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A Lyapunov-Krasovskii Methodology for ISS of Time-Delay Systems

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***Abstract*— This paper presents a Lyapunov-Krasovskii methodology for studying the input-to-state stability of nonlinear time-delay systems. The methodology is feasible by the use, for instance, of the M_2 norm (that is the norm induced by the inner product in the Hilbert space known in literature as M_2, or Z) in the space of continuous functions, and by the use of functionals which have a suitable (simple) integral term with strictly increasing kernel. The proposed results can be seen as a preliminary step towards extending some existing stability criteria to nonlinear time-delay systems with disturbance inputs.**

Index Terms: **Input-to-State Stability (ISS), Functional Differential Equations, Nonlinear Time-Delay Systems, Lyapunov-Krasovskii Theorem.**

I. INTRODUCTION

In the literature of time-delay systems, Lyapunov-Krasovskii theorems have played a role of paramount importance for both the input-output stability (which considers zero-state response) and the asymptotic stability (which considers zero-input response) (see, e.g., Proposition 8.3, pp. 288 in [7]).

In the seminal paper [19], the notion of input-to-state stability [16] has been generalized to nonlinear time delay systems. Sufficient conditions are stated in the setting of Lyapunov-Razumikhin theorems to yield the input-to-state stability of nonlinear time delay systems. It is well known, as far as the stability of time delay systems is concerned, that the Razumikhin method can be considered as a particular case of the method of Lyapunov-Krasovskii functionals (see Section 4.8, pp. 254 in [11]). The primary objective of our paper is to address input-to-state stability from a perspective of Lyapunov-Krasovskii functionals for time-delay systems.

As seen from the tutorial paper [18], the notion of input-to-state stability has had a great impact on the analysis and synthesis of nonlinear delayless systems. Following the seminal paper [19], it is very natural to believe that this concept will play an important role in the case of controlling nonlinear time-delay systems too. The motivation behind our work is to further extend the ISS theory to nonlinear time-delay systems.

In this paper, we propose a general tool, based on Lyapunov functionals, for checking stability properties of nonlinear time-delay systems. In particular, we give Lyapunov-Krasovskii theorems for the input-to-state stability of time-delay systems. The methodology is feasible by the use, for instance, of the M_2 norm (that is the norm induced by the inner product in the Hilbert space known in literature as M_2, or Z, see [2,4,6]) in the space of continuous functions, and by the use of functionals which have a suitable (simple) integral term with strictly increasing kernel.

II. PRELIMINARIES

Let us consider the following nonlinear time-delay system

$$\begin{aligned} &\dot{x}(t) = f(x_t, u(t)), \qquad t \geq 0, \qquad a.e., \\ &x(\tau) = \xi_0(\tau), \qquad \tau \in [-\Delta, 0], \end{aligned} \tag{1}$$

where $x(t) \in R^n$, $u(t) \in R^m$ is the input function, for $t \geq 0$ $x_t : [-\Delta, 0] \to R^n$ is the standard function (see Section 2.1, pp. 38 in [8]) given by $x_t(\tau) = x(t+\tau)$, Δ is the maximum involved delay, f is a function from $C([-\Delta, 0]; R^n) \times R^m$ to R^n, $C([-\Delta, 0], R^n)$ denotes the set of the continuous functions which are defined on $[-\Delta, 0]$ and take values in R^n, $\xi_0 \in C([-\Delta, 0]; R^n)$. Without loss of generality we suppose that $f(0,0) = 0$, thus ensuring that $x(t) = 0$ is the trivial solution for the unforced system $\dot{x}(t) = f(x_t, 0)$. Multiple discrete non commensurate as well as distributed delays can appear in (1).

The symbol $|\cdot|$ stands for the Euclidean norm of a real vector, or the induced Euclidean norm of a matrix. We indicate the essential supremum norm of an essentially bounded function with the symbol $\|\cdot\|_\infty$. A function u is said to be *essentially bounded* if $ess\sup_{t\geq 0} |u(t)| < \infty$. For given times $0 \leq T_1 < T_2$, we indicate with $u_{[T_1,T_2)} : [0, +\infty) \to R^m$ the function given by $u_{[T_1,T_2)}(t) = u(t)$ for all $t \in [T_1, T_2)$ and $= 0$ elsewhere. An input u is said to be *locally essentially bounded* if, for any $T > 0$, $u_{[0,T)}$ is essentially bounded. A function $w : [0, b) \to R$, $0 < b \leq +\infty$,

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is said to be locally absolutely continuous if it is absolutely continuous in any interval $[0,c]$, $0<c<b$. Moreover, we indicate with the symbol $\|\cdot\|_{M_2}$ the well known norm (see [2,4,6]) induced by the inner product in the Hilbert space $M_2 = R^n \times L_2([-\Delta,0];R^n)$, given by, for $Y = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}$, with $y_0 \in R^n$, $y_1 \in L_2([-\Delta,0];R^n)$,

$$\|Y\|_{M_2} = \left(y_0^T y_0 + \int_{-\Delta}^{0} y_1^T(\tau) y_1(\tau) d\tau \right)^{\frac{1}{2}} \tag{2}$$

$L_2([-\Delta,0];R^n)$ is the space of square Lebesgue integrable functions from $[-\Delta,0]$ to R^n. Again, we indicate with the same symbol $\|\cdot\|_{M_2}$ the norm in the space of continuous functions $C([-\Delta,0],R^n)$, given by $\|\phi\|_{M_2} = \left(\phi^T(0)\phi(0) + \int_{-\Delta}^{0} \phi^T(\tau)\phi(\tau)d\tau\right)^{\frac{1}{2}}$. Recall that a function $\gamma : R^+ \to R^+$ is: of class K if it is zero at zero, continuous and strictly increasing; of class K_∞ if it is of class K and it is unbounded; of class L if it decreases to zero as its argument tends to $+\infty$. A function $\beta : R^+ \times R^+ \to R^+$ is of class KL if it is of class K in the first argument and is of class L in the second argument. A function $\eta : R^+ \to R^+$ is positive definite if it is zero at zero and positive elsewhere.

With the symbol $\|\cdot\|_a$ we indicate any norm in $C([-\Delta,0];R^n)$ such that, for some positive reals $\gamma_a, \bar{\gamma}_a$, the following inequalities hold

$$\gamma_a|\phi(0)| \le \|\phi\|_a \le \bar{\gamma}_a\|\phi\|_\infty, \ \forall \phi \in C([-\Delta,0];R^n) \tag{3}$$

For example, the $\|\cdot\|_{M_2}$ norm satisfies the following inequalities

$$|\phi(0)| \le \|\phi\|_{M_2} \le (1+\Delta)^{\frac{1}{2}}\|\phi\|_\infty, \quad \forall \phi \in C([-\Delta,0];R^n) \tag{4}$$

and, therefore, it is a $\|\cdot\|_a$ norm.

As usual, by ISS we mean both input-to-state stable and input-to-state stability.

The following standard hypothesis (see [8,11]) is assumed throughout the paper:

Hp_0) *The function $f : C([-\Delta,0];R^n) \times R^m \to R^n$ and the input function $u : R^+ \to R^m$ are such that the function*

$$g : C([-\Delta,0];R^n) \times R^+ \to R^n \tag{5}$$

given, for $(\phi,t) \in C([-\Delta,0];R^n) \times R^+$, by $g(\phi,t) = f(\phi,u(t))$, is bounded on any bounded set $U \in C([-\Delta,0];R^n) \times R^+$ (the set $C([-\Delta,0];R^n)$ being endowed with the supremum norm), and satisfies the Carathéodory conditions in $C([-\Delta,0];R^n) \times R^+$. •

Remark 1: As is well known, from the hypothesis Hp_0 it follows that the system (1) admits a unique solution on a maximal interval $[0,b)$, $0 < b \le +\infty$, which is (componentwise) locally absolutely continuous and, if $b < +\infty$, is unbounded in $[0,b)$ (see Section 2.6, pp. 58 in [8], and Sections 2.2 and 2.4, pp. 96, 100 in [11]). •

III. MAIN RESULTS

In the following, the continuity of a functional $V : C([-\Delta,0];R^n) \to R^+$ is intended with respect to the supremum norm. Given a continuous functional $V : C([-\Delta,0];R^n) \to R^+$, the upper right-hand derivative D^+V of the functional V is given by (see [3], Definition 4.2.4, pp. 258)

$$D^+V(\phi,v) = \limsup_{h\to 0^+} \frac{1}{h}\left(V(\phi_h^\star) - V(\phi)\right), \tag{6}$$

where $\phi_h^\star \in (C([-\Delta,0];R^n)$ is given by

$$\phi_h^\star(s) = \begin{cases} \phi(s+h), & s \in [-\Delta,-h], \\ \phi(0) + f(\phi,v)(h+s), & s \in [-h,0], \end{cases} \tag{7}$$

The functional D^+V is generalized because it can take infinite values (see [11], pp. 205). Throughout the paper, the following hypothesis is assumed for the functional V involved in the Lyapunov-Krasovskii methodology presented below:

Hp_1) *the functional $V : C([-\Delta,0];R^n) \to R^+$ is continuous and is such that, for any (componentwise) locally absolutely continuous solution $x(t)$ of system (1) over a maximal interval $[0,b)$, $0 < b \le +\infty$, the following facts hold for the function $w : [0,b) \to R^+$, given by $w(t) = V(x_t)$:*

1) the function w is locally absolutely continuous in $[0,b)$;

2) the upper right-hand derivative of the function w, $D^+w(t) = \limsup_{h\to 0^+} \frac{w(t+h)-w(t)}{h}$, is such that, for almost all $t \in [0,b)$,

$$D^+w(t) = D^+V(x_t,u(t)) \tag{8}$$

Remark 2: As is well known, the absolute continuity property of w assures that, when the upper right-hand derivative $D^+w \le 0$ almost everywhere, then w is non-increasing, and the condition (8) is fundamental for practical use because D^+V is calculated without knowing the solution, by means of (6), (7). If the function f and the input u were continuous (this is clearly not the case of this paper, see the hypothesis Hp_0), then: 1) the absolute continuity property of w would not be needed, since w continuous and $D^+w \leq 0$ everywhere yields w to be non-increasing (see [15], Theorem 2.3, App. I); 2) the condition (*8*) would be satisfied by locally Lipschitz functionals (Driver 1962, see [3], Theorem 4.2.3, pp. 258, and, for the delayless case, Yoshizawa 1966, see [15], Theorem 4.3, App. I). Note that, the absolute continuity of w is not guaranteed by the local Lipschitz property of the functional V. For example, consider the globally Lipschitz functional V given by $V(\phi) = |\phi(\tau)|$, with $-\Delta \leq \tau < 0$, and take into account that the initial condition ξ_0 in general is not absolutely continuous. •

As said before, the definition of input-to-state stability for time-delay systems has been given in [19]. For reader's convenience, and to make our work self-contained, we reproduce the definition of ISS for time delay systems parallel to the one given for nonlinear delayless systems in [16].

Definition 3: *The system (*1*) is said to be input-to-state stable (ISS) if there exist a KL function β and a K function γ such that, for any initial state ξ_0 and any measurable, locally essentially bounded input u, the solution exists for all $t \geq 0$ and furthermore it satisfies*

$$|x(t)| \leq \beta\left(\|\xi_0\|_\infty, t\right) + \gamma\left(\|u_{[0,t)}\|_\infty\right) \qquad (9)$$

It is evident that the trivial solution of a time delay system which is input-to-state stable is globally asymptotically stable. In the linear case, the converse holds too. Let us consider the following linear time-delay system (time invariant case of system 1.31, pp. 16 in [7], or system 5.15, pp. 547, in [11])

$$\begin{aligned}
\dot{x}(t) = A_0 x(t) + \sum_{i=1}^{p} A_i x(t-\Delta_i) + \\
\int_{-\Delta}^{0} A_{0,1}(\theta) x(t+\theta) d\theta + Bu(t), \qquad t \geq 0, \qquad a.e., \\
x(\tau) = \xi_0(\tau), \qquad \tau \in [-\Delta, 0],
\end{aligned} \qquad (10)$$

where $B \in R^{n\times m}, A_j \in R^{n\times n}, j = 0,1,\ldots,p$, $A_{0,1}$ is a $n \times n$ matrix of piecewise continuous functions which are defined in $[-\Delta, 0]$ and take values in R, $0 < \Delta_1 < \Delta_2 < \cdots < \Delta_p = \Delta$ are the arbitrary (non-commensurate) time delays, $\xi_0 \in C([-\Delta,0];R^n)$, the input u is measurable, locally essentially bounded.

Proposition 4: *The linear time-delay system (*10*) is input-to-state stable if and only if the trivial solution of the unforced system is asymptotically stable.*

Proof. By a steps procedure with sufficiently small step, it follows that there exist two positive reals L and M such that, in $[0, \Delta]$, for any initial state ξ_0 and for any input u, the following inequality holds

$$\|x(t)\| \leq L\|\xi_0\|_\infty + M\|u_{[0,t)}\|_\infty \qquad (11)$$

Being x_Δ absolutely continuous and being its derivative square Lebesgue integrable, by Theorem 2.1 in [2], the solution of system (10) can be expressed, for $t \geq \Delta$, by

$$X(t) = S(t-\Delta)X(\Delta) + \int_{\Delta}^{t} S(t-\tau)\begin{bmatrix} Bu(\tau) \\ 0 \end{bmatrix} d\tau, \quad (12)$$

where $X(t) = \begin{bmatrix} x_t(0) \\ x_t \end{bmatrix} \in M_2$, and $S(t): M_2 \to M_2$ is a C_0-semigroup. If the trivial solution of the unforced system is asymptotically stable, then there exist two positive constants M and ω such that (see Theorem 2.3 in [6])

$$\|S(t)\| \leq Me^{-\omega t}, \qquad t \geq 0, \qquad (13)$$

where $\|S(t)\| = \sup_{Y\in M_2} \frac{\|S(t)Y\|_{M_2}}{\|Y\|_{M_2}}$.

The following inequalities hold, for $t \geq \Delta$,

$$\begin{aligned}
&|x(t)| \leq \|X(t)\|_{M_2} = \\
&\left\| S(t-\Delta)X(\Delta) + \int_{\Delta}^{t} S(t-\tau)\begin{bmatrix} Bu(\tau) \\ 0 \end{bmatrix} d\tau \right\| \leq \\
&\|S(t-\Delta)\|\|X(\Delta)\|_{M_2} + \int_{\Delta}^{t} \|S(t-\tau)\|\|B\||u(\tau)|d\tau,
\end{aligned} \qquad (14)$$

Therefore, by using the inequality (13), by $\|X(\Delta)\|_{M_2} = \|x_\Delta\|_{M_2} \leq (1+\Delta)^{\frac{1}{2}}\|x_\Delta\|_\infty$, and taking into account (11), an inequality of the form (*9*) for the system (10) can be derived. •

Theorem 5: *If there exist a functional $V : C([-\Delta,0];R^n) \to R^+$, functions α_1, α_2 of class K_∞, and functions α_3, ρ of class K such that:*

H_1) $\alpha_1(|\phi(0)|) \leq V(\phi) \leq \alpha_2(\|\phi\|_a)$, $\forall\, \phi \in C([-\Delta,0];R^n)$;

H_2) $D^+V(\phi,u) \leq -\alpha_3(\|\phi\|_a)$, $\qquad \forall\, \phi \in C([-\Delta,0];R^n)$, $\forall\, u \in R^m : \ \|\phi\|_a \geq \rho(|u|)$;

*then, the system (*1*) is input-to-state stable with $\gamma = \alpha_1^{-1} \circ \alpha_2 \circ \rho$.*

Proof. The lines of a part of the proof of the main theorem in [16] will be followed here. Let the input u be such that $ess\sup_{t\geq 0}|u(t)| = v$, for a suitable $v \in R^+$. Let $c = \alpha_2(\rho(v))$ and introduce the set $S = \{\xi \in C([-\Delta,0];R^n) : V(\xi) \leq c\}$.

Claim 1: If the solution $x(t)$ is such that, for a certain time $t_0 \geq 0$, $x_{t_0} \in S$, then $x_t \in S$ for $t \geq t_0$.

Proof: see [16] for details. It suffices to note that, because of the hypothesis Hp_1, the locally absolutely continuous function $w(t) = V(x_t)$ is non-increasing when $D^+V(x_t, u(t))$ is non-positive almost everywhere. When $x_t \in S$, necessarily $|x(t)| \leq \gamma(v)$, with $\gamma = \alpha_1^{-1} \circ \alpha_2 \circ \rho$.

Claim 2: There exists a KL function β such that, for each ξ_0 and each bounded control u, there exists a time instant $T > 0$ (possibly $T = +\infty$) such that

1) $|x(t)| \le \beta(\|\xi_0\|_\infty, t)\ \forall\ t < T$;

2) $x_t \in S\ \forall\ t \ge T$.

Proof: see [16]. Just take into account Lemma 4.4 in [12], inequalities (3) and note that, for almost all t in some interval $[0, T)$ (where $V(x_t) > c$), by the inequality in H_2, the following inequality holds, for $w(t) = V(x_t)$,

$$D^+ w(t) \le -\alpha_3 \circ \alpha_2^{-1}(w(t)),\ a.e.\ in\ [0, T), \tag{15}$$

since, by the hypothesis Hp_1,

$$\begin{aligned} D^+ w(t) = D^+ V(x_t, u(t)) \le -\alpha_3(\|x_t\|_a) \le \\ -\alpha_3 \circ \alpha_2^{-1}(V(x_t))) = -\alpha_3 \circ \alpha_2^{-1}(w(t)),\ a.e.\ in\ [0, T) \end{aligned} \tag{16}$$

Note that the inequality (15) guarantees that the locally absolutely continuous function $w(t) = V(x_t)$ is non-increasing in $[0, T)$ and this, together with Claim 1, guarantees that the solution $x(t)$ is defined for all $t \ge 0$ (see Remark 1). Causality arguments can be used for locally essentially bounded inputs. •

Remark 6: Theorem 5 turns out to be useful for studying a large class of nonlinear systems, when using the M_2 norm and Lyapunov-Krasovskii functionals which have one integral with strictly increasing kernel. Indeed, by the strictly increasing kernel, it is possible to generate a negative integral term, that is needed for satisfying the inequalities which involve the input. That is, such a negative integral term is very useful in order to satisfy $D^+ V(\phi, u) \le -\alpha_3(\|\phi\|_{M_2})$, whenever $|u|$ is upper bounded by a K function of the M_2 norm of ϕ. The norms used in the inequalities involved in the Lyapunov-Krasovskii theorem developed in this paper are different from the ones involved in the Lyapunov-Krasovskii theorem for asymptotic stability. It is the first time that sufficient Lyapunov-Krasovskii conditions, with appropriate norms, are explicitly proposed for ISS of a large class of time-delay systems. •

IV. ILLUSTRATIVE EXAMPLE

Let us consider the following example, taken from [5],

$$\begin{aligned} \dot{x}_1(t) =& \frac{x_2(t)}{1 + x_1^2(t-\Delta)} + \sigma x_2(t-\Delta) \\ \dot{x}_2(t) =& x_1(t) x_2(t-\Delta) + u(t) + w(t), \end{aligned} \tag{17}$$

where $x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \in R^2$, $u(t) \in R$ is the control input, $w(t) \in R$ is a measurable, locally essentially bounded disturbance, σ is a constant parameter, $\Delta > 0$ is an arbitrarily large, known time-delay. The initial condition is given by a function $\xi_0 \in C([-\Delta, 0]; R^n)$.

In [5] the case with no disturbance is studied: it is shown that, if the parameter is such that $|\sigma| < 1$, the system (17) is globally stabilized by the following nonlinear control law, which is a continuous time difference equation,

$$\begin{aligned} & u(t) = -x_1(t) x_2(t-\Delta) + (1 + x_1^2(t-\Delta)) \cdot \\ & \cdot \left(-k^T \begin{bmatrix} x_1(t) \\ \frac{x_2(t)}{1+x_1^2(t-\Delta)} + \sigma x_2(t-\Delta) \end{bmatrix} \right. \\ & -\sigma x_1(t-\Delta) x_2(t-2\Delta) + \\ & \left. + \frac{2 x_2(t) x_1(t-\Delta)}{(1 + x_1^2(t-\Delta))^2} \left(\frac{x_2(t-\Delta)}{1 + x_1^2(t-2\Delta)} + \sigma x_2(t-2\Delta) \right) \right) \\ & - \sigma(1 + x_1^2(t-\Delta)) u(t-\Delta), \end{aligned} \tag{18}$$

where k^T is a suitable row vector such that the matrix $A_B - B_B k^T$ has prescribed eigenvalues in the left-half complex plane, A_B, B_B being a Brunovskii couple (see [9], pp. 153, 231). Note that, because of its dependence on $u(t-\Delta)$ and $x(t-2\Delta)$, such a control law can be applied only for $t \ge \Delta$. If, as in [5], the input is chosen equal to zero in $[0, \Delta)$, then the solution of the system (17) exists in $[0, \Delta]$ and, by the Bellman-Gronwall Lemma, it satisfies the inequality

$$\begin{aligned} & |x(t)| \le e^{(1 + \|\xi_0\|_\infty)\Delta} \left((1 + |\sigma|\Delta) \|\xi_0\|_\infty + \Delta \|w_{[0,t)}\|_\infty \right), \\ & \quad t \in [0, \Delta] \end{aligned} \tag{19}$$

When $|\sigma| > 1$, the internal dynamics of the closed loop system (17)(18) is unstable so that the control law (18) does not work (see [5] for details). In this paper we shall consider non-negative values of the parameter σ. By the methodology here presented, we shall prove that, when a measurable, locally essentially bounded disturbance w adds to the control law, the closed-loop system is ISS with respect to such disturbance if $\sigma = 0$, while the closed-loop system can become unstable if $0 < \sigma < 1$.

As in [5], define, for $t \ge 0$,

$$z(t) = \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} = \begin{bmatrix} x_1(t) \\ \frac{x_2(t)}{1+x_1^2(t-\Delta)} + \sigma x_2(t-\Delta) \end{bmatrix}, \tag{20}$$

so that the control law (18) yields the following equation, for $t \ge \Delta$,

$$\dot{z}(t) = H z(t) + \begin{bmatrix} 0 & 0 \\ \frac{1}{1+z_1^2(t-\Delta)} & \sigma \end{bmatrix} \begin{bmatrix} w(t) \\ w(t-\Delta) \end{bmatrix}, \tag{21}$$

where $H = A_B - B_B k^T$ is a Hurwitz matrix in canonical controllable form, with arbitrarily preassigned eigenvalues up to a suitable choice of gain k^T. Let us study, by the methodology of this paper, the ISS of system (21) with respect to a measurable, locally essentially bounded disturbance $d(t) \in R^2, t \geq \Delta$ (which in (21) takes the place of $[\, w(t) \quad w(t-\Delta)\,]^T$). The hypothesis Hp_0 is satisfied. Let us apply Theorem 5 with the M_2 norm and the following Lyapunov-Krasovskii functional (with strictly increasing kernel) which satisfies the hypothesis Hp_1

$$V(\phi) = \phi^T(0)P\phi(0) + \int_{-\Delta}^{0} \phi^T(\tau) \left(\frac{-\tau}{\Delta} Q_1 + \frac{\tau+\Delta}{\Delta} Q_2 \right) \phi(\tau) d\tau, \tag{22}$$

where $\phi = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} \in C([-\Delta, 0]; R^n)$, P, Q_1, Q_2 are symmetric positive definite matrices, $Q_2 > Q_1$.

We obtain, for $\|\phi\|_{M_2} \geq \rho|d|$, where ρ is a suitable positive real,

$$\begin{aligned} D^+V(\phi, d) &= \phi^T(0)(H^T P + PH)\phi(0) + \\ & 2\phi^T(0)P \begin{bmatrix} 0 & 0 \\ \frac{1}{1+\phi_1^2(-\Delta)} & \sigma \end{bmatrix} d + +\phi^T(0)Q_2\phi(0) \\ & - \phi^T(-\Delta)Q_1\phi(-\Delta) \\ & - \frac{1}{\Delta}\int_{-\Delta}^{0} \phi^T(\tau)(Q_2 - Q_1)\phi(\tau)d\tau \\ & \leq \phi^T(0)\left(H^T P + PH + Q_2 + \right. \\ & \left. \left\| P \begin{bmatrix} 0 & 0 \\ 1 & |\sigma| \end{bmatrix} \right\| \frac{3}{\rho} I \right) \phi(0) + \\ & + \left\| P \begin{bmatrix} 0 & 0 \\ 1 & |\sigma| \end{bmatrix} \right\| \frac{1}{\rho} \cdot \\ & \int_{-\Delta}^{0} \phi^T(\tau)\phi(\tau)d\tau - \phi^T(-\Delta)Q_1\phi(-\Delta) - \\ & \quad - \frac{1}{\Delta}\int_{-\Delta}^{0} \phi^T(\tau)(Q_2 - Q_1)\phi(\tau)d\tau \end{aligned} \tag{23}$$

Let us choose $Q_1 = I, Q_2 = 2I$ (I is the identity matrix). Let us choose P such that

$$H^T P + PH = -4I \tag{24}$$

and let ρ be sufficiently large such that $\left\| P \begin{bmatrix} 0 & 0 \\ 1 & |\sigma| \end{bmatrix} \right\| \frac{3}{\rho} < min\left\{1, \frac{1}{\Delta}\right\}$. Then we obtain

$$\begin{aligned} & D^+V(\phi, d) \leq -\phi^T(0)\phi(0) \\ & - \frac{2}{3\Delta}\int_{-\Delta}^{0} \phi^T(\tau)\phi(\tau)d\tau - \phi^T(-\Delta)Q_1\phi(-\Delta) \leq \\ & - min\left\{1, \frac{2}{3\Delta}\right\} \|\phi\|_{M_2}^2 \end{aligned} \tag{25}$$

Thus, by Theorem 5, it is proved that system (21) is ISS with respect to the disturbance $d(t)$, that is, for each $\sigma \in (-1, 1)$ there exist a KL function β_σ and a K function γ_σ such that, for $t \geq \Delta$,

$$|z(t)| \leq \beta_\sigma(\|z_\Delta\|_\infty, t - \Delta) + \gamma_\sigma(\|d_{[\Delta,t)}\|_\infty), \tag{26}$$

where, as usual, z_Δ is the function in $C([-\Delta, 0]; R^n)$ given by $z_\Delta(\tau) = z(\Delta + \tau)$ (take into account (19) and (20) in $[0, \Delta]$). In the case $\sigma = 0$, there is no dependence on $w(t-\Delta)$ in (21) and therefore, for $d(t) = [\, w(t) \quad 0\,]^T$, the following inequality holds for $t \geq \Delta$, by the KL function β_0 and the K function γ_0,

$$|z(t)| \leq \beta_0(\|z_\Delta\|_\infty, t - \Delta) + \gamma_0(\|w_{[\Delta,t)}\|_\infty) \tag{27}$$

Let us now go back to the variables x. From (20), we obtain, for $t \geq \Delta$,

$$x(t) = \begin{bmatrix} z_1(t) \\ z_2(t)(1 + z_1^2(t-\Delta)) - \sigma(1 + z_1^2(t-\Delta))x_2(t-\Delta) \end{bmatrix} \tag{28}$$

From the equation in the second variable, $x_2(t)$, it follows that, if $\sigma > 0$, the closed-loop system (17),(18) in general cannot satisfy the inequality (*9*). For instance, assume $\sigma > 0$ and let the gain k^T in the control law (18) be chosen such that the matrix H has real asymptotically stable eigenvalues $\lambda_2 < \lambda_1 < 0$ (which is the case of the simulated example, without disturbance, in [5]). Choose the disturbance to be constant and equal to a positive real $\bar{w}$ in $[0, +\infty)$. From (21), (19) and (20) it follows that, for any initial conditions ξ_0, there exist a positive real $\rho > 1$, a positive real $\bar{w}$ and a positive real $\bar{t} \geq \Delta$, such that, for all $t \geq \bar{t}$,

$$\sigma(1 + z_1^2(t-\Delta)) \geq \rho \tag{29}$$

and the scalar continuous time difference equation in the variable $x_2(t)$ in (28), evaluated for $t \geq \bar{t}$, with initial conditions in $[\bar{t} - \Delta, \bar{t}]$, is unstable.

While, if $\sigma = 0$, from (28) it follows that

$$\begin{aligned} |x(t)| &\leq |z(t)| + |z(t)|(1 + |z(t-\Delta)|^2) \leq \\ & 2|z(t)|(1 + |z(t-\Delta)|^2), \qquad t \geq \Delta \end{aligned} \tag{30}$$

and, taking into account (27) and that $|z(t)| \leq |x(t)|\ \forall t \geq 0$, it follows that

$$\begin{aligned} & |x(t)| \leq 2(1 + \|x_\Delta\|_\infty^2) \cdot \\ & \cdot \left(\beta_0(\|x_\Delta\|_\infty, t - \Delta) + \gamma_0(\|w_{[\Delta,t)}\|_\infty)\right), \ t \in [\Delta, 2\Delta), \end{aligned} \tag{31}$$

$$
\begin{aligned}
|x(t)| \leq & 2(1+2\beta_0^2(\|x_\Delta\|_\infty, t-2\Delta)+2\gamma_0^2(\|w_{[\Delta,t)}\|_\infty))\cdot \\
& \cdot \left(\beta_0(\|x_\Delta\|_\infty, t-\Delta)+\gamma_0(\|w_{[\Delta,t)}\|_\infty)\right), \\
& t \in [2\Delta, +\infty)
\end{aligned} \tag{32}
$$

From (31),(32) it follows that the closed-loop system (17),(18), $t \geq \Delta$, with initial conditions equal to x_Δ, is ISS with respect to $w_{[\Delta,+\infty)}$.

From (19), (31) and (32), by standard properties of K and KL functions, it follows for system (17) with $\sigma = 0$, with input u equal to zero in $[0, \Delta)$ and given by the control law (18) in $[\Delta, +\infty)$, that there exist a KL function β and a K function γ such that, for all $t \geq 0$, the following inequality holds

$$
|x(t)| \leq \beta(\|\xi_0\|_\infty, t) + \gamma(\|w_{[0,t)}\|_\infty). \tag{33}
$$

Remark 7: Note that, when $\sigma = 0$, the control law (18) is no more a continuous time difference equation in the variable $u(t)$ (the term $u(t-\Delta)$ disappears in (18)), therefore in this case the problem of instability of the controller (see [5]) does not occur. While, if $\sigma > 0$, the controller can well be unstable. Indeed, it would be sufficient that $\sigma(1 + x_1^2(t-\Delta)) \geq \rho > 1$ for all t sufficiently large. •

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