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# Improving Transient Performance in Tracking Control for a Class of Nonlinear Discrete-time Systems with Input Saturation

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*Abstract*— **This paper addresses a target tracking control problem for a class of SISO discrete-time partially linear composite systems with actuator saturation. Quick response and small overshoot are two desired transient performances of target tracking control. While most of the design schemes compromise between these two performances, we try to achieve both simultaneously by using a composite nonlinear feedback (CNF) control technique. The CNF control, consisting of a linear feedback law and a nonlinear feedback law, aims to improve the transient performance of the closed-loop systems. The linear feedback part is designed to yield a closed-loop system with a small damping rate for quick response, while the nonlinear feedback part is used to increase the damping ratio of the closed-loop system as the system output approaches the target reference to reduce the overshoot. We also show that the closed-loop system with improved transient performance preserves the stability of the nonlinear part of the partially linear composite system.**

## I. Introduction

The tracking control problems, such as target tracking [4] and output regulation [8], are extensively studied in the literature. Settling time and overshoot are two important transient performance indices, and quick response and small overshoot are desirable in the most of the target tracking control problems. However, it is well known that, in general, quick response results in a large overshoot. Thus, most of the design schemes have to make a trade-off between these two transient performance indices. To improve the transient performance, Lin *et al.* [14] proposed a composite nonlinear feedback (CNF) control technique for a class of second order linear systems. The CNF control law consists of a linear feedback law and a nonlinear feedback law without any switching element. The linear feedback part is designed to yield a closed-loop system with a small damping rate for quick response, while the nonlinear feedback part is used to increase the damping ratio of the closed-loop system as the system output approaches the target reference to reduce the overshoot. Turner *et al.* [20] later extended the results of [14] to higher order and multiple input systems under a restrictive assumption on the system. However, both [14] and [20] considered only the state feedback case. Recently, Chen *et al.* [2] have developed a CNF control design to a more general class of systems with measurement feedback, and successfully applied the technique to solve a hard disk servo problem. The extension of this idea to general linear continuous MIMO systems is found in a recent paper [6]. The CNF control techniques for linear discrete-time systems can be found in [7] and [21].

This work aims to design a CNF control law for discrete-time partially linear composite systems with input saturation. The results for its continuous-time counterpart have been reported in [12]. In recent two decades, the nonlinear control problems for partially linear composite systems have been extensively studied by many researchers such as [9], [10], [11], [15], [17] and [18], to name just a few. It was shown in [17] that a nonlinear system which is zero input globally asymptotically stable (GAS) will preserve its GAS property if its input decreases to zero with a very fast exponential rate. However a bad transient performance may destroy the stability of the nonlinear part before the output rapidly decays to zero. This is also true for discrete-time systems since the inter-sampling behavior is equivalent to the response of a continuous-time system with unchanging input. Based on the linear part of the composite system, the CNF control is designed such that the closed-loop system has desired performances, e.g., quick response and small overshoot. Moreover, we show that the closed-loop system with improved transient performance preserves the stability of the nonlinear part of the partially linear composite system.

## II. Problem Formulation and Preliminaries

Consider a partially linear composite discrete-time systems with input saturation, $\Sigma$, characterized by

$$\begin{aligned} \xi(k+1) &= f(\xi(k), y(k)), \quad \xi(0) = \xi_0 && (1)\\ x(k+1) &= Ax(k) + B\mathrm{sat}(u(k)), \quad x(0) = x_0 && (2)\\ y(k) &= Cx(k) && (3) \end{aligned}$$

where $(\xi, x) \in \mathbb{R}^m \times \mathbb{R}^n$, $u \in \mathbb{R}$ and $y \in \mathbb{R}$ are respectively the state, control input and control output of the given system $\Sigma$, $f$ is a $C^1$ function, $A$, $B$, $C$ are appropriate dimensional constant matrices, and the saturation function is defined by

$$\mathrm{sat}(u) = \mathrm{sign}(u)\ \min(|u|,\ u_{\max}), \tag{4}$$

where $u_{\max}$ is the maximum amplitude of the control channel.

Our aim is to design a certain controller, with all the state information known, which renders the whole (closed-loop) system track a step function with amplitude of $r$ under the input constraint. Without loss of generality, we assume $f(0, r) = 0$. In fact, if $f(\xi^*, r) = 0$ with $\xi^* \neq 0$, the state transformation $\tilde{\xi} = \xi - \xi^*$ gives

$$\dot{\tilde{\xi}} = f(\tilde{\xi} + \xi^*, r) := \tilde{f}(\tilde{\xi}, r)$$

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then, we have $\tilde{f}(0,r) = 0$. For tracking purpose, the following assumptions on the given system are required:

**A1:** $(A,B)$ is controllable;

**A2:** $(A,B,C)$ is invertible and has no invariant zeros at $z=1$; and

**A3:** There exists a $C^1$ positive definite function $V_\xi(\cdot)$ and class $K_\infty$ functions $\alpha_1$ and $\alpha_2$ such that

$$\alpha_1(||\xi(k)||) \leq V_\xi(\xi(k)) \leq \alpha_2(||\xi(k)||), \tag{5}$$

$$V_\xi(\xi(k+1)) - V_\xi(\xi(k)) < 0, \tag{6}$$

where $\xi(k) \in \mathbb{R}^m$ is the solution of

$$\xi(k+1) = f(\xi(k), r), \quad \xi(0) = \xi_0. \tag{7}$$

*Remark 2.1:* Our objective here is to design control laws that are capable of achieving fast tracking of target references under input saturation. As such, it is well understood in the literature that assumptions **A1**-**A2** are standard and necessary. Assumption **A3** is to ensure that the nonlinear system (1) is asymptotically stable when the system output $y$ tracks exactly the step command input $r$.

*Lemma 2.1:* Consider the nonlinear control system of the form

$$\xi(k+1) = f(\xi(k), r + \eta(k)), \tag{8}$$

which satisfies Assumption **A3**. Given any $\gamma > 0$ and $0 < a < 1$, there exists a scalar $\beta > 0$ such that for any

$$|\eta(k)| \leq \beta \cdot a^k, \quad k \geq 0, \tag{9}$$

the solution $\xi(k)$ of (8) exists and is bounded for all $k \geq 0$ provided that $\xi(0) \in \Omega_\gamma := \{\xi \ : \ ||\xi|| \leq \gamma\}$.

*Remark 2.2:* In Sussmann and Kokotović [17], they say $a$ is good for $(\gamma,\beta)$ if $a$ satisfies Lemma 2.1. Here, however, we propose a similar discrete-time version. In fact, our lemma considers when $\gamma$ and $a$ are given, there indeed exists a $\beta$ such that $a$ is good for $(\gamma,\beta)$.

## III. Design of the CNF Control Law

In this section, we proceed to develop a composite nonlinear feedback (CNF) control technique for the case when all the state variables of the linear part of the plant $\Sigma$ are measurable. The design will be done in four steps described in the following step-by-step design procedure which is a natural extension of the results of [2].

Step s.1: Design a linear feedback law,

$$u_{\mathrm{L}}(k) = Fx(k) + Gr, \tag{10}$$

where $r \in \mathbb{R}$ is the step reference. The state feedback gain matrix $F \in \mathbb{R}^{1\times n}$ is chosen such that

1) the output of the closed-loop system (2) and (3) under the state feedback $u = Fx$ is such that $A + BF$ is Schur,
2) the resulting closed-loop system transfer matrix, *i.e.*, $C(zI - A - BF)^{-1}B$, has certain desired properties, *e.g.*, having a small dominating damping ratio.

Let $G$ be a scalar constant and is given by

$$G := [C(I - A - BF)^{-1}B]^{-1}. \tag{11}$$

Here we note that $G$ is well defined because $A+BF$ is stable, and $(A,B,C)$ is right invertible and has no invariant zeros at $z=1$, which implies $(A+BF,B,C)$ is right invertible and has no invariant zeros at $z=1$ (see *e.g.*, Theorem 3.8.1 of Chen *et al.* [3]).

Step s.2: Next, we compute

$$H := \left[I + F(I - A - BF)^{-1}B\right]G \tag{12}$$

and

$$x_{\mathrm{e}} := G_{\mathrm{e}}\, r := (I - A - BF)^{-1}BG\, r. \tag{13}$$

Note that the definitions of $H$, $G_{\mathrm{e}}$ and $x_{\mathrm{e}}$ would become transparent later in our derivation. Given a positive definite matrix $W \in \mathbb{R}^{n\times n}$, solve the following Lyapunov equation:

$$P = (A+BF)'P(A+BF) + W, \tag{14}$$

for $P > 0$. Such a $P$ exists since $A + BF$ is asymptotically stable. Then, the nonlinear feedback control law $u_{\mathrm{N}}$ is given by

$$u_{\mathrm{N}}(k) = \rho(r,y)B'P(A+BF)(x(k) - x_{\mathrm{e}}), \tag{15}$$

where $\rho(r,y)$ is some nonpositive function, locally Lipschitz in $y$, which is used to change the closed-loop system damping ratio as the output approaches the target. The choice of this nonlinear function will be discussed at the end of this section.

Step s.3: Given $\gamma > 0$ and $a = \sqrt{1 - \frac{\lambda_{\min}(W)}{\lambda_{\max}(P)}}$, select $\beta$ such that $a$ is good for $(\gamma,\beta)$.

Step s.4: The linear and nonlinear feedback laws derived in the previous steps are now combined to form a CNF controller:

$$\begin{aligned} u(k) &= u_{\mathrm{L}}(k) + u_{\mathrm{N}}(k) \\ &= Fx(k) + Gr + \\ &\quad \rho(r,y)B'P(A+BF)(x(k) - x_{\mathrm{e}}). \end{aligned} \tag{16}$$

This completes the design of the CNF controller.

*Theorem 3.1: Consider the given system (1) to (3) satisfying assumptions A1 to A3. Define*

$$\mathcal{N} := \left\{ x \in \mathbb{R}^n : ||x|| \leq \frac{\beta}{\|C\|}\left(\frac{\lambda_{\min}(P)}{\lambda_{\max}(P)}\right)^{1/2} \right\} \tag{17}$$

*For any $\delta \in (0,1)$, let $c_\delta > 0$ be the largest positive scalar satisfying the following property:*

$$|\ Fx\ | \leq (1-\delta)u_{\max}, \tag{18}$$

*for all $x \in \boldsymbol{X}_\delta$, where*

$$\boldsymbol{X}_\delta := \Big\{x : x'Px \leq c_\delta, x \in \mathcal{N}\Big\}. \tag{19}$$

*Then, for any nonpositive function $\rho(r, y)$, locally Lipschitz in $y$ and $|\rho(r, y)| \leq \rho^* := 2(B'PB)^{-1}$, the solution of the closed-loop system under the CNF control law (16) exists and is bounded for all $k \geq 0$, provided that the initial state $x_0 = x(0)$ and $r$ satisfy:*

$$\tilde{x}_0 = \tilde{x}(0) := (x_0 - x_{\mathrm{e}}) \in \boldsymbol{X}_\delta, \quad |H\, r| \leq \delta u_{\max}. \tag{20}$$

*Moreover, the system output $y$ tracks asymptotically the step command input of amplitude $r$.*

**Proof.** The closed-loop system compromising the given plant (1)–(3) and the CNF control law (16) is given by

$$\begin{aligned} \xi(k+1) &= f(\xi(k), y(k)) && (21)\\ x(k+1) &= Ax(k) + B\mathrm{sat}(u(k)) && (22)\\ y(k) &= Cx(k). && (23) \end{aligned}$$

where

$$u(k) = Fx(k) + Gr + \rho(r, y)B'P(A + BF)(x(k) - x_{\mathrm{e}})$$

Let $\tilde{x}(k) = x(k) - x_{\mathrm{e}}$. The closed-loop system (21)-(23) can be expressed as

$$\begin{aligned} \xi(k+1) &= f(\xi(k), r + C\tilde{x}(k)) && (24)\\ \tilde{x}(k+1) &= (A + BF)\tilde{x}(k) + Bw, && (25) \end{aligned}$$

where

$$\begin{aligned} w &= \mathrm{sat}(F\tilde{x}(k) + Hr + \rho(r, y)B'P(A + BF)\tilde{x}(k))\\ &\quad -F\tilde{x}(k) - Hr. \end{aligned}$$

Define a Lyapunov function $V(\tilde{x}) = \tilde{x}'P\tilde{x}$, then we have

$$\lambda_{\min}(P)||\tilde{x}||^2 \leq V(\tilde{x}) \leq \lambda_{\max}(P)||\tilde{x}||^2 \tag{26}$$

where $\lambda_{\min}(P)$ and $\lambda_{\max}(P)$ are the minimal and maximal eigenvalues of $P$, respectively. Then,

$$\begin{aligned} \triangle V(\tilde{x}(k)) &= V(\tilde{x}(k+1)) - V(\tilde{x}(k))\\ &= -\tilde{x}'(k)W\tilde{x}(k) + w'(k)B'PBw(k) +\\ &\quad 2\tilde{x}'(k)(A + BF)'PBw(k). \end{aligned}$$

It has been shown in [21] that,

$$2\tilde{x}'(k)(A + BF)'PBw(k) + w'(k)B'PBw(k) \leq 0, \tag{27}$$

for all $\tilde{x} \in \boldsymbol{X}_\delta$, $|Hr| \leq \delta u_{\max}$ and $-\rho^* \leq \rho(r, y) \leq 0$. Thus

$$\triangle V(\tilde{x}(k)) \leq -\tilde{x}'(k)W\tilde{x}(k) \leq 0, \tag{28}$$

which implies that $\boldsymbol{X}_\delta$ is an invariant set of the closed-loop system in (25). Thus the solution of (25) exists and is bounded for all $k \geq 0$ and $\tilde{x}_0 \in \boldsymbol{X}_\delta$. Nothing that $x(k) = x_{\mathrm{e}} + \tilde{x}(k)$, $x(k)$ exists and is bounded for all $k \geq 0$ and $x_0$ satisfies (20).

To show the existence and boundedness of the solution $\xi$ of (21), it suffices to show that

$$||\tilde{y}(k)|| = ||y(k) - Cx_e|| = ||C\tilde{x}(k)|| \leq \beta \cdot a^k$$

To this end, by recalling a lemma from [16], page 447, Lemma 13.2, we have $0 < \lambda_{\min}(W) \leq \lambda_{\max}(P)$ and $V(\tilde{x}(k+1)) \leq \varrho \cdot V(\tilde{x}(k))$ where $\varrho = 1 - \frac{\lambda_{\min}(W)}{\lambda_{\max}(P)}$.

Therefore, we get

$$V(\tilde{x}(k+1)) \leq \varrho^k \cdot V(\tilde{x}(0)) \tag{29}$$

and then

$$\lambda_{\min}(P)||\tilde{x}(k+1)||^2 \leq \varrho^k \cdot V(\tilde{x}(0)) \tag{30}$$

so that

$$||\tilde{x}(k+1)|| \leq \left(\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}\right)^{1/2} \cdot ||\tilde{x}(0)|| \cdot (\sqrt{\varrho})^k$$

for all $k \geq 0$. Finally, note that $a = \sqrt{\varrho}$,

$$||\tilde{y}(k+1)|| = ||C\tilde{x}(k+1)|| \leq \beta \cdot a^k,$$

for all $\tilde{x}(0) \in \boldsymbol{X}_\delta$. By Lemma 2.1, the solution of (21) exists and is bounded for all $k \geq 0$.

Moreover, noting that $W > 0$, all trajectories of (25) starting from inside $\boldsymbol{X}_\delta$ will converge to the origin. This, in turn, indicates that, for all initial state $x_0$ and the step command input $r$ that satisfy (20), we have

$$\lim_{k\to\infty} x(k) = x_{\mathrm{e}}, \tag{31}$$

Hence,

$$\lim_{k\to\infty} y(k) = C \lim_{k\to\infty} x(k) = Cx_{\mathrm{e}} = r.$$

This completes the proof of Theorem 3.1. ∎

Next, we note that the key component in designing the CNF controllers is the selection of $\rho$ and $W$. The freedom in choosing the nonlinear function $\rho$ is used to tune the control laws so as to improve the performance of the closed-loop system as the controlled output $y$ approaches the target reference. Since the main purpose of adding the nonlinear part to the CNF controller is to speed up the settling time and to reduce the overshoot, or equivalently to contribute a significant value to the control input when the tracking error, $r - y$, is small, it is appropriate for us to select a nonlinear gain function such that the nonlinear part will be in action when the control signal is far away from its saturation level, and thus it will not cause the control input to hit its limits. Under such a circumstance, it is straightforward to verify that the closed-loop system comprising the linear part of the plant, i.e., (2), and the CNF control law (16) can be expressed as

$$\tilde{x}(k+1) = (A + BF)\tilde{x}(k) + \rho BB'P(A + BF)\tilde{x}(k). \tag{32}$$

It is clear that eigenvalues of the closed-loop system in (32) can be changed by the nonlinear function $\rho$. Assuming that $y(0) \neq r$ (for the trivial case when $y = r$, there is no need to add any nonlinear gain to the control), we propose the following nonlinear gain

$$\rho(r, y) = \frac{-\kappa_1(B'PB)^{-1}}{|y(0) - r|^{\kappa_2}}\Big||y(k) - r|^{\kappa_2} - |y(0) - r|^{\kappa_2}\Big|, \tag{33}$$

with

$$0 \leq \kappa_1 \leq 1$$

which starts from 0 and gradually increases to a final gain of $-\kappa_1(B'PB)^{-1}$ as $y$ approaches to the target reference $r$. The parameter $\kappa_2$ is used to determine the speed of change in $\rho$. It can be shown that the closed-loop poles of (32) are related to the invariant zeros of an auxiliary system characterized by

$$\begin{aligned} G_{\rm aux}(z) &:= C_{\rm aux}(zI - A_{\rm aux})^{-1}B_{\rm aux} \\ &:= B'P(zI - A - BF)^{-1}B \end{aligned}$$

which is obviously stable, and which was shown in [7] to be a square, invertible and uniform rank system with one infinite zero of order 1 and with $n-1$ stable invariant zeros. In fact, if we select $\kappa_1 = 1$, the closed-loop poles of (32) in the steady state when $y = r$ are precisely given by the invariant zeros of $G_{\rm aux}(z)$ together with additional one at $z = 0$. Generally, the invariant zeros of $G_{\rm aux}(z)$ can be pre-assigned by the appropriate choice of $W$, which can also be selected using a trial and error approach by limiting it to be in a diagonal matrix and adjusting its diagonal weights through simulation. We refer interested readers to [7] for details.

## IV. Illustrative Examples

*Example 4.1:* (Target tracking problem.) Consider a system characterized by

$$\begin{aligned} \xi_1(k+1) &= \frac{4\xi_1(k)\xi_2(k)}{1+\xi_1^2(k)\xi_2^2(k)}\xi_1(k) && (34) \\ \xi_2(k+1) &= y(k) - r && (35) \\ x(k+1) &= Ax(k) + B\ \mathrm{sat}(u) && (36) \\ y(k) &= Cx(k) && (37) \end{aligned}$$

with $u_{\max} = 1$, where

$$A = \begin{bmatrix} 1 & 0.1 & 0 & 0 & 0 \\ -0.1 & 1 & 0.1 & 0 & 0 \\ 0 & 0 & 1 & 0.1 & 0 \\ 0 & 0 & -0.1 & 1 & 0.1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0.1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}^T$$

The nonlinear part (34)-(35) of the system is taken from [19]. It is shown in [19] that the nonlinear part (34)-(35) is globally exponential stable when the tracking error $y - r = 0$, that is, Assumption A3 is satisfied. However, even the tracking error converges exponentially to zero with

$$y(k) - r = \xi_2(0)(1/2)^{k+1}$$

the nonlinear part of the system is unstable when $\xi_1(0) \neq 0$ and $\xi_2(0) = \xi_1^{-1}(0)$.

The tracking target is a step function with amplitude $r = 0.2$. Our aim is to design an appropriate CNF controller with state feedback to improve the transient performance of the closed-loop system while maintains the exponential stability of the nonlinear part of the system. It is not difficult to verify that Assumptions A1-A2 are also satisfied for the system (34)-(37). A linear feedback control law is firstly designed by using the low gain feedback technique [13]. We obtain a linear control law $u_{\rm L}(k) = Fx(k) + Gr$ with

$$\begin{aligned} F &= [\,0.7851 \quad 0.1370 \quad -0.0432 \quad -4.1191 \quad -5.7906\,] \\ G &= 5.0487. \end{aligned}$$

The nonlinear function $\rho(r, y)$ is chosen as in (33) with $\kappa_1 = 0.25$ and $\kappa_2 = 16$. Finally, the CNF control law is given by

$$u(k) = Fx(k) + Gr + \rho(r, y)B'P(A + BF)(x(k) - x_{\rm e}). \quad (38)$$

where $x_{\rm e} = (I - A - BF)^{-1}BGr$, and $P$ is the positive solution of $P = (A + BF)'P(A + BF) + I_5$. Using Simulink in Matlab, we obtain the simulation result in Figure 1, which is done under the following initial condition $\xi_1(0) = -0.8$, $\xi_2(0) = \xi_1^{-1}(0) = -1.25$ and $x(0) = 0$. The simulation result shows that the control law with the nonlinear components, *i.e.*, the CNF controller, improved the transient performance significantly. Specifically, Figure 1.(a) and 1.(b) show the trajectories of the closed-loop systems under the linear control law and the CNF control law respectively. All the states of the closed-loop system under the CNF control law convergence to the steady state quickly in 15 seconds with much smaller amplitude. However, under the linear control law, more than 45 seconds are required for all the trajectories convergence to the steady state. Figure 1.(c) and 1.(d) compare the system outputs of the closed-loop systems and the control inputs under the linear control and the CNF control respectively. The overshoot under the linear control is 21.93%, but for the CNF control, it is only $0.46\%$.

*Example 4.2:* (Output regulation for RTAC system.) The normalized motion equation of the Rotational/Translatinal Actuator (RTAC) system ( see, e.g., [1], [8]) is given by

$$\begin{aligned} \ddot{\varsigma} + b\dot{\varsigma} + \varsigma &= \epsilon(\dot{\theta}\sin\theta - \ddot{\theta}\cos\theta) && (39) \\ \ddot{\theta} &= -\epsilon\ddot{\varsigma}\cos\theta + \upsilon && (40) \end{aligned}$$

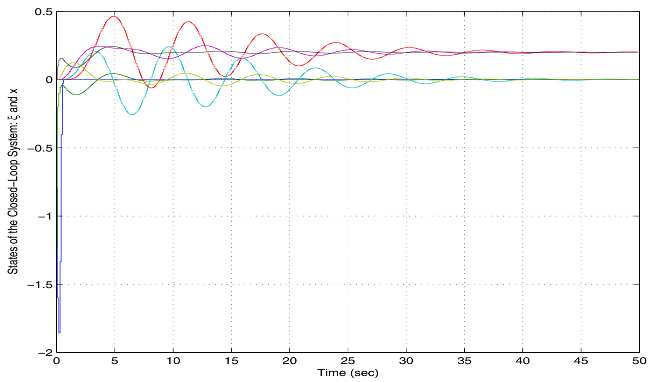
where $\varsigma$ is the normalized displacement of the cart, $\theta$ the angular position of the eccentric mass, $\upsilon$ the normalized control input, $\epsilon$ the coupling between the translational and rotational motion, and $b$ the coefficient of viscous friction for motion of the cart.

Let

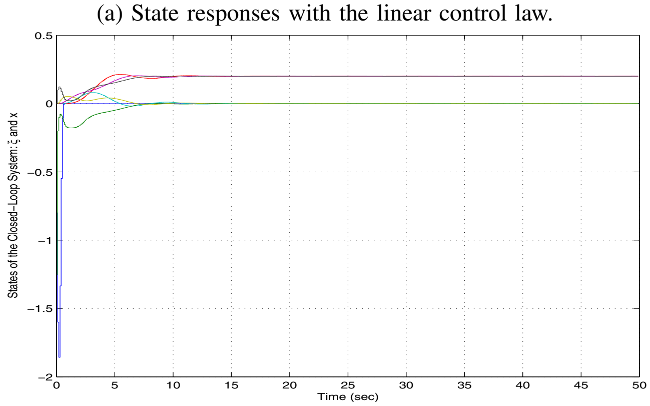
$$y_1 = \theta, \ \ y_2 = \dot{\theta}, \ \ \xi_1 = \varsigma + \epsilon\sin\theta, \ \ \xi_2 = \dot{\varsigma} + \epsilon\dot{\theta}\cos\theta$$

the state space representation of (39) and (40) is given by

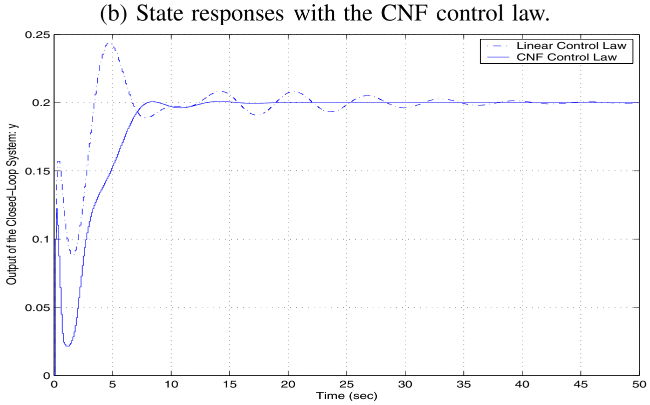
$$\begin{aligned} \dot{\xi}_1 &= \xi_2 && (41) \\ \dot{\xi}_2 &= -\xi_1 + \epsilon\sin y_1 - b(\xi_2 - \epsilon y_2\cos y_1) && (42) \\ \dot{y}_1 &= y_2 && (43) \\ \dot{y}_2 &= u && (44) \end{aligned}$$



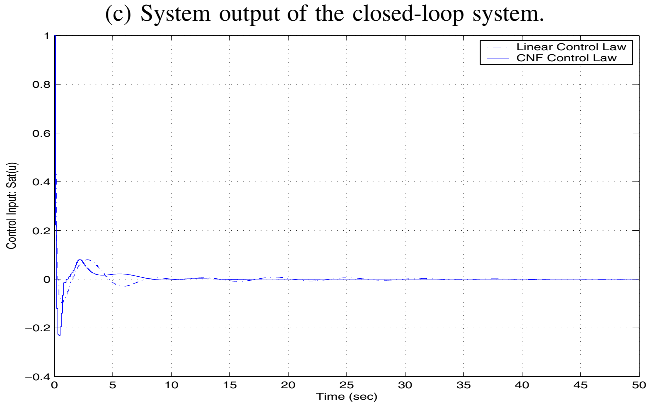

(a) State responses with the linear control law.




(b) State responses with the CNF control law.

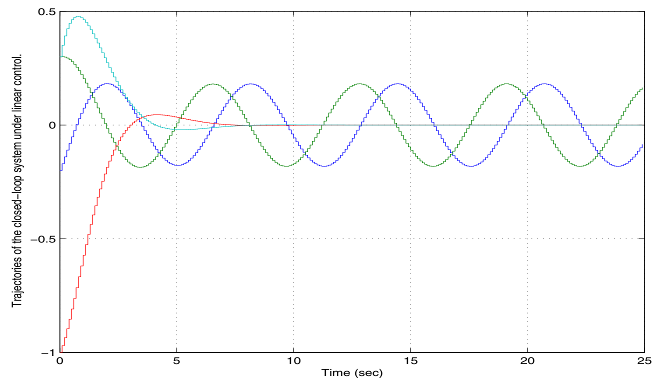



(c) System output of the closed-loop system.

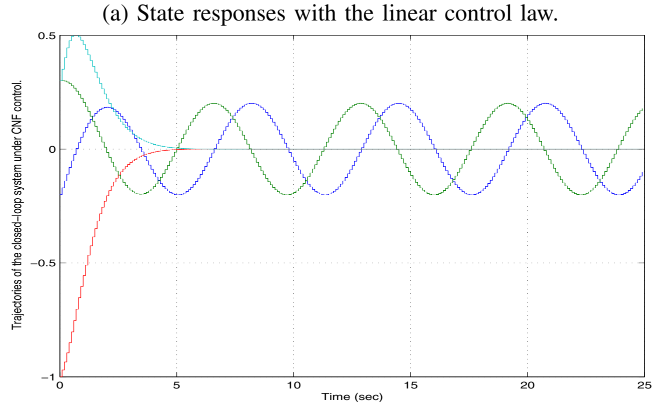



(d) Control input of the closed-loop system.

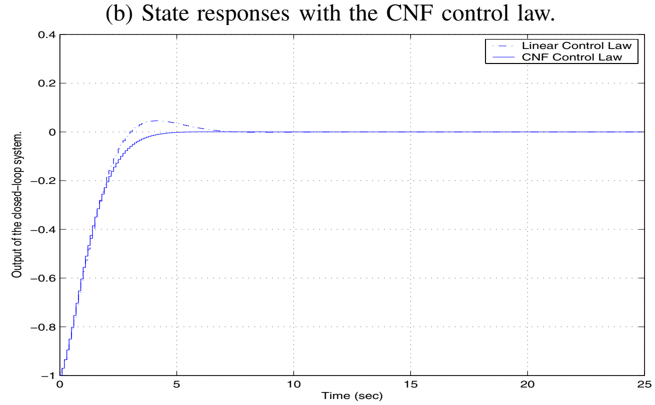
Fig. 1. Target tacking control: Linear Control vs. CNF control.



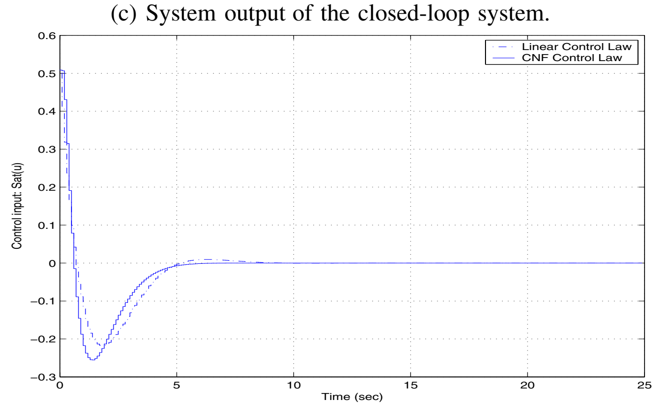

(a) State responses with the linear control law.




(b) State responses with the CNF control law.




(c) System output of the closed-loop system.




(d) Control input of the closed-loop system.

Fig. 2. Output regulation for RTAC system.

where

$$
\begin{aligned}
u &= \frac{\epsilon \cos y_1}{1-\epsilon^2\cos^2 y_1}(\xi_1 - (1+y_2^2)\epsilon \sin y_1) \\
&\quad + \frac{1}{1-\epsilon^2\cos^2 y_1}\upsilon \qquad (45)
\end{aligned}
$$

Discretizing the continuous time model (41)–(44) via Euler's method with $T$ as sampling period, and considering the input saturation, we obtain

$$
\begin{aligned}
\xi_1(k+1) &= \xi_1(k) + T\xi_2(k) && (46)\\
\xi_2(k+1) &= \xi_2(k) - T(\xi_1(k) - \epsilon \sin y_1(k)) \\
&\quad -Tb(\xi_2(k) - \epsilon y_2(k)\cos y_1(k))) && (47)\\
y_1(k+1) &= y_1(k) + Ty_2(k) && (48)\\
y_2(k+1) &= y_2(k) + T\mathrm{sat}(u(k)) && (49)
\end{aligned}
$$

Our objective is to design a CNF control law for the system (46)–(49) to regulate the output $y_1$ to zero quickly without any overshoot. To this end, let $r = 0$, and assume that $T = 0.1$ and $b = 0.1$, $\epsilon = 0.2$, and $u_{\max} = 1$. It is easy to verify that the zero dynamics of the system (46)–(49)

$$
\begin{aligned}
\xi_1(k+1) &= \xi_1(k) + T\xi_2(k) \\
\xi_2(k+1) &= -T\xi_1(k) + (1-Tb)\xi_2(k)
\end{aligned}
$$

is stable, that is, the system (46)–(47) satisfies assumption A3. Moreover, we have

$$
A = \begin{bmatrix} 1 & 0.1 \\ 0 & 1 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix}\ C = \begin{bmatrix} 1 \\ 0 \end{bmatrix}'
$$

then, assumptions A1 and A2 are also satisfied. A linear feedback gain is obtained by using ITAE prototype design [5] which gives

$$
F = [-0.9318 \quad -1.4109] \qquad (50)
$$

Then the CNF control law is given by

$$
u(k) = Fx(k) + \rho(r,y)B'P(A+BF)x(k) \qquad (51)
$$

where $P$ is the positive definite solution of $P = (A + BF)'P(A+BF) + I_2$, and $\rho(r,y)$ is given by (33) with $y = y_1$, $\kappa_1 = 0.1$ and $\kappa_2 = 5$. Let the initial conditions $\xi_1(0) = -0.2$, $\xi_2(0) = 0.3$, $y_1(0) = -1$ and $y_2(0) = 0.3$, the simulation result is shown in Figure 2. Under the ITAE linear control law, the output of the closed-loop system is regulated to zero with visible overshoot. However, the CNF control law regulates the output of the closed-loop system to zero without any overshoot. Moreover, the rise time under the CNF control law is almost the same as the one under linear control law.

## V. Conclusions

We have extended the so-called CNF control techniques for linear input-saturated discrete-time systems to a class of SISO partially linear composite discrete-time systems with actuator saturation. The closed-loop system is able to track step function signals yet the whole system is stable. It has been shown that the transient performance is improved comparing to normal linear approaches. Both CNF and linear controllers avoid adverse effect of peaking-phenomenon. Further extension to MIMO case can be established similarly by provoking the results of CNF control for linear MIMO discrete-time systems (see [7]).

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