fact which is non classical in relaxation techniques. The main goal of this paper is to provide a suboptimal solution to the guaranteed cost control problem (14), determined in only one shot, avoiding thus convergence difficulties. The quality of the proposed suboptimal solution will be illustrated by means of a numerical example.

III. MAIN RESULT

In this section, before we give the main result of this paper, we need to introduce the following partitions of matrices $\mathcal{P}$, $\mathcal{P}^{-1}$ and $\mathcal{T}$ used towards the linearization of constraints appearing in problem (14). Let

$$\mathcal{P} := \begin{bmatrix} X & U \\ U' & \hat{X} \end{bmatrix}$$
$$\mathcal{P}^{-1} := \begin{bmatrix} Y & V \\ V' & \hat{Y} \end{bmatrix} \qquad (15)$$
$$\mathcal{T} := \begin{bmatrix} Y & I \\ V' & 0 \end{bmatrix}$$

where all matrix blocks are square $n \times n$ real matrices, $X, \hat{X}, Y, \hat{Y}$ are symmetric and U, V are non singular. This last assumption is introduced with no loss of generality.

Theorem 1: Assume that there exist a scalar γ, symmetric matrices W, X, Y and matrices F, L, G, M of compatible dimensions satisfying the following $N+1$ linear matrix inequalities

$$\begin{bmatrix} W & \bullet & \bullet \\ XB_1 + FD_2 & X & \bullet \\ B_1 & I & Y \end{bmatrix} > 0 \qquad (16)$$

$$\begin{bmatrix} \mathcal{L}_i + \mathcal{L}_i' & \bullet & \bullet & \bullet & \bullet \\ A_i' + M' & \mathcal{H}_i + \mathcal{H}_i' & \bullet & \bullet & \bullet \\ C_1Y + D_1L & C_1 & -\gamma I & \bullet & \bullet \\ 0 & X & 0 & -I & \bullet \\ \mathcal{L}_i + G & 0 & 0 & 0 & -I \end{bmatrix} < 0 \qquad (17)$$

where $\mathcal{L}_i := A_iY + B_{2i}L$ and $\mathcal{H}_i := XA_i + FC_2$ for all $i = 1, \cdots, N$. The full order dynamic output feedback controller

[1]This problem should be stated with inf instead of min. All feasible sets of problems expressed in terms of LMIs must be considered closed from the interior within a precision defined by the user.

with state space representation defined by matrices

$$\begin{aligned} A_c &= U^{-1}(M' + XG - FC_2Y)V'^{-1} \\ B_c &= U^{-1}F \\ C_c &= LV'^{-1} \end{aligned} \qquad (18)$$

with U and V satisfying $XY + UV' = I$ is such that $\|H_{zw}(s)\|_2^2 < \gamma \mathrm{trace}(W)$ for all pairs $(A, B_2) \in \mathbb{D}$.

Proof: For each $i = 1, \cdots, N$ fixed, it is readily verified that the inequality

$$\begin{bmatrix} -(\mathcal{L}_i + G)'(\mathcal{L}_i + G) & \bullet & \bullet \\ \mathcal{L}_i + G & -I & \bullet \\ 0 & 0 & 0 \end{bmatrix} \leq 0 \qquad (19)$$

holds. Hence, performing the Schur complement with respect to the two last columns and rows of (17) and adding the result to (19) multiplied both sides by the symmetric matrix $\mathrm{diag}(I, X, I)$, we get

$$\begin{bmatrix} \mathcal{L}_i + \mathcal{L}_i' & \bullet & \bullet \\ A_i' + M' + X(\mathcal{L}_i + G) & \mathcal{H}_i + \mathcal{H}_i' & \bullet \\ C_1Y + D_1L & C_1 & -\gamma I \end{bmatrix} < 0 \quad (20)$$

which together with (15) and (18) can be rewritten in the form

$$\begin{bmatrix} \mathcal{T}'(\mathcal{A}_i'\mathcal{P} + \mathcal{P}\mathcal{A}_i)\mathcal{T} & \bullet \\ \mathcal{C}\mathcal{T} & -\gamma I \end{bmatrix} < 0 \qquad (21)$$

for each $i = 1, \cdots, N$ allowing the conclusion that the Lyapunov inequality associated to the closed loop system $\mathcal{A}'\mathcal{P} + \mathcal{P}\mathcal{A} + \gamma^{-1}\mathcal{C}'\mathcal{C} < 0$ holds for all $\mathcal{A} \in \mathcal{D}$. Once again, using (15) it is seen that inequality (16) is equivalent to $W > \mathcal{B}'\mathcal{P}\mathcal{B}$ and $\mathcal{P} > 0$ yielding

$$\begin{aligned} \|H_{zw}(s)\|_2^2 &= \mathrm{trace}\left(\mathcal{B}' \int_0^\infty e^{\mathcal{A}'t}\mathcal{C}'\mathcal{C}e^{\mathcal{A}t}dt\ \mathcal{B}\right) \\ &< \gamma\ \mathrm{trace}(\mathcal{B}'\mathcal{P}\mathcal{B}) \\ &< \gamma\ \mathrm{trace}(W)\ ,\quad \forall\ \mathcal{A} \in \mathcal{D} \end{aligned} \qquad (22)$$

which proves the proposed theorem. ■

An important feature of the guaranteed cost provided by Theorem 1 is that it is determined from any feasible solution of a set of $N + 1$ LMIs. Once a feasible solution is determined, the associated guaranteed cost is given by the nonlinear function $\mu(\gamma, W) := \gamma\ \mathrm{trace}(W)$ allowing the conclusion that the robust full order dynamic output feedback controller associated to the guaranteed cost problem (7) is given by (18) where the involved matrices variables are provided by

$$\min_{\gamma,W,X,F,Y,L,G,M} \{\mu(\gamma, W)\ :\ (16) - (17)\} \qquad (23)$$

The solution of this nonconvex design problem will be addressed in the sequel. For the moment, we want to stress that the variable $\gamma > 0$ appearing linearly in the linear matrix inequalities (17) is of capital importance to get the next result which states that in the case of known systems characterized by $N = 1$ the solution of problem (23) provides the H_2 optimal full order output feedback controller.

Corollary 1: Assume $N = 1$. The optimal solution of the guaranteed cost problem (23) provides the H_2 optimal full order output feedback controller.

Proof: Adopting, with the same notation, the change of variables

$$\begin{aligned} (\gamma W, \gamma X, \gamma F) &\rightarrow (W, X, F) \\ (\gamma^{-1}Y, \gamma^{-1}L, \gamma^{-1}G) &\rightarrow (Y, L, G) \\ M &\rightarrow M \end{aligned} \qquad (24)$$

and noticing that as far as the feasibility of (17) is concerned we can set $G = -A_iY - B_{2i}L$ without introducing any conservatism, then the guaranteed cost provided by Theorem 1 satisfies $\|H_{zw}(s)\|_2^2 < \mathrm{trace}(W)$ where the involved variables verifies (16) and

$$\begin{bmatrix} \mathcal{L}_i + \mathcal{L}_i' & \bullet & \bullet \\ A_i' + M' & \mathcal{H}_i + \mathcal{H}_i' + \gamma^{-1}X^2 & \bullet \\ C_1Y + D_1L & C_1 & -I \end{bmatrix} < 0 \quad (25)$$

For $\gamma \rightarrow +\infty$ it is seen that the linear matrix inequalities (16) and (25) together with (18) provide the H_2 optimal full order output feedback controller [7]. ■

As a consequence of Corollary 1, without uncertainty ($N = 1$), the H_2 norm optimal solution is recovered by the present result. This is of course the minimum to be asked to a constructive approach since it guarantees that there exist a neighborhood around the nominal model in the parameter space where the inequalities introduced in the design step correspond to feasible constraints. Furthermore, it is interesting to put in evidence that the linear matrix inequalities (17) provided by Theorem 1 have been determined from the observation that the bilinear inequalities (21) can be generated by completing the squares to get quadratic constraints with opposite sign to that of the constraints (19), for each $i = 1, \cdots, N$.

The problem has been developed for C_2 constant and $(A, B_2) \in \mathbb{D} := \mathrm{co}\{(A_i, B_{2i})\ ,\ \ i = 1, \cdots, N\}$. It is almost straightforward to extend the result to the case when B_2 is constant and $(A, C_2) \in \mathbb{D} := \mathrm{co}\{(A_i, C_{2i})\ ,\ \ i = 1, \cdots, N\}$, as formally presented in the next theorem.

Theorem 2: Assume that there exist a scalar γ, symmetric matrices W, X, Y and matrices F, L, G, M of compatible dimensions satisfying the following $N + 1$ linear matrix inequalities

$$\begin{bmatrix} W & \bullet & \bullet \\ XB_1 + FD_2 & X & \bullet \\ B_1 & I & Y \end{bmatrix} > 0 \qquad (26)$$

$$\begin{bmatrix} \mathcal{L}_i + \mathcal{L}_i' & \bullet & \bullet & \bullet & \bullet \\ A_i' + M' & \mathcal{H}_i + \mathcal{H}_i' & \bullet & \bullet & \bullet \\ C_1Y + D_1L & C_1 & -\gamma I & \bullet & \bullet \\ Y & 0 & 0 & -I & \bullet \\ 0 & \mathcal{H}_i' + G' & 0 & 0 & -I \end{bmatrix} < 0 \quad (27)$$

where $\mathcal{L}_i := A_iY + B_2L$ and $\mathcal{H}_i := XA_i + FC_{2i}$ for all $i = 1, \cdots, N$. The full order dynamic output feedback controller with state space representation defined by matrices

$$\begin{aligned} A_c &= U^{-1}(M' + GY - XB_2L)V'^{-1} \\ B_c &= U^{-1}F \\ C_c &= LV'^{-1} \end{aligned} \tag{28}$$

with U and V satisfying $XY + UV' = I$ is such that $\|H_{zw}(s)\|_2^2 < \gamma\text{trace}(W)$ for all pairs $(A, C_2) \in \mathbb{D}$.

Proof: Follows the same pattern as the proof of Theorem 1, being thus omitted. ■

The same can be done when polytopic uncertainty is present in all matrices (A, B_2, C_2). As well, the discrete time case is solved in much the same way with only slight differences in the matrix inequalities.

It has to be stressed that the determination of a dynamic output feedback for uncertain dynamical systems is indeed a difficult problem. Whatever the type of linear time invariant uncertainty (polytopic or norm bounded), in the quadratic approach, where a single Lyapunov matrix is used for stability check, necessary and sufficient conditions are written in terms of bilinear matrix inequalities. Up to now, there is no numerical algorithm with guaranteed convergence able to provide a feasible solution to such inequalities. In the case of norm bounded uncertainty problem (equivalent to the H_∞ one) bilinearity in the matrix inequalities comes from the fact that they imply a multiplier matrix R together with the inverse. The classical choice $R = \gamma I$ with one directional search with respect to the scalar $\gamma > 0$ may lead to quite conservative results. In the polytopic uncertainty case, the matrix inequalities associated to each vertex of the uncertain domain are also bilinear with respect to the unknown matrix variables. By over bounding some terms, sufficient LMI conditions have been obtained. Up to now, it is difficult to make a fair comparison between the two uncertainty formulations (there are cases where both can be used to describe the same uncertainty domain).

Finally, it is important to notice that the criterion to be handled is nonlinear but subjected only to LMI constraints. A possible way to solve the associated guaranteed cost problem (23) is to perform a one dimensional search along the scalar γ axis. This is clearly time consuming and provides a suboptimal solution whose accuracy depends on the discretization step. It is possible to avoid this one line search by choosing a log-type criterium as discussed in the next section.

IV. Numerical Issues

In order to keep free the variable γ in LMIs (17), we propose to solve the problem (23) by means of the Frank-Wolfe algorithm which can take advantage of the fact that its feasibility set is convex. Since the objective function is positive then, with no loss of generality it can be replaced by its natural logarithm yielding a concave function such that

$$\ln\, \mu(\gamma, W) \le \ln\, \mu(\gamma_k, W_k) - 2 + \theta_k(\gamma, W) \tag{29}$$

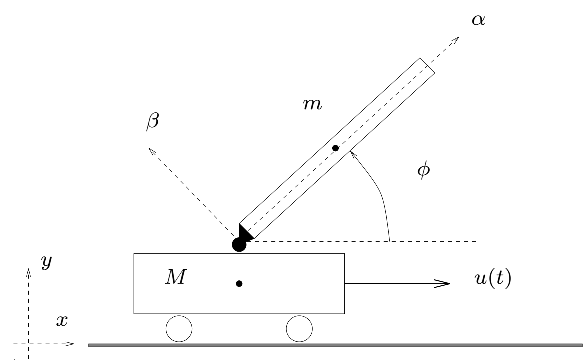

Fig. 1. Inverted pendulum

holds for all pairs $(\gamma > 0, W > 0)$ and $(\gamma_k > 0, W_k > 0)$, where

$$\theta_k(\gamma, W) := \frac{\gamma}{\gamma_k} + \frac{\text{trace}(W)}{\text{trace}(W_k)} \tag{30}$$

The Frank-Wolfe algorithm is well adapted to deal with this particular class of problem since, due to the concavity of the objective function, the step size in the descent direction does not need to be determined. Indeed, it can be shown that the convex programming problems defined for all $k = 0, 1, \cdots$, with linear objective function

$$(\gamma_{k+1}, W_{k+1}) = \arg\min\{\theta_k(\gamma, W) \ : \ (16) - (17)\} \tag{31}$$

generate a sequence that converges to a solution (possibly local) of the problem under consideration (23) and, at the same time, enables us to estimate an actual convergence gap which gives a measure of the distance to the optimal solution. Indeed, using (29) we have that

$$\begin{aligned} \mu(\gamma_{k+1}, W_{k+1}) \quad \le \quad & \mu(\gamma_k, W_k) \\ & \times\, \exp(\theta_k(\gamma_{k+1}, W_{k+1}) - 2) \end{aligned} \tag{32}$$

from which it is seen that its right hand side is an upper bound of the current cost value attained by the feasible solution provided by (31) at iteration $k + 1$ for all $k = 0, 1, \cdots$. In several examples solved, it has been verified that the proposed algorithm performed well and provided a solution in a few number of iterations. In addition, for $N = 1$ in all these cases the global optimum has always been attained.

V. Example

Figure 1 shows an inverted pendulum mounted on a small car moving horizontally due the action of an external force $u(t)$. The inverted pendulum is constituted by a bar with uniformly distributed mass. The goal is to determine the control action $u(t)$ in order to bring the pendulum to the vertical position $\theta = \phi - \pi/2 = 0$ from any initial small deviation. Assuming that the friction coefficient between the air and the car f_c and the air and the bar f_b are not exactly known but belong to the box $(f_c, f_b) = [0.15,\ 0.25] \times [0.15,\ 0.25]$, following [6] the linearized model can the written as (1)-(3)

where matrix $A = E^{-1}A_f$ is determined from

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3/2 & -1/4 \\ 0 & 0 & -1/4 & 1/6 \end{bmatrix}$$

$$A_f = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -(f_c + f_b) & f_b/2 \\ 0 & 5/2 & f_b/2 & -f_b/3 \end{bmatrix}$$

matrix $B_2 = E^{-1}B_{20}$ follows from

$$B_{20} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

and the remaining matrices are

$$B_1 = \begin{bmatrix} 0 & 0 \\ \pi/6 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

and

$$C_2 = \begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix}, \ D_2 = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

which indicates that the symmetric horizontal displacement of the free end of the pendulum with respect to the vertical is the measured variable, that is $x_{meas} = x - \ell\cos(\phi) \approx x + \ell\theta$ where $\ell = 1$ is the length of the bar. Moreover, matrix B_1 makes clear that the impulsive external perturbation is equivalent to an initial condition characterized by $\theta(0) = \pi/6$ and $\dot{\theta}(0) = 0$ with the car being at rest on the origin. Finally, the controlled variable is defined by matrices

$$C_1 = \begin{bmatrix} 20 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \ D_1 = \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix}$$

It is important to observe that the four vertices of the uncertain system under consideration are open loop unstable. First, we solved problem (23) associated to the nominal system with parameters $(f_c, f_b) = (0.20,\ 0.20)$. The Frank-Wolfe algorithm given in (31) reached the global optimal controller (5)-(6) whose transfer function is given by

$$H_{cn}(s) = -0.9273 \cdot 10^3 \frac{(s + 5.338)(s + 0.2996)}{(s - 4.606)(s + 2.83)} \times \frac{(s - 0.0857)}{(s^2 + 21.41s + 277.8)} \quad (33)$$

This is an immediate consequence of Corollary 1 which states that for the nominal system, the optimal solution of the guaranteed cost problem (23) provides the H_2 optimal full order output feedback controller.

The same algorithm has been used to determine the associated robust output feedback controller corresponding to the parameter uncertainty belonging to the box $(f_c, f_b) = [0.15,\ 0.25] \times [0.15,\ 0.25]$. The behavior of the proposed algorithm (31) is shown in Figure 2. Only eight iterations

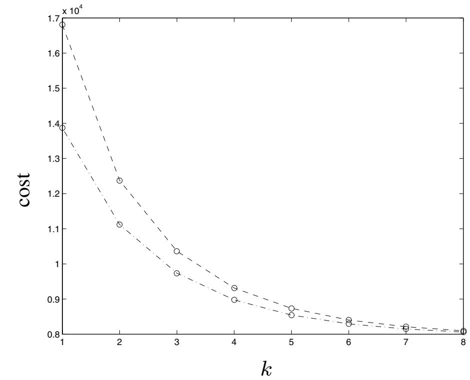

Fig. 2. Frank-Wolfe iterations

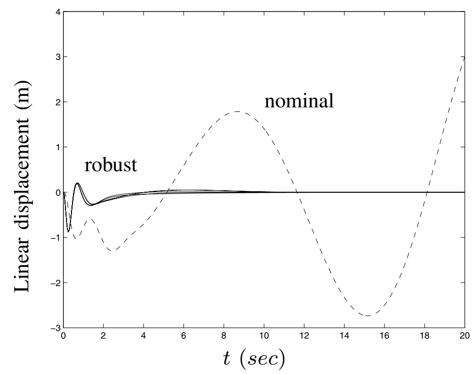

Fig. 3. Time simulation - car displacement

have been necessary to reach a local optimal solution which provided the controller

$$H_c(s) = -1.0264 \cdot 10^4 \frac{(s + 4.524)(s + 0.3021)}{(s - 12.08)(s + 2.61)} \times \frac{(s - 0.255)}{(s^2 + 53.32s + 1327)} \quad (34)$$

It is interesting to observe that both controllers $H_{cn}(s)$ and $H_c(s)$ are unstable. However, as it it is clearly verified in Figure 3 (dashed line) the nominal one is not robust since the closed loop system is unstable for small friction coefficients, namely $(f_c, f_b) = (0.15,\ 0.15)$. On the other hand, Figure 3 and 4 show that the performance of the robust controller is quite good (solid lines) for all parameters (f_c, f_b) belonging to the uncertainty box. Even under friction uncertainty, the robust control is able to bring the pendulum to the vertical position while simultaneously the car is positioned at the origin. The simulation also shows that the system under consideration is highly sensitive to variations on the friction coefficients. Even though, we have verified numerically that problem (23) was feasible for the box $(f_c, f_b) = [0.10,\ 0.30] \times [0.10,\ 0.30]$ which corresponds to parameter uncertainty magnitude of 50% around each nominal value.

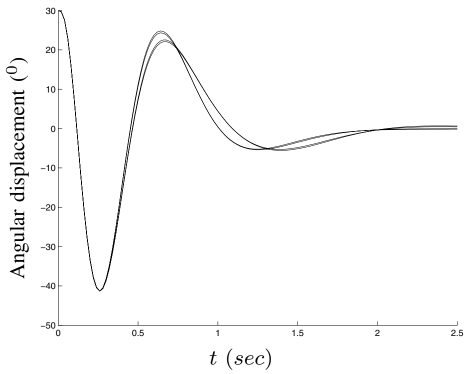

Fig. 4. Time simulation - bar displacement

VI. Conclusions

Partial information dynamic output feedback is a complex problem for systems with model uncertainty, especially when the uncertainty is structured as for the polytopic case. Generally, its solution is written through bilinear matrix inequalities which can be processed by relaxation type techniques but with no convergence guarantee. We proposed for the H_2 guaranteed cost problem a new formulation based on a LMI sufficient condition, whose conservatism is tackled by introducing two degree of freedom : a matrix variable which, in some sense, tries to attenuate the absence of a nominal model and a scalar multiplier of classical use in robust control determination. The development of a Frank-Wolfe algorithm enables to find the solution in one run and reveals itself very efficient in terms of convergence and speed. A numerical example has been given for performance illustration.

References

[1] S. P. Boyd, L. El Ghaoui, E. Feron and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*, SIAM, Philadelphia, 1994.

[2] Doyle, J. C., and Stein G., "Multivarible feedback design - Concepts for a classical modern synthesis", *IEEE Trans. Automat. Contr.*, vol. 26, No 1, pp. 4-16, 1981.

[3] Doyle, J. C., Glover K., Khargonekar, P. P., and Francis, B. A., "State space solutions to standard H_2 and H_∞ control problems", *IEEE Trans. Automat. Contr.*, vol. 34, No 8, pp. 831-847, 1989.

[4] Doyle, J. C., Zhou, K. M., Glover K., and Bodenheimer, B., "Mixed H_2 and H_∞ performance objectives - optimal control", *IEEE Trans. Automat. Contr.*, vol. 39, No 8, pp. 1575-1587, 1994.

[5] Gahinet, P., and Apkarian, P., "A linear matrix inequality approach to H_∞ control", *Inter Jour of Robust and Nonlinear Contr*, vol. 4, No 4, pp. 421-448, 1994.

[6] J. C. Geromel, e Alvaro G. B. Palhares, *Análise Linear de Sistemas Dinâmicos : Teoria, Ensaios Práticos e Exercícios* (in portuguese), Editora Edgard Blucher LTDA, São Paulo, Brazil, 2004.

[7] Geromel, J. C., Bernussou, J., and de Oliveira, M. C., "H-2-norm optimization with constrained dynamic output feedback controllers: Decentralized and reliable control", *IEEE Trans. Automat. Contr.*, vol. 44, No 7, pp. 1449-1454, 1999.

[8] Zames, G., "Feedback and optimal sensitivity - model reference transformations, multiplicative seminorms and approximate inverses", *IEEE Trans. Automat. Contr.*, vol. 26, No 2, pp. 301-320, 1981.

[9] Zames, G., and Francis, B. A., "Feedback, minimax sensitivity and optimal robustness", *IEEE Trans. Automat. Contr.*, vol. 28, No 5, pp. 585-601, 1983.