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A systematic approach to the design of robust diagonal dominance based MIMO controllers

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***Abstract*— A new method, based on the use of LMIs, is presented for the design of dynamic pre-compensators to achieve high levels of diagonal dominance. This is combined with a new technique for designing diagonal loop-shaping controllers such that the resulting closed loop system will be guaranteed to satisfy some prescribed mixed-sensitivity constraints on its singular values. The effectiveness of the combined methods is illustrated by application to the design of a simply structure dominance-based controller for an aircraft gas-turbine engine.**

***Index Terms*— Diagonal dominance, SVD analysis, loop-shaping, linear multivariable controller design**

I. Introduction

THE classical techniques of frequency domain design for single-input single-output systems have been generalized and applied to multivariable feedback systems by Rosenbrock [1], his co-workers Hawkins [2] and Munro [3], and separately by MacFarlane and his co-worker Kouvaritakis [4]. Rosenbrock's Nyquist array design method consists essentially of determining a multivariable pre-compensator matrix K, constant or dynamic, such that the resulting forward path transfer function $G(s) \times K$ is diagonal dominant. When this condition is fulfilled, a set of single loop compensators are designed to meet the over all closed-loop specifications.

There have been various attempts to formulate a systematic technique for designing such pre-compensators. The pseudo-diagonalisation technique by Hawkins [2], the ALIGN algorithm by MacFarlane and Kouvaritakis [4], and a scaling algorithm by Edmunds [5], are examples of some of the widely used techniques for the design of pre-compensators. More recently novel methods have been proposed for the design of the precompensator, which include the LMI method of Chughtai and Munro [6] and the Evolutionary optimisation approach of Nobakhti et al [7].

However, due to the structure of these being static, there is a limitation on the performance they are able to achieve, and it was recognised that systematic tools were needed for the effective design of dynamic pre-compensators. In this area, there has been less success due to the added complexity of the problem. Examples of some of the techniques developed are that of Chen and Munro [8] for a PID based compensator, the technique of Ford and Daly [9] for dynamic pseudo-diagonalisation, and that of Bryant and Yeung [10]. A common problem with all of these techniques is that the structure of the pre-compensator has to be fixed before hand and this means that either the pre-compensator will have less than sufficient dynamics in some elements, or unneeded dynamics on some others. For more details on this problem see [11]. Recently, a technique was proposed by Nobakhti and Munro [12] for the design of a dynamic pre-compensator where the structure of the pre-compensator would only emerge at the end of the design process, and there are no restrictions on the dynamical form and structure of each pre-compensator element.

In this paper we propose an LMI formulation for the design of dynamic preocmpensators which is able to achieve systematically high amounts of diagonal dominance. Once the proposed LMI technique has been used to design the pre-compensator, the diagonal-dominance controller design technique will then proceeds to the design of a set of n single loop controllers. Until recently, the design of these controllers was carried out purely on a 'single-loop' basis; that is, the 'multivariable' aspect of the problem is ignored and only once the design of these controllers was finished, could the over all 'multivariable' properties of the closed loop system be assessed. Thus, one was unable to design the single loops controller to address issues such as robustness. Recently, a technique was proposed [13] which allowed the diagonal dominance controller to be designed with robustness in mind.

In this work, alternative tighter bound are proposed and the technique is extended to deal with mixed-sensitivity specifications. Together with the LMI approach to design the precompensators, the combination makes a powerful and systemcatic approach to the design of diagonal dominance based robust multivariable controllers.

II. LMI Formulation of the dynamic pre-compensator

A simple and most effective precompensator to achieve dominance would be the inverse of the system. However for the non-minimum phase systems or systems with large pole excess this is might not be possible. However, under these conditions we can proceed in two steps:

Step 1.Find a dominant term of $G(s)$ which is minimum phase and proper.

Step 2.Invert the dominant term.

To achieve first step we can use the Fundamental dominance Criteria defined as [10],

Definition 1: Fundamental Dominance.
Let $A \in \mathbf{C}^{n\times n}$ be a complex matrix, and let $A = B + D$,

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where $B \in \mathbf{C}^{n\times n}$ and $D \in \mathbf{C}^{n\times n}$, such that B is non-singular. Then, the term B is said to be the dominant term of A if and only if,$\rho(B^{-1}D) < 1$
Let $G(s)$ be a transfer function matrix (TFM)with splitting given as:

$$G(s) = X(s) + Y(s) \tag{1}$$

such that,

$$\rho(X(s)^{-1}Y(s)) < 1 \tag{2}$$

Then, from Step 2. the required pre-compensator will be$K(s) = X(s)^{-1}$

Since $\rho(.) < \bar{\sigma}(.)$, condition 2 for $G(s) \in \mathbf{C}^{n\times n}$ will be given as,

$$\parallel G(s)K(s) - I \parallel_\infty < 1 \tag{3}$$

Remark 1: Indeed 3 is a standard open loop model matching problem in H_∞ frame work with I as reference model. But this is what we were aiming for. As Rosenbrock's diagonal dominance requires compensated system to be diagonal. The only drawback of 3 is its conservatism. However, this can be relaxed by replacing I with a suitable matrix as presented in [6].
Since, the minimisation of ∞-norm of a system can be presented in the form of LMIs using the well known bounded real lemma.

A. Linearisation by change of Variables.

Let the system $G(s)$ is given in state space form as,

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned} \tag{4}$$

If $K(s)$ is given as:

$$\begin{aligned} \dot{x}_k &= A_k x_k + B_k u \\ v &= C_k x_k + D_k u \end{aligned} \tag{5}$$

Then, the $G(s)K(s)$ will be given as,

$$\begin{aligned} \dot{x}_{gk} &= \begin{bmatrix} A & BC_k \\ 0 & A_k \end{bmatrix} x_{gk} + \begin{bmatrix} BD_k \\ B_k \end{bmatrix} u \\ y &= \begin{bmatrix} C & DC_k \end{bmatrix} x_{gk} + DD_k u \end{aligned} \tag{6}$$

Which makes (**??**) nonlinear. However, it can linearized by use using a technique outlined in [16], from which the resulting LMI will be,

$$\begin{bmatrix} QA^T + AQ + B\tilde{C}_k + (B\tilde{C}_k)^T & * & * & * \\ \tilde{A}_k + A^T & A^T R + RA & * & * \\ \tilde{D}_k^T B^T & \tilde{B}_k^T & -\gamma I & * \\ CQ + D\tilde{C}_k & C & D\tilde{D}_k - I & -\gamma I \end{bmatrix} < 0$$

$$\text{for} \quad \begin{bmatrix} Q & I \\ I & R \end{bmatrix} > 0 \tag{7}$$

where

$$\begin{aligned} \tilde{A}_k &= RAQ + RBC_k M^T + NA_k M^T, \\ \tilde{B}_k &= (RBD_k + NB_k), \\ \tilde{C}_k &= C_k M^T, \\ \tilde{D}_k &= D_k. \end{aligned} \tag{8}$$

Once the LMIs of (7) are solved the pre-compensator matrices can be obtained by (8).

III. Single-loop controller design

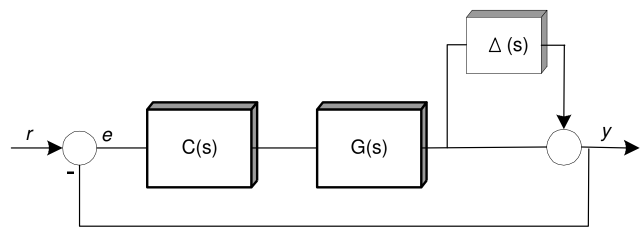

Fig. 1. Closed-loop control system

Consider an uncertain plant as shown in Figure 1. The robust stability and robust performance requirements place constraints on the singular values of open loop plant. In [13] it was shown that for a set of matrices $A = [a_{ij}] \in F$, $\tilde{A} = [a_{ii}] \in F$ and $E = [a_{ij,\ i\neq j}] \in F$, $F \subseteq \mathbf{C}^{n\times n}$, the quantity $D_{max}(A)$ defines an upper bound for the singular values of the matrix, such that the following inclusion region would always hold true for its singular values $\forall x \geq 0$,

$$\{\min(a_{ii}) - D_{max}(A)\} \leq x \leq \{\max(a_{ii}) + D_{max}(A)\} \tag{9}$$

where,

$$D_{max}(A) = \max\{C_{max}(A) \quad , \quad R_{max}(A)\} \tag{10}$$

$$C_{max}(A) = \max_j \sum_{\substack{i=1 \\ i\neq j}}^{m} | a_{ij} | \quad , \quad R_{max}(A) = C_{max}(A^T)$$

In this section we will present a new, tighter, inclusion theorem for singular values of a complex matrix. Which is as follows:

Theorem 1: Let $A = [a_{ij}] \in F$, $\tilde{A} = [a_{ii}] \in F$ and $E = [a_{ij,\ i\neq j}] \in F$ $F \subseteq \mathbf{C}^{n\times n}$. Then $\sqrt{n}C_{max}(A)$ defines an alternative upper singular value bound. In other words,

$$| \sigma_1(A) - \max_i | a_{ii} | | \leq \sqrt{n} C_{max}(A), \tag{11}$$

Proof: From $A = \tilde{A} + E$, it maybe shown that

$$| \parallel A \parallel - \parallel \tilde{A} \parallel | \leq \parallel E \parallel. \tag{12}$$

Choosing the norm above to be the spectral norm, it may be shown that,

$$| \sigma_1(A) - \max_i | a_{ii} | | \leq \parallel E \parallel_2, \tag{13}$$

From which, it is possible to show

$$
\begin{aligned}
\| E \|_2^2 \le tr(E^H E) &= \sum_{i,j=1}^{n} | e_{ij} |^2 \\
&\le \sum_{j=1}^{n} \left[\sum_{i=1}^{n} | e_{ij} | \right]^2 \\
&\le n \left[\max_{\forall j \in \mathbb{N}} \sum_{i=1}^{n} | e_{ij} | \right]^2 \\
&= n \| E \|_1^2.
\end{aligned} \tag{14}
$$

Hence,

$$
\| E \|_2 \le \sqrt{n} \| E \|_1, \tag{15}
$$

where $\| E \|_1$ is the maximum column sum norm, as defined previously. Note, however, that

$$
\begin{aligned}
\| E \|_1 &= \max_{\forall j \in \mathbb{N}} \sum_{i=1}^{n} | e_{ij} | \\
&= \max_{\forall j \in \mathbb{N}} \left[| e_{jj} | + \sum_{\substack{i=1 \\ i \neq j}}^{n} | e_{ij} | \right].
\end{aligned} \tag{16}
$$

Since E consists of the off diagonal terms of A, then $e_{jj} = 0,\ \forall j \in \mathbb{N}$ and for $j \neq i \Rightarrow a_{ij} = e_{ij}$. Hence,

$$
\begin{aligned}
\| E \|_1 &= \max_{\forall j \in \mathbb{N}} \sum_{\substack{i=1 \\ i \neq j}}^{n} | a_{ij} | \\
&= \max_{\forall j \in \mathbb{N}} C_j(A) = C_{max}(A).
\end{aligned} \tag{17}
$$

In other words, $\| E \|_1$ equals the largest column Gershgorin disk of A. Inserting this back into (13), we arrive at the final inequality

$$
| \sigma_1(A) - \max_i | a_{ii} | | \le \sqrt{n} C_{max}(A), \tag{18}
$$

which shows that if the largest *column* Gershgorin disk of A is inflated by $\sqrt{n}$, then it bounds the largest singular value of A with respect to the absolute value of its largest diagonal element. Similarly a lower bound may be obtained for the smallest singular value. ■

This result can be used directly to design the diagonal controller to give closed loop robust stability and or performance.

IV. Rolls-Royce Spey Jet Engine

The methods outlined above are now applied to a locally linearized model of the Rolls-Royce Spey Jet Engine [17], [18] to design closed loop robust MIMO controller. The model contains 18 states, including three states of the actuators. The model also have two RHP transmission zeros. It has three inputs, which are fuel flow (FF), inlet guide vanes (IGV), and exit nozzle area (NA). The outputs are low-pressure spool speed (NL), high-pressure spool speed (NH), and surge margin (SM). The paring was obtained through use of dynamic RGA-number analysis. The output (SM), which here is used as the surge margin indicator, is in fact a ratio of outlet air pressures at the high and low pressure spools. The surge line itself is not easily measurable and it is commonly controlled through such indicators. As with all turbofans, the Spey engine is a highly interacting and non-linear system. It is common practice to linearize the model at several operating point, and to design a linear controller for each one. These controllers will then be scheduled by some means, to cover the whole of the flight envelop [19], [20]. In this example, one such linear controller is designed for the 85% NH sea-level condition. The DNA for the open loop uncompensated engine is shown in Figure 2, for frequencies $\omega \in [10^{-2} - 10^{2}]$ rad./sec. This shows the presence of significant interactions in this engine, since the open loop transfer function matrix is clearly not diagonal dominant.

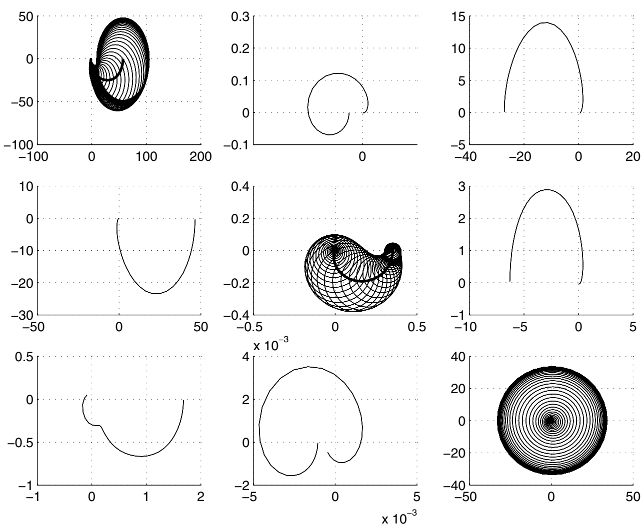

Fig. 2. Direct Nyquist array with column dominance discs for Spey engine

An 18^{th} order pre-compensator was designed using the LMIs of (7), and resulted in the DNA shown in Figure 3. The levels of dominance are so high that the system may be considered as practically decoupled.

At this point the design may proceed with the full-order pre-compensator. However, in spite of its much higher performance, there may be circumstances where a simpler pre-compensator is preferred. These can include, economical, implementational, maintenance and other such related reasons. The order reduction may be effected through many means, in this instance, the simple yet effective curve fitting method was used to find a low order approximation to this 18^{th} order pre-compensator. The resulting low order pre-compensator so found was,

$$
K_{sj} = \begin{bmatrix} \frac{.0127(s+.7)}{.1s+1} & \frac{.008(s+.3)(s+100)}{(.1s+1)(s+40)} & \frac{.014(s+50)}{.1s+1} \\ \frac{-2.202(s+1)}{.1s+1} & \frac{3.3026(s+1.5)}{.1s+1} & \frac{-1.606(s+40)}{.1s+1} \\ \frac{-.0087(s+7)}{.1s+1} & \frac{-.005(s-0.6)}{.1s+1} & \frac{.0265(s+60)}{.1s+1} \end{bmatrix} \tag{19}
$$

The Bode plots of the full order pre-compensator and the reduced-order compensator $K_{sj}(s)$ are given in Figure 4.

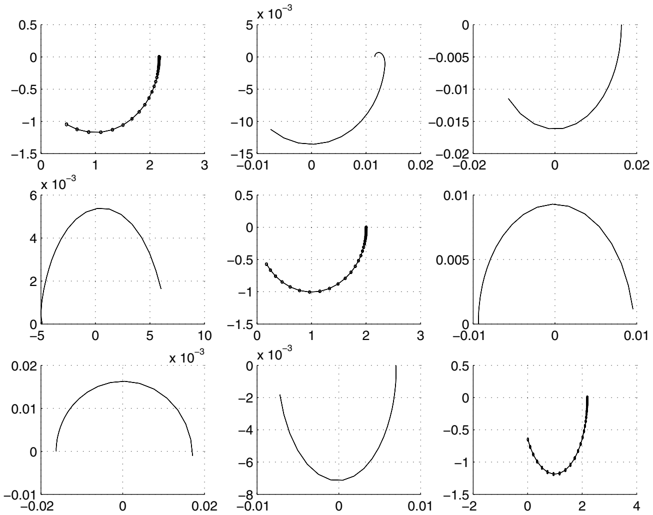

Fig. 3. Direct Nyquist array with column dominance discs for compensated system using 18^{th} order pre-compensator.

Figure 5 shows the DNA of the compensated system, using

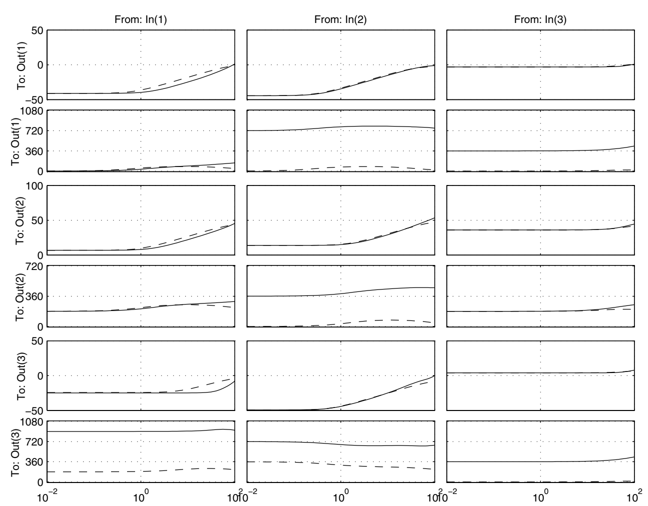

Fig. 4. Bode plots of the full order pre-compensator (Solid) and K_{sj} (Dashed)

the reduced order pre-compensator $K_{sj}(s)$. Now a diagonal loop tuning compensator can be designed to achieve the required closed-loop system performance. The design objectives which in here are *directly* specified, i.e. not specified through weighting functions, are to achieve,

- Peak overshoot of no more than 3.5dB on $S(s)$.
- Bandwidth of 20 rad/sec on $T(s)$.
- Roll off of 20 dB/dec for both $S(s)$ and $T(s)$.

Clearly the design objectives are imposed on both $S(s)$ and $T(s)$, thus this constitutes a mixed-sensitivity design objective.

Figure 6 shows the singular value bounds for the open-loop compensated system obtained from using the inclusion Theorem 1. Using these bounds, together with the approach

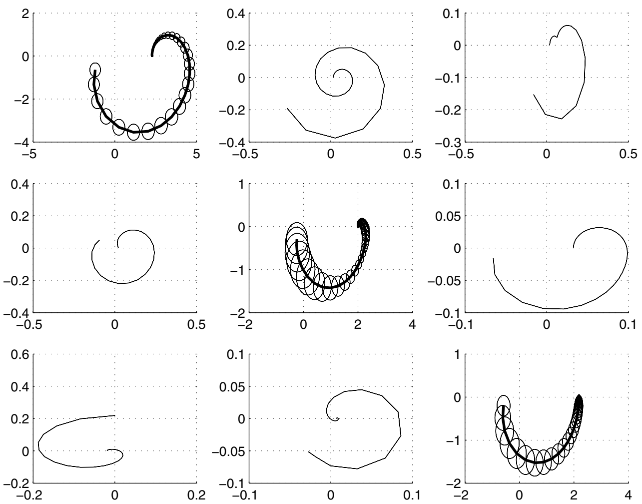

Fig. 5. DNA of the compensated system using K_{sj}

outlined in this section the following set of loop-shaping controllers were designed for the Spey engine,

$$C(s) \quad = \quad (0.5s+1)diag\left\{\frac{3}{s}, \frac{5}{s}, \frac{5}{s}\right\} \qquad (20)$$

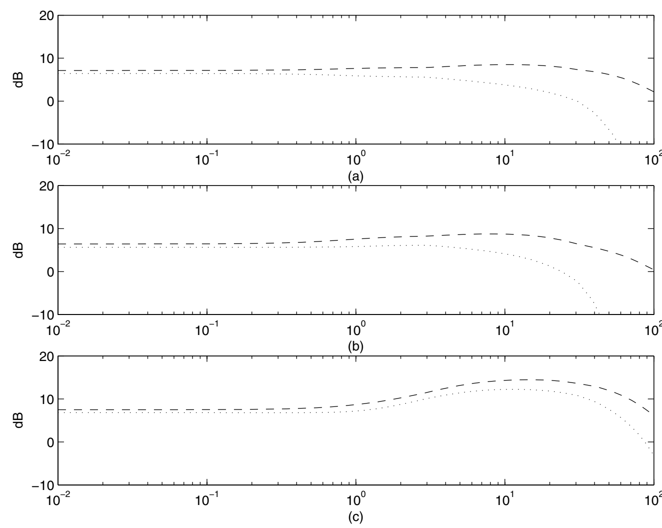

Fig. 6. Singular value bounds of the open-loop compensated system

Figures 7 and 8 show the resulting upper and lower singular value bounds for $S(s)$ and $T(s)$ respectively, from which it may be seen that all the design objectives are fulfilled.

The resulting closed-loop system DNA is shown in Figure 9. The step responses of the closed-loop system, for a unit step applied separately to each reference input, are given in Figure 10, which shows that the cross-channel interactions remain less than about 10% at all times.

V. Conclusions

A new approach to the design of dominance based robust controllers has been presented, where LMIs are used in the

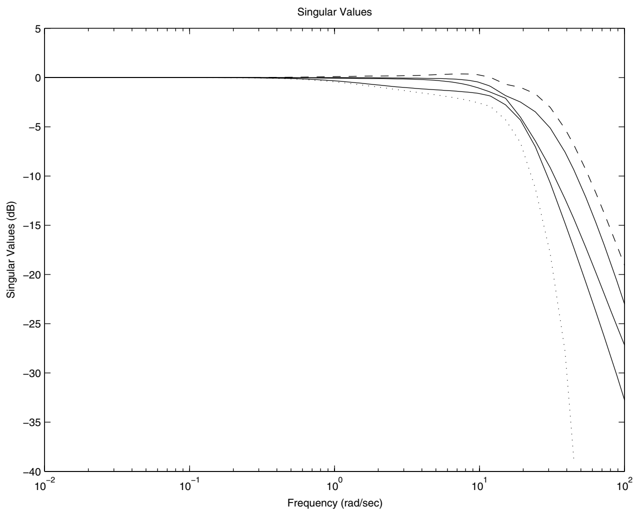

Fig. 7. Singular value bounds of the complementary sensitivity

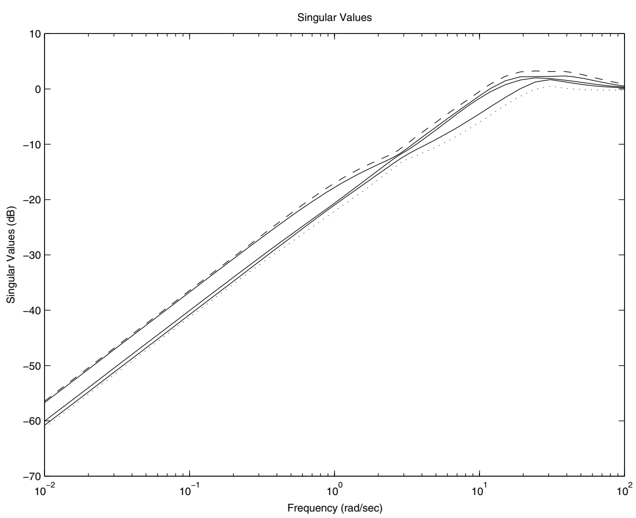

Fig. 8. Singular value bounds of the sensitivity

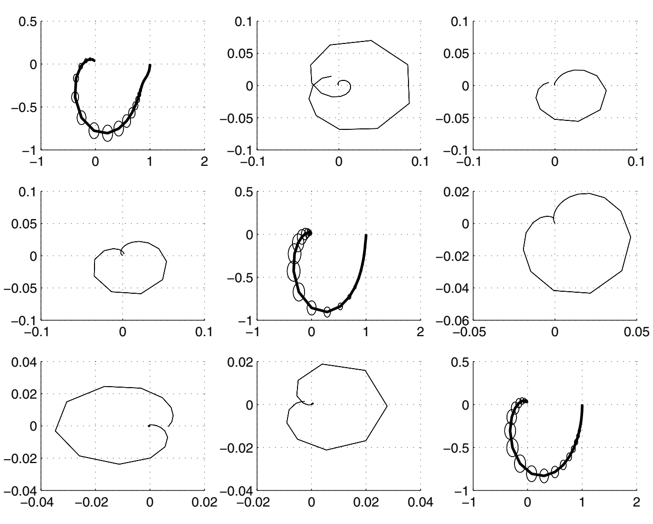

Fig. 9. DNA of the closed loop system

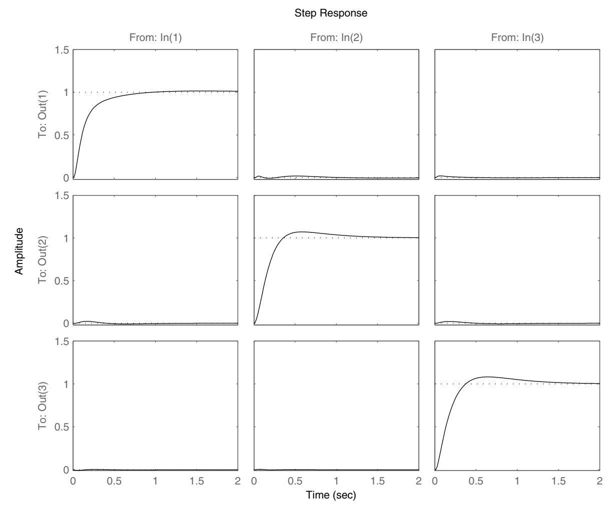

Fig. 10. Step response of the closed loop system

design of pre-compensators for achieving diagonal dominance and Gershgorin based bound on the singular values are used to design the single-loop controllers.

The diagonal dominance design technique was initially admired for the simplicity of its resulting controllers. However, the use of high order pre-compensators, of the type developed here, moves the dominance-based controller into the league of complex high order controllers. Nonetheless, there remains a critical advantage with the dominance approach, which is that the controller is broken down into two parts whose structural complexity are not coupled, with the function of each being independent of the other and well defined. Thus, for example, even though the pre-compensator has the same order as that of the plant, what the eventual operator of the controller will see as the 'tuning' knobs are the diagonal controllers which are typically still PI, leads, lags or other basic compensators. Therefore, irrespective of the order of the pre-compensator, diagonal dominance allows 'post-design' tuning of the full multivariable controller. This inherent allowance of retuning the controllers if the specifications change means that a dominance-based controller may remain advantageous even if it has a full order structure.

However, there may be situations where the designer requires a controller with a simpler structure, limited for example in dynamical order. It has been shown in this paper how the dominance approach can address this requirement, through the use of fitting curves to the magnitude parts of the Bode plots of the elements of the higher order pre-compensator. It is important to note that order reduction, as applied here does not have the ill-effects it is commonly known for. Typically high-order controllers are reduced in order *after* they are deigned, meaning it is likely that the specifications they were design for are no longer completely satisfied. With the diagonal dominance approach, order reduction is applied to the pre-compensator, *before* the design of the final loop shaping controllers. This means, where-as before the diagonal controllers were designed so that the full order system would meet the desired specifications, now they will be designed such that the system with the reduced order pre-compensator meets the same specifications.

The methods presented here have been applied to the Rolls Royce Spey jet engine and the results have shown that the algorithm considerably reduces the interactions present in this system. The levels of diagonal dominance achieved by the full order precompensator is several fold higher than any reported previously in the literature. In addition to the obvious high levels of diagonal dominance obtained from this technique, the main advantage of expressing the diagonal dominance problem in LMI form is that the problem becomes more flexible, and the LMI approach can be used to find pre-compensators, which fulfil other constraints such as robustness against parameter variations and pole region constraints. However, it is important to point out that if order-reduction is applied to the resulting pre-compensator, it may no longer satisfy specifications any such constraints imposed by the original LMI formulation.

This new technique of designing dynamic pre-compensators was combined with a novel method of designing the diagonal loop-shaping controller such that the resulting closed loop system will be guaranteed to satisfy some prescribed singular value behaviour constraints, and thus a prescribed level of system robustness and performance. Through the Spey engine example, the effectiveness of this combined approach was demonstrated, where a high performance simply structured mixed-sensitivity controller was designed.

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