Proceedings of the
44th IEEE Conference on Decision and Control, and
the European Control Conference 2005
Seville, Spain, December 12-15, 2005

A Hybrid, PDE-ODE Controller for Intercepting an Intelligent, Well-informed Target in a Stationary, Cluttered Environment

Ahmad A. Masoud
Electrical Engineering Department, KFUPM, P.O. Box 287, Dhahran 31261, Saudi Arabia, e-mail: masoud@kfupm.edu.sa

Abstract - A new class of intelligent controllers that can semantically embed an agent in a spatial context constraining its behavior in a goal-oriented manner was suggested in [1,11]. A controller of such a class can guide an agent in a stationary unknown environment to a fixed target zone along an obstacle-free trajectory. Here, an extension is suggested that would enable the interception of an intelligent target that is maneuvering to evade capture amidst stationary clutter (i.e. the target zone is moving). This is achieved by forcing the differential properties of the potential field used to induce the control action to satisfy the wave equation. Theoretical developments, as well as, proofs of the ability of the modified control to intercept the target along an obstacle-free trajectory are supplied. Simulation results are also provided.

I. Introduction

The survival of a specie is dependent, among other things, on its ability to develop its navigation competence to successfully handle a prey-hunter situation. The developed capabilities are subjective in nature, that is: the structure of the capture scheme is built around cues that are meaningful to the hunter and related to aspects of the prey's personality. In a situation where an intelligent prey is being hunted, the prey may be aware of the hunter. Even more, it may be aware of what the hunter expects from it. In such a situation, the prey may initiate a deception scheme that is masked by its expected behavior with the goal of engaging in an interactive message exchange with the hunter. The structure of this exchange is based on the model which the prey has for the hunter's decision making process. Its aim is to out-maneuver its pursuer and evade capture. Through this exchange, the prey may even acquire a soft control of the hunter that could reverse the role of each. Here, the nesting of action release, or equivalently message exchange, has the form of I KNOW that IT KNOWS that I KNOW etc. [2]. The deeper the nesting is, the more likely that the actions released by the party concerned drive it to the desired conclusion. Such subjectivity seems inherent in the structure of these kind of problems. This has convinced the majority of researchers in the area of the need to use evolutionary techniques (e.g. neural networks, genetic algorithms, etc.) to "absorb" the personality profile of the target in order to derive a successful capture scheme. While profiling may work most of the time, the situation is dramatically different for the case of active intelligent targets. These targets are capable of adjusting their trajectories to enhance their chance of survival by extending their domain of awareness to include that of their pursuers. To the best of this author's knowledge, this class of problems was not addressed in the literature [3].

This paper hypothesizes the existence of objective techniques for target interception whose success is directly tied to the potential of achieving a desired conclusion instead of being dependant on the psychology of the prey. Testing the above hypothesis is carried out with the help of a newly introduced class of intelligent motion controllers [1,11].Up-till-now such controllers are implemented using harmonic potential fields (HPFs). HPFs possess many useful properties that make them excellent tools for navigation [13]. Most notably, is that an HPF is also a Morse function (see appendix). Despite the efficiency of HPF-based methods in tackling environments with sophisticated geometry, they can only engage simple, stationary targets (a sitting duck). In this paper a modified version of the control that utilizes the wave potential is suggested for engaging active, intelligent targets in a provably-correct manner.

This paper is organized as follows: in section II the problem is formulated. Section III provides a background on PDE-ODE controllers. Sections IV and V presents the PDE and the ODE components needed for constructing the control. Section VI contains motion analysis. Simulation results and conclusions are in sections VII and VIII respectively.

II. Problem Formulation

Designing a method for intercepting a moving target can be a challenging task, especially if an active intelligent target is engaged. The target may attempt to intelligently maneuver to evade capture using full knowledge of its surroundings which may be as complex as a maze. It may also be well-informed about the movements of its pursuer. Here, planning the movements of the pursuer in a manner that can cope with the above situation is carried out using the nonlinear, dynamical system:

$$\dot{x} = g(V(x(t), x_p(t), \Gamma)) \qquad x(0) \in \Omega. \qquad 1$$

The above system is required to make:

$$\lim_{t \to \infty} |x(t) - x_p(t)| \to 0, \qquad 2$$

and $x(t) \in \Omega \;\; \forall\, t$,

where Ω is an admissible subset of the N-dimensional state space, Γ is the boundary of the non-admissible subset of state space (O, $\Gamma = \partial O$), $x_p(t)$ is the trajectory of the target ($x_p(t) \in \Omega \;\; \forall\, t$), $x(t) \in R^N$, g: $R \to R^N$, V: $R^N \times R^N \times R^M \to R$, where $\Gamma \in R^M$, $M \leq N-1$. The above implies the followings:

1- the target may be intelligent,

2- the target may have an accurate model of its environment, as

well as information about the movements of its pursuer,
3- no psychological profile of the target, its tendencies, and habits or, for that matter, a statistical model of the target's behavior are needed ,
4- the pursuer has a good model of the environment, and full information about the movements of the target.
The target is assumed to have limited power so that it cannot instantly change its position or orientation (i.e. $x_p(t) \in C^L$, $L \geq 2$).

III A Background

The controller of interest (figure-1) functions to convert the goal, constraints on behavior, and the available representation of an environment into a sequence of actions $\{u_0,..,u_L\}$. These actions must yield a corresponding sequence of states $\{X_0,...,X_L\}$ so that the final state X_L is the goal state of the agent, and all the transient states satisfy the constraints on behavior. The action sequence is called a plan which is a member of a field of plans (action field) that densely covers state space so that regardless of the starting point, a plan always exists to move the agent to its goal.

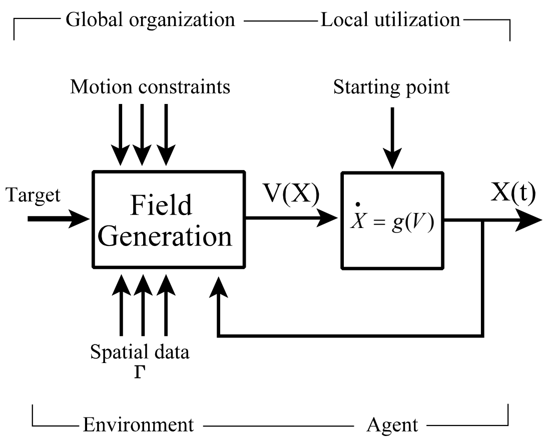

Figure-1: A PDE-ODE Controller.

The information needed for synthesizing the action field from a PDE-ODE Controller is generated by the synergetic interaction of a massive number of differential systems (micro-agents), figure-2. A micro-agent has a structure that is identical to the agent being controlled, with the exception that its state is stagnant and immobilized to an *a priori* known location in state space.

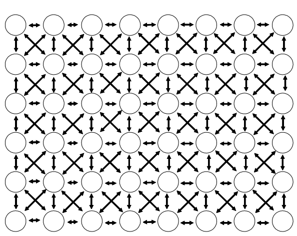

Figure-2: A collective of micro-agents.

The actions of the micro-agent collective are emulated using a potential field that is operated on by a vector partial differential operator. The synergetic interaction needed to convert the individual actions of the micro-agents into a guidance-capable group action is achieved using the two steps:
1- the individual micro-agents are informationally coupled (figure-3). This is achieved using the proper partial differential governing relation,
2- the process of morphogenesis [4], which is responsible for guiding the group structure of the action field, is activated by factoring the influence of the agent's environment in the behavior generation process. This influence is factored-in using boundary control action. The overall control structure is induced by solving the boundary value problem (BVP) that is constructed using the partial differential governing relation from step-1, and the boundary control action.

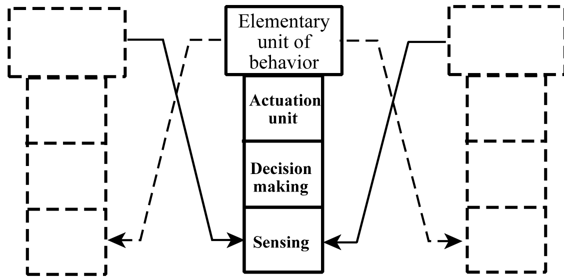

Figure-3: An informationally-coupled micro-agent.

The above mode of behavior synthesis is in conformity with the artificial life approach to behavior generation [5]. Figure-4 shows the evolution of the action field for an HPF-based controller.

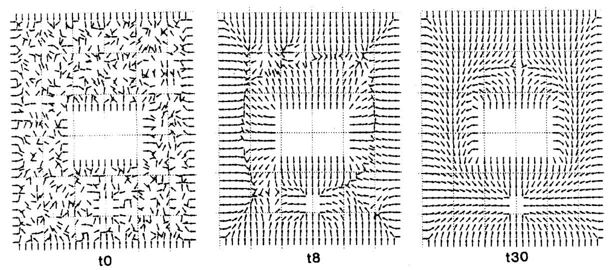

Figure-4: Evolution of action structure

IV. The PDE Component

Scalar potential fields that describe changes in both space and time are suitable for synthesizing action fields that can be used for tracking moving targets. The partial differential relation that may be used to govern the differential properties of such surfaces is the wave equation (WE):

$$\nabla^2 V = \frac{1}{a^2}\frac{\partial^2 V}{\partial t^2} \qquad 3$$

where a is a positive constant, ∇ is the gradient operator, ∇^2 is Laplace operator. The reason the WE is chosen over other spatio-temporal governing relations (e.g the diffusion equation) is mainly due to the nature of its solution. From the method of separation of variables, the solution of a field that is dependent on both space and time (V(x,t)), and is governed by the WE may be written in the form:

$$V(x,t) = R(x)T(t) \qquad 4$$

where R is the position-only dependent component of the solution, and T is time-only dependent component. Therefore, the WE may be placed in the form:

$$T\,\nabla^2 R - \frac{1}{a^2}\,R\,\frac{\partial^2 T}{\partial^2 t} = 0\ . \qquad 5$$

The only way equation 5 can be satisfied is for the position and time dependent terms to be equal to the same constant which is, for convenience, chosen equal to $-\lambda^2$:

$$\frac{\nabla^2 R}{R}=a^2\frac{\partial^2 T/\partial t^2}{T}=-\lambda^2 \ . \qquad 6$$

As a result, T(t), and R(x) may be computed by solving the following Helmholtz equations (HEs):

$$\nabla^2 R+\lambda^2 R=0 \qquad \text{N-D HE} \qquad 7$$

$$\frac{\partial^2 T}{\partial t^2}+(a\lambda)^2 T=0 \qquad \text{1-D HE}$$

It is well-known that the fundamental solution of an N-D HE provides N orthogonal basis functions capable of representing an arbitrary scalar function that is defined on that space. Therefore, the above set of equations yields N+1 orthogonal basis enough to represent (using the generalized Fourier series expansion) any piecewise continuous function in both time and space.

A. *BVP-1, The Dirichlet Case:*
The generating BVP is (Figure-5): solve

$$\nabla^2 V = \frac{1}{a^2}\frac{\partial^2 V}{\partial t^2} \ . \qquad 8$$

subject to $V(x,x_p(t),\Gamma)=C$ for $x\in\Gamma$, and $V(x_p(t),x_p(t),\Gamma)=0$, where C is a positive constant, and $x_p(t)$ is the target's trajectory.

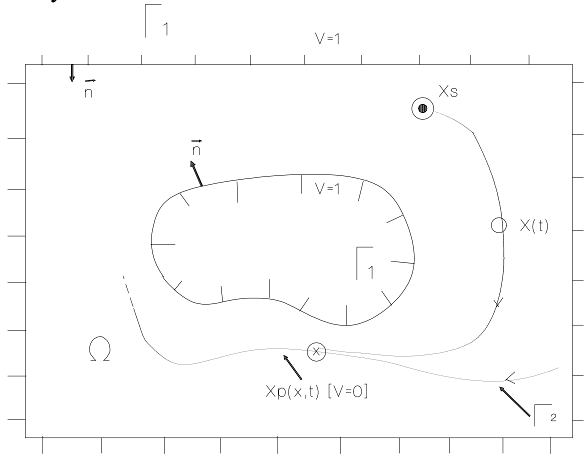

Figure-5: Boundary Conditions, the Dirichlet case.

B. *BVP-2, The Homogeneous Neumann Case:*
The generating BVP is: solve

$$\nabla^2 V = \frac{1}{a^2}\frac{\partial^2 V}{\partial t^2} \ . \qquad 9$$

subject to $V(x(0),x_p(t),\Gamma)=C$, $\partial V(x,x_p(t),\Gamma)/\partial n=0$ for all $x\in\Gamma$, and $V(x_p(t),x_p(t),\Gamma)=0$, where **n** is a unit vector normal to Γ. Existence and uniqueness of the solution of the above BVPs were proven in [6,7].

V. The ODE System

The interception trajectory is generated using the first order nonlinear differential system:

$$\dot{x}=g(x(t),x_p(t),\Gamma) \ , \qquad 10$$

$$g(x,x_p,\Gamma)=-[\nabla V(x(t),x_p(t),\Gamma)+\frac{\nabla V(x(t),x_p(t),\Gamma)}{\|\nabla V(x(t),x_p(t),\Gamma)\|^2}\frac{\partial\nabla V(x(t),x_p(t),\Gamma)}{\partial t}] \ ,$$

It can be shown that the above ODE system is capable of satisfying condition 2. Unfortunately, the system encounters a singularity when $x(t)=x_p(t)$. To remedy this problem, the condition in 2 is relaxed to:

$$\lim_{t\to\infty}|x(t)-x_p(t)|\leq \rho, \qquad 11$$

where $1>>\rho>0$. A singularity-free ODE system that can accomplish the above is:

$$\dot{x}=g(x(t),x_p(t),\Gamma) \ ,$$

$$g(x,x_p,\Gamma)=-[\nabla V(x(t),x_p(t),\Gamma)+\frac{\nabla V(x(t),x_p(t),\Gamma)}{\beta(\|\nabla V(x(t),x_p(t),\Gamma)\|^2)}\frac{\partial V(x(t),x_p(t),\Gamma)}{\partial t}]$$

where $$\beta(x)=\begin{cases} x & x\geq\rho \\ \eta(x) & x<\rho \end{cases} \ , \qquad 12$$

and $\eta(x)$ is a monotonically increasing function that satisfies the followings: $\eta(0)=\epsilon$, $\eta(\rho)=\rho$ $\rho>\epsilon>0$,

$$\frac{d\eta(x)}{dx}|_{x=\rho}=1 \ , \qquad \frac{d\eta(x)}{dx}|_{x=0}=0 \ . \qquad 13$$

A form for $\eta(x)$ that satisfies the above conditions is:

$$\eta(x)=\epsilon+\frac{2\rho-3\epsilon}{\rho^2}x^2+\frac{2\epsilon-\rho}{\rho^3}x^3 \ . \qquad 14$$

It ought to be noticed that the partial, implicit dependence of $\partial V/\partial t$ on dx_p/dt does not imply that the velocity of the target is needed for its computation. Any numerical procedure for the solution of the time dependent potential computes $\partial V/\partial t$ using the a finite difference approximation:

$$\frac{V(x(t),x_p(t),\Gamma)-V(x(t-dt),x_p(t-dt),\Gamma)}{dt} \ . \qquad 15$$

Constructing the above approximation requires only that the time-dependant position of the target be estimated.

A. *The solution for the ODE System in 12 Exists*
The nonlinear ODE system in 12 satisfies the global Lipschitz condition [8]. In other words, for every time interval $\tau\in[0,\infty)$, ∃ the constants $k\tau<\infty$, and $h\tau<\infty$, so that

$$\|g(x,x_p(t),\Gamma)-g(y,x_p(t),\Gamma)\|\leq k\tau\|x-y\| \qquad 16$$

and $$\|g(x(0),x_p(t),\Gamma)\|\leq h\tau \ ,$$

$\forall\ t\in[0,\tau]$, and $\forall\ x,y\in R^n$. This means that the solution of 12 does exist, and is unique. This in turns imply:

1- x is differentiable almost everywhere (i.e. dx/dt exists),

2- the relation: $\dot{x}=g(x(t),x_p(t),\Gamma)$

holds for t where dx/dt exists,

3- x(t) satisfies: $$x(t)=x(0)+\int_0^t g(x(t_1),x_p(t_1),\Gamma)dt_1 \ . \qquad 17$$

VI. Motion Analysis

In this section, the ability of the suggested PDE-ODE system to enforce the convergence and avoidance constraints in 2 is examined. The proof is based on Liapunov second method [9]. A key element of the proof is showing that the wave potential in section IV is a Liapunov function candidate (LFC).

A. The Wave Potential is an LFC

It is well-known that the solution of the WE is analytic. This satisfies the requirement that a LFC be differentiable, or at least continuous. It can also be shown that surfaces with differential properties that are governed by the linear, elliptic, WE partial differential operator satisfy the maximum Principle (i.e they are free of local extreme, where minima or maxima of such functions can only occur on the boundary of the space on which V is defined (Γ)), [10]. This in turn leads to the satisfaction of the second condition required by an LFC, that is:

1- $V(x(t_i), x_p(t_i), \Gamma) = 0$ at, and only at $x=x_p$, 18
2- $V(x(t_i), x_p(t_i), \Gamma) > 0$ $x \in \Omega$, $\forall t_i \in t$.

B. Liapunov's Direct Method

A point x_p is considered to be an equilibrium point of the system in 1 if

$$g(x_p(t),x_p(t), \Gamma) = 0 \qquad \forall t \geq 0 \ . \qquad 19$$

This equilibrium point is considered to be globally, asymptotically stable (i.e. $x(t) \to x_p(t)$ as $t \to \infty$) if $\exists$ a LFC $\ni$:

1- $dV(x(t_i), x_p(t_i),\Gamma)/dt = 0$ at and only at $x=x_p$, 20
2- $dV(x(t_i), x_p(t_i),\Gamma)/dt < 0$ $x \in \Omega$, $\forall t_i \in t$.

C. Convergence Analysis

The time derivative of the wave potential is:

$$\dot{V}(x(t),x_p(t),\Gamma) = \nabla V(x(t),x_p(t),\Gamma)^t \dot{x} + \frac{\partial V(x(t),x_p(t),\Gamma)}{\partial t} . \qquad 21$$

Substituting the value of dx/dt from 10, we have:

$$\dot{V}(x(t),x_p(t),\Gamma) = -\|\nabla V(x(t),x_p(t),\Gamma)\|^2 \ . \qquad 22$$

Since, from the maximum principle, V contains no local extrema in Ω (note that Γ, and $x_p \notin \Omega$) where, by design, the minimum is placed at x_p and the maximum at Γ, the time derivative of the wave potential satisfies the conditions in 20. Therefore, V is a valid Liapunov Function. Hence, global asymptotic convergence to x_p from anywhere in Ω is guaranteed.

In a similar way, it can be shown that for the singularity-free, dynamical system in 12 $dV/dt < 0 \ \forall \ x \in \Omega - B_\rho$, where

$$B_\rho(x) = \{x: \|x-x_p\| < \rho\} \ . \qquad 23$$

This implies that: $$\lim_{t \to \infty} x(t) \in B_\rho(x) \ . \qquad 24$$

D.1. Avoidance Analysis (the Dirichlet Case)

Let Γδ be a thin region surrounding Γ. Proving that motion will not be steered towards Γ inside Γδ is sufficient to prove that the forbidden regions will not be entered.

Assuming that x is initially inside Γδ, let x_n be the distance between x and Γ:

$$x_n = x^t \mathbf{n} , \qquad 25$$

where **n** denotes a unit vector normal to Γ. Since **n** is not a function of time, the time derivative of x_n is equal to:

$$\dot{x}_n = \dot{x}^t \boldsymbol{n} = -(1+\frac{1}{\|\nabla V\|^2}\frac{\partial V}{\partial t})\nabla V^t \boldsymbol{n} = -(1+\frac{1}{\|\nabla V\|^2}\frac{\partial V}{\partial t})\frac{\partial V}{\partial n} \ . \qquad 26$$

Since the state is initially assumed to be outside Γ ($x_n > 0$), proving that a measure of the length of x_n is always non-decreasing is sufficient to prove that the state will never enter the forbidden regions.

Let Va be a measure of x_n:

$$V_a = x_n^2 \ . \qquad 27$$

The time derivative of Va may be computed as:

$$\dot{V}_a = 2x_n \dot{x} = -2x_n(1+\frac{1}{\|\nabla V\|^2}\frac{\partial V}{\partial t})\frac{\partial V}{\partial \boldsymbol{n}} \ . \qquad 28$$

Since the value of V on Γ is constrained to a constant C,

$$\frac{\partial V}{\partial t} = 0 \qquad x \in \Gamma \ . \qquad 29$$

Since x is assumed to be very close to Γ ($x \in \Gamma\delta$), the following approximation may be constructed:

$$\frac{\partial V}{\partial t} \approx 0 \qquad x \in \Gamma\delta \ . \qquad 30$$

Also, from the maximum principle, we have

$$V(x1,x_p,\Gamma) > V(x2,x_p,\Gamma) \qquad x1 \in \Gamma \ , \ x2 \in \Gamma\delta \ . \qquad 31$$

Therefore $$\frac{\partial V}{\partial n} < 0 \qquad x \in \Gamma\delta \ . \qquad 32$$

Using the above results it can be seen that the value of the time derivative of the Liapunov function inside Γδ is equal to

$$\dot{V} \approx -2x_n \frac{\partial V}{\partial n} > 0 \qquad 33$$

Therefore, the guidance field will always push x away from Γδ, steering it away from Γ. In other words, the forbidden regions will be avoided.

D.2. Avoidance Analysis (the Neumann Case)

In the Neumann case ∂V/∂n is set to zero at Γ. Substituting this in 28, we have:

$$\dot{V} = 0 \qquad x \in \Gamma \qquad 34$$

In other words, motion cannot proceed towards Γ; hence the forbidden regions will not be entered.

VII Simulation Results

In this section, the tracking and region avoidance capabilities of the wave potential approach are tested. Three types of partial differential relations are used to govern the point properties of the potential field:

1-The Lpalace Equation

$$\nabla^2 V = 0, \tag{35}$$

applied in a quasi stationary manner with the ODE system:

$$\dot{x} = -\nabla V. \tag{36}$$

2-The Diffusion Equation $$\nabla^2 V = \frac{1}{a^2}\frac{\partial V}{\partial t}, \tag{37}$$

the same ODE system in 36 is used.

3- The suggested Wave Equation approach.

The results from the three approaches are compared for a stationary, linearly moving, and slowly moving targets . It was observed that all techniques exhibit equivalent capabilities in terms of converging to the target and avoiding the forbidden regions. However, the disparity between the performances of the above techniques greatly widened when the linear motion of the target was augmented with rapid, sinusoidal oscillation. Figure-6 shows the response of the quasi stationary Laplace approach. As can be seen, the rapid fluctuations which the target superimposed along its escape path confused the tracker leaving it undecided whether to exit the corridor from the right side or the left side (figure-7). The total failure of the quasi-stationary Laplace strategy is a consequence of its total disregard to the temporal dependence of events and its reliance on spatial information only in guiding the tracker to the target. The absence of the dimension of time from the strategy of the interceptor opens a "hatch" for the target through which this neglected dimension is exploited not only to evade capture by the interceptor, but even to totally paralyze it. Here, the target is aware that the interceptor is tracking it along the minimum distance path with no regard to where the target is moving to next, or how fast it is going there. Therefore, once the target observes a move of the tracker, it quickly repositions itself so that the distance of the interceptor to the new location of the target is shorter if the interceptor moves along a direction that is opposite to the one it has. By continuously repeating this relatively simple maneuver, the target is able to bring the motion of the interceptor to a standstill along the x-axis trapping it in a limit cycle along the y-axis (figure-7). Although the ODE system for the Diffusion strategy uses only spatial information to lay an interception trajectory to the target, temporal information is indirectly utilized in encoding the target and environment information in the potential field. For this case (figure-8), despite the ability of the tracker to proceed in the general direction of the target, it fails to keep up with its rapid fluctuating movements.

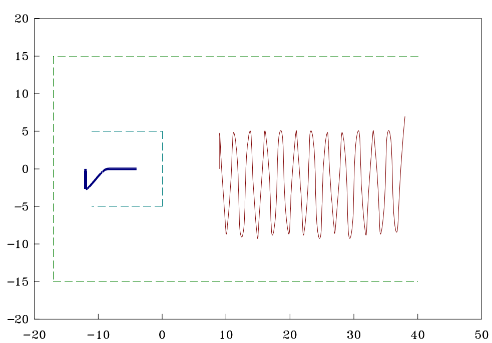

Figure-6: Response of the tracker, Lapalace.

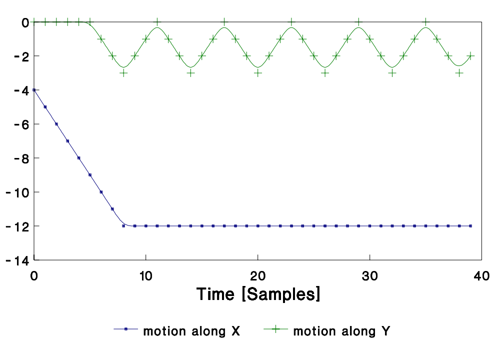

Figure-7: x,y components of trajectory vs time, Lapalce.

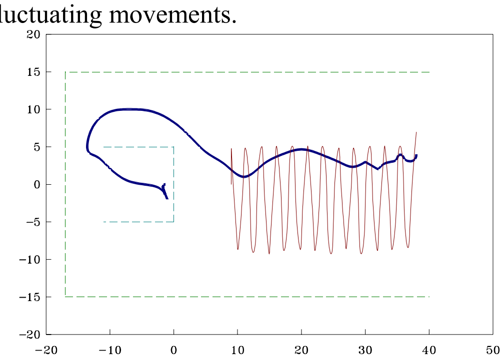

Figure-8: Response of the tracker, Diffusion .

As for the Wave Equation strategy, the interceptor was able to closely follow the target (figure-9).

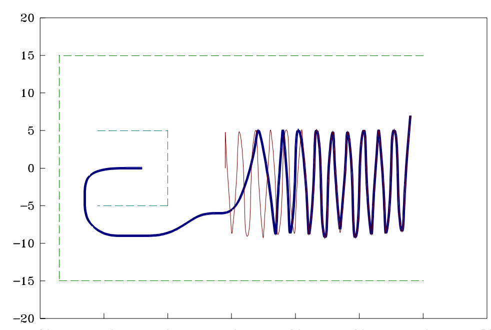

Figure-9: Response of the tracker, Wave.

VIII Conclusion

This paper focuses on a special class of pursuit-evasion problems concerned with intercepting a well-informed, active, intelligent target that is maneuvering to escape capture in a known, stationary, cluttered environment. The paper examines the possibility of the existence of a tracking method that is not dependent on the psychological profile of the prey or the manner it may utilize the knowledge it has to evade capture. It seems that the suggested wave potential-based controller has the ability to exhibit such objectivity in goal-oriented, action synthesis. There are still many questions that need to be answered about the behavior of the suggested approach. For example, the ability of the tracker to always intercept the target regardless of the amount of information which the target has, or the efficiency of its utilization needs to be carefully examined. Another question is concerned with the effect of placing dynamical constraints on the tracker's ability to intercept a target. Another issue has to do with the effect of delays caused by the potential field synthesis process on the interception ability of the pursuer. These are some of the important questions which future investigations of this approach should carefully answer.

Acknowledgment

The author thanks KFUPM for its support of this work.

Appendix

A. *Definition*: Let V(X) be a smooth (at least twice differentiable), scalar function ($V(X): R^N \rightarrow R$). A point Xo is called a critical point of V if the gradient vanishes at that point ($\nabla V(Xo)=0$); otherwise, Xo is regular. A critical point is Morse, if its Hessian matrix (H(Xo)) is nonsingular. V(X) is Morse if all its critical points are Morse [12].

B. *Proposition*: If V(X) is a harmonic function defined in an N-dimensional space (R^N) on an open set Ω, then the Hessian matrix at every critical point of V is nonsingular, i.e. V is Morse.

*Proof***:** There are two properties of harmonic functions that are used in the proof:

1- a harmonic function (V(X)) defined on an open set Ω contains no maxima or minima, local or global in Ω. An extrema of V(X) can only occur at the boundary of Ω,

2- if V(X) is constant in any open subset of Ω, then it is constant for all Ω.

Other properties of harmonic functions may be found in [13].

Let Xo be a critical point of V(X) inside Ω. Since no maxima or minima of V exist inside Ω, Xo has to be a saddle point. Let V(X) be represented in the neighborhood of Xo using a second order Taylor series expansion:

$$V(X) = V(Xo) + \nabla V(Xo)^T (X - Xo) + \frac{1}{2}(X - Xo)^T H(Xo)(X - Xo) \qquad |X\text{-}X0| << 1. \qquad 38$$

Since Xo is a critical point of V, we have:

$$V' = V(X) - V(Xo) = \frac{1}{2}(X - Xo)^T H(Xo)(X - Xo) \qquad |X\text{-}X0| << 1. \qquad 39$$

Notice that adding or subtracting a constant from a harmonic function yields another harmonic function , i.e. V\` is also harmonic. Using eigenvalue decomposition:

$$V' = \frac{1}{2}(X - Xo)^T U^T \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & . & 0 \\ . & . & . & . \\ 0 & 0 & . & \lambda_N \end{bmatrix} U(X - Xo) \qquad 40$$

$$= \frac{1}{2}\xi^T \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & . & 0 \\ . & . & . & . \\ 0 & 0 & . & \lambda_N \end{bmatrix} \xi \quad = \quad \frac{\mathbf{1}}{\mathbf{2}}\sum_{\mathbf{i=1}}^{\mathbf{N}} \lambda_{\mathbf{i}} \xi_{\mathbf{i}}^2$$

where U is an orthonormal matrix of eigenvectors, λ's are the eigenvalues of H(Xo), and $\xi=[\xi_1\ \xi_2\ ..\xi_N]^T = U(X\text{-}Xo)$. Since V\` is harmonic, it cannot be zero on any open subset Ω; otherwise, it will be zero for all Ω, which is not the case. This can only be true if and only if all the λ_i's are nonzero. In other words, the Hessian of V at a critical point Xo is nonsingular. This makes the harmonic function V also a Morse function.

It ought to be mentioned that a navigation function defined in [14] is a special case of a harmonic potential field. According to [14] a navigation function must satisfy the following properties:

1- it is smooth (at lest C^2),
2- it contains only one minimum located at the target point,
3- it is a Morse function,
4- it is maximal and constant on Γ.

A harmonic function (V) is C^∞ and Morse. Harmonic functions are extrema-free in Ω. Their maxima and minima can only happen at the boundary of Ω. In the harmonic approach Γ and the target point (X_T) are treated as the bounary of Ω. Through applying the appropriate boundary conditions, the minimum of V is forced to occur on X_T. Also by the application of the Drichlet boundary conditions, the value of V is forced to be maximal and constant at Γ. The Drichlet condition (constant potential on the boundary) is one of many settings used in constructing a harmonic potential that may be used for navigation.

References

[1]A. Masoud, "An Informationally-Open, Organizationally-Closed Control Structure for Navigating a Robot in an Unknown, Stationary Environment" 2003 IEEE International Symposium on Intelligent Control, October 5-8, 2003, Houston, Texas, USA, pp. 614-619. .

[2] V. Lefebvre,"The Structure of Awareness: Toward a Symbolic Language of Human Reflexion", Volume 41, Sage Library of Social Research, Sage Publications, Beverly Hills, London, 1977.

[3] E. Rodin,"A Pursuit-Evasion Bibliography-Version 2", Computers and Mathematics with Applications, Vol. 18, No. 1-3, 1989, pp. 245-320.

[4] R. Thom, "Structural Stability and Morphogenesis", W. A. Benjamin Inc., 1975.

[5] C. Langton,"Artificial Life", Artificial Life, SFI Studies in the Sciences of Complexity, Ed. C. Langton, Addison-Wesley, 1988, pp. 1-47.

[6] J. Cooper, C. Bardos, "A nonlinear Wave Equation in a Time-Dependent Domain", Journal of Mathematical Analysis and Applications, Vol. 42, No. 1, April 1973, pp. 29-60.

[7] K. LEE, "A Mixed Problem for Hyperbolic Equations with Time-Dependent Domaini", Journal of Mathematical Analysis and Applications, Vol. 16, 1966, pp. 455-471.

[8] A. Kolmogrov, S. Fomin, "Elements of the Theory of Functions and Functional Analysis", Graylock Press, Rochester, New York, 1963.

[9] M. Vidyasagar, "Nonlinear System Analysis", Prentice-Hall Inc., Englewood Cliffs, N. J. 1978.

[10] R. McOwen, "Partial Differential Equations: Methods and Applications", Prentice Hall, 1996.

[11] S. Masoud, A. Masoud, " Motion Planning in the Presence of Directional and Obstacle Avoidance Constraints Using Nonlinear Anisotropic, Harmonic Potential Fields: A Physical Metaphor", IEEE Transactions on Systems, Man, & Cybernetics, Part A: systems and humans, Vol 32, No. 6, November 2002, pp. 705-723.

[12] J. Milnor, "Morse Theory", Princeton University Press, 1963.

[13] S. Axler, P. Bourdon, W. Ramey, "Harmonic Function Theory", Springer, 1992.

[14] E. Rimon and D. Koditschek. Exact robot navigation using artificial potential fields. IEEE Trans. Robot. & Autom., 8(5):501-518, October 1992.