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Two-level numerical procedure for optimal design of digital modal controllers

K.Yu. Polyakov, E.N. Rosenwasser and B.P. Lampe

Abstract— **The problem of optimal modal synthesis of digital controllers is considered. The goal is to minimize a cost function over the set of controllers that place closed-loop poles in a specified region $\mathbb{D}$. A two-level numerical optimization procedure is developed for design of optimal reduced-order modal controllers. An important feature of the method is that the original cost function is not modified. For the case of a quadratic cost function, a new method of optimization over polynomials is developed on the basis of a Diophantine equation.**

Index Terms— **Sampled-data systems, Modal control, Direct digital control, Optimization, Numerical methods, Random search**

I. Introduction

Many requirements to control systems, e.g., decay rate of transients and oscillation damping, can be formulated in terms of locations of the closed-loop poles. The classical *pole placement* problem supposes exact assignment of these poles [1]. Nevertheless, it often suffices to place these poles in a prescribed region $\mathbb{D}$ of the complex plane, and in addition, to minimize a cost function $J(C)$ depending on the controller C. Then, the minimization problem can be written in the form

$$C_* = \arg \min_{C \in \mathcal{C}} J(C)\,, \qquad (1)$$

where $\mathcal{C}$ denotes the set of admissible controllers, which place all closed-loop poles in $\mathbb{D}$. Problem (1) will be called the *optimal modal controller design problem*. So far no analytic solution is known to this problem for an arbitrary form of the region $\mathbb{D}$.

Problem (1) is very important for applications. For instance, it is known that formal use of the $\mathcal{H}_2$-minimization procedure for sampled-data systems yields in some cases marginally stable systems [2]. Therefore, we naturally arrive at the problem (1), the solution of which gives a suboptimal, but stabilizing controller.

In the simplest case, the degree of stability is constrained, so that the region $\mathbb{D}$ is the open half-plane $\Re s < -\alpha$ ($\alpha > 0$) for continuous-time systems and the region outside the disk of radius $\mathrm{e}^{\alpha T}$ in the ζ-plane for sampled-data systems with sampling period T. Hereinafter we use the variable $\zeta = \mathrm{e}^{-sT}$ associated with the backward shift operator [2].

One of the first design methods that guarantees some prescribed degree of stability α was proposed by Anderson and Moore [3], [4]. The idea is to modify the linear quadratic cost function in such a way that all processes in the closed-loop system decay faster than $\mathrm{e}^{-\alpha t}$. A discrete version of this approach can be found in [1]. It is important that the optimization is actually performed with respect to a *modified* criterion, and it is not guaranteed that the system will be close to the optimum with respect to the original cost functional. Moreover, in many cases the method appears to be too conservative, i.e., the obtained degree of stability is greater than required.

Other methods based on a modification of a quadratic cost function by an appropriate selection of weighting matrices for different stability regions can be found in [5-11] and references therein. They are based on one-to-one mappings of simply connected regions $\mathbb{D}$ onto the open left half-plane, while an optimization procedure serves mainly as a technique for pole placement and eigenstructure assignment.

In [12], an original method was proposed, which is based on the Youla-Kučera parameterization of the set of stabilizing controllers. The idea consists in approximating the optimal (with respect to the original functional) Youla parameter by a function having poles only in $\mathbb{D}$. It is shown that the optimal cost can be approached as closely as desired using a sequence of controllers of increasing order, which place all closed-loop poles in $\mathbb{D}$. Nevertheless, this method often leads to high-order controllers, which are undesirable in applications.

In this paper we propose a new two-level numerical procedure for the optimal design of modal digital controllers. It is based on a parameterization of the set of all controllers of order ℓ associated with a fixed characteristic polynomial. The procedure developed below is suitable for any region $\mathbb{D}$ and allows to restrict the order of the controller. As distinct from the simplest parametric optimization of controller transfer function, the present method guarantees that the closed-loop poles remain in $\mathbb{D}$ for all trial controllers.

The paper is organized as follows. In Sec. II the standard sampled-data system with a scalar (2,2)-block is introduced. The set of all controllers associated with a fixed characteristic polynomial is described in Sec. III. In Sec. IV, we develop a parameterization of all controllers of a given order ℓ that yield a fixed characteristic polynomial. The main result in the form of a two-level optimization algorithm is formulated in Sec. V, in Sec. VI we investigate its application to quadratic optimization problems for discrete and sampled-data systems. For this class of problems, a new effective algorithm for optimization over fixed-degree polynomials is developed on the basis of a Diophantine

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K. Polyakov and E. Rosenwasser are with the Department of Automatic Control, State University of Ocean Technology, St. Petersburg, Russia

B.P. Lampe is with the Institute of Automation, University of Rostock, Germany. bernhard.lampe@uni-rostock.de

equation. The method is illustrated by a numerical example in Sec. VII, where we consider an $\mathcal{L}_2$-optimal control problem for a sampled data system with restrictions imposed both on the degree of stability and the damping ratio.

II. Standard sampled-data system

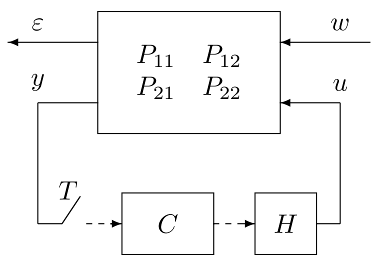

Fig. 1. Standard sampled-data system

Consider the standard sampled-data system [13] shown in Fig. 1, where ε, y, w, and u denote the output, feedback, input and control signal, respectively. The continuous-time plant is described by the following operator equations:

$$\begin{aligned} \varepsilon &= P_{11}(s)w + P_{12}(s)u \\ y &= P_{21}(s)w + P_{22}(s)u\,. \end{aligned}$$

The feedback signal is sampled with period T, and the block H denotes a hold element with transfer function $H(s)$.

Hereinafter, we will assume that $P_{22}(s)$ is a scalar transfer function so that the controller discrete transfer function $C(\zeta)$ is scalar as well. Also, we suppose that the standard system is stabilizable.

III. Systems with a fixed characteristic polynomial

The input-output behavior of the plant inside the control loop at the sampling instants is described by the discrete transfer function $D_{22}(\zeta) = \mathcal{D}_{P_{22}H}(T,\zeta,0)$, where $\mathcal{D}_F(T,\zeta,t)$ denotes the discrete Laplace transform for the function $F(s)$ [2]. Let

$$D_{22}(\zeta) = \mathcal{D}_{P_{22}H}(T,\zeta,0) = \frac{n(\zeta)}{d(\zeta)} \tag{2}$$

where $n(\zeta)$ and $d(\zeta)$ are coprime polynomials. By $\delta(\cdot)$ we denote the degree of a polynomial, assuming that $\delta(0) = -\infty$. Hereinafter, we denote $p = \max\{\delta(n), \delta(d)\}$.

The transfer function $C(\zeta)$ can be written as a ratio of coprime polynomials

$$C(\zeta) = \frac{a(\zeta)}{b(\zeta)}\,. \tag{3}$$

The number $\operatorname{ord} C = \max\{\delta(a), \delta(b)\}$ will be called the *order of the controller*.

The characteristic polynomial of the closed-loop system with controller (3) has the form[1]

$$\Delta(\zeta) = an - bd\,. \tag{4}$$

[1]Hereinafter, function arguments are often omitted for brevity.

Let us choose a characteristic polynomial Δ. Then, a controller (3) satisfying (4) will be called a Δ-*controller*. As follows from the polynomial equation theory [14], there exists a set of polynomial pairs satisfying (4):

$$a(\zeta) = a_0 + d\xi\,, \quad b(\zeta) = b_0 + n\xi \tag{5}$$

where $\{a_0, b_0\}$ is an arbitrary solution of (4), and ξ is an arbitrary polynomial. Among all solutions (5), there exists a solution with a of minimal degree such that $\delta(a) \le \delta(d)-1$, and a solution with b of minimal degree such that $\delta(b) \le \delta(n)-1$. If $\delta(\Delta) < \delta(n)+\delta(d)$, both the minimal solutions coincide.

IV. Controllers of order ℓ

Assume that we have a characteristic polynomial Δ and a desired controller order ℓ such that the following assumptions hold:

A1 $\delta(\Delta) \le \ell + p$;
A2 There exists a solution $\{a_0, b_0\}$ of (4) such that $\delta(a_0) \le \ell$ and $\delta(b_0) \le \ell$.

Comparing the polynomial degrees at both sides of (4), it is easy to see that A1 is a necessary condition for this equation to be solvable. On the other hand, condition A2 is necessary for existence of a controller of order ℓ satisfying (4).

Let $\{a_0, b_0\}$ be the solution of (4) with a of minimal degree for $\delta(d) \ge \delta(n)$ and the solution with b of minimal degree for $\delta(d) < \delta(n)$. Then, it can be shown that the controller $C_0(\zeta) = a_0/b_0$ has the minimal order among all controllers satisfying (4).

Theorem 1: Let Δ be given such that A1–A2 hold, and the pair $\{a_0, b_0\}$ define the minimal order Δ-controller. Then, the set of all Δ-controllers such that $\operatorname{ord} C \le \ell$ can be parameterized as

$$C(\zeta) = \frac{a_0 + d\xi}{b_0 + n\xi} \tag{6}$$

where $\xi(\zeta)$ is an arbitrary polynomial such that $\delta(\xi) \le \delta_\xi = \ell - p$.

Proof: First, we consider the case $p = \delta(d) \ge \delta(n)$. Let $C(\zeta)$ be a Δ-controller such that $\operatorname{ord}\ C \le \ell$. We will show that under the conditions of Theorem 1 the polynomial ξ satisfies the estimate $\delta(\xi) \le \ell - \delta(d)$.

Let $\delta(\Delta) < \delta(n) + \delta(d)$. Then, for the solution $\{a_0, b_0\}$ of (4) with a_0 of minimal degree, we have

$$\delta(a_0) \le \delta(d) - 1\,, \quad \delta(b_0) \le \delta(n) - 1\,.$$

Therefore,

$$\begin{aligned} \delta(a) &= \delta(a_0 + d\xi) \le \delta(d) + \delta(\xi) \le \ell \\ \delta(b) &= \delta(b_0 + n\xi) \le \delta(n) + \delta(\xi) \le \ell\,. \end{aligned}$$

Hence, $\delta(\xi) \le \ell - \delta(d)$.

If $\delta(\Delta) \ge \delta(n) + \delta(d)$, with account for A1 we obtain

$$\delta(a_0) \le \delta(d) - 1\,, \quad \delta(b_0) \le \delta(\Delta) - \delta(d) \le \ell\,.$$

Therefore,

$$\begin{aligned}\delta(a) &= \delta(a_0 + d\xi) \le \delta(d) + \delta(\xi) \le \ell \\ \delta(b) &= \delta(b_0 + n\xi) \le \max\{\ell, \delta(n) + \delta(\xi)\} \le \ell\,.\end{aligned}$$

Hence, $\delta(\xi) \le \ell - \delta(d)$.

Now let a polynomial ξ satisfy the estimate $\delta(\xi) \le \ell - \delta(d)$. We will show that $\operatorname{ord} C \le \ell$. From (5) it immediately follows that $\delta(a) \le \ell$. If $\delta(\Delta) < \delta(n) + \delta(d)$, we have

$$\delta(b) \le \delta(n) + \delta(\xi) \le \delta(n) + \ell - \delta(d) \le \ell\,.$$

If $\delta(\Delta) \ge \delta(n) + \delta(d)$, due to A1 we obtain

$$\delta(b) \le \delta(\Delta) - \delta(d) \le \ell$$

so that $\operatorname{ord} C \le \ell$.

The proof for the case $\delta(d) < \delta(n)$ is similar. ∎

We note that for a fixed $\Delta(\zeta)$ and $\ell \ge p$ there is a freedom in selecting the polynomial ξ that can be utilized for optimization purposes. For $\ell < p$ there exists at most a single controller of order ℓ satisfying (4), namely $C_0(\zeta)$.

V. Two-level optimization procedure

Since C is uniquely determined by the pair $\{\xi, \Delta\}$ and vice versa, the cost functional can be written as $J(\xi, \Delta)$.

Let us choose a desired controller order ℓ and a characteristic polynomial Δ_0 satisfying A1. Then, the maximal degree δ_ξ of ξ can be found by Theorem 1. For $\delta_\xi \ge 0$, among all polynomials ξ such that $\delta(\xi) \le \delta_\xi$ we can choose ξ_0 that minimizes the cost function for the given $\Delta = \Delta_0$:

$$\xi_0(\zeta) = \arg \min_{\xi\colon \delta(\xi) \le \delta_\xi} J(\xi, \Delta_0)\,. \tag{7}$$

This optimization problem constitutes the lower level of the proposed algorithm and often can be solved analytically. The controller obtained by (6) with $\xi = \xi_0$ will be called the *suboptimal modal controller* associated with Δ_0.

For the complete solution of the problem we have to variate (at the upper level) the polynomial Δ in such a way that its roots remain inside the region $\mathbb{D}$. Then, the proposed two-level optimization procedure can be written as

$$\min_{p_i \in \mathbb{D}} \min_{\xi\colon \delta(\xi) \le \delta_\xi} J(\xi, \Delta)\,,$$

where $p_i\,(i = 1, \ldots, m)$ are the roots of $\Delta(\zeta)$. This procedure yields the *optimal modal controller of order* ℓ constructed by (6).

Since the dependence of the cost functional on the roots of Δ can be non-convex and might possess numerous local minima, it is appropriate to use randomized search algorithms [15-17]. In essence, the proposed procedure is similar to the Q-parameterization approach of [17], but, as distinct from the latter, controller complexity can be effectively restricted.

All computations in this paper were performed using the DirectSD Toolbox for Matlab [18], where an adaptive random search technique is realized. The algorithm starts with an initial step size h_0. The step is decreased by a factor $0 < \gamma < 1$ after each n_f failed attempts. The algorithm stops after k_{max} iterations or if the step size h became smaller than h_{min}.

Algorithm

Input: Polynomials $n(\zeta)$ and $d(\zeta)$
Desired controller order ℓ
Cost functional $J(C)$
Search algorithm parameters
h_0, h_{min}, γ, n_f, and k_{max}

Output: Transfer function $C(\zeta)$ of the optimal modal controller of order ℓ

Step 1: Find $\delta_\xi = \ell - p$.
Step 2: Set $k := 0$, $f := 0$, $h := h_0$, and $J_* := \infty$.
Step 3: Set $k := k + 1$.
Step 4: Choose a trial characteristic polynomial Δ_k such that $\delta(\Delta_k) \le \ell + p$, all its roots are inside the region $\mathbb{D}$, and A2 holds.
Step 5: Construct the basic solution $\{a_0, b_0\}$ of (4) with a_0 of minimal degree for $\delta(d) \ge \delta(n)$ or with b_0 of minimal degree for $\delta(d) \le \delta(n)$.
Step 6: If $\delta_\xi \ge 0$, construct the polynomial ξ_0 of degree δ_ξ as the solution of the optimization problem (7), else set $\xi_0 = 0$.
Step 7: Construct C_k by (6) with $\xi = \xi_0$, and find the associated cost $J_k = J(C_k)$.
Step 8: If $J_k < J_*$, then set $C_* := C_k$, $J_* := J_k$, and $f := 0$, else set $f := f + 1$.
Step 9: If $f = n_f$, then set $h := \gamma h$ and $f := 0$.
Step 10: If $k < k_{max}$ and $h > h_{min}$, then goto Step 3, else stop.

Remark 1. If $\ell \ge p - 1$, then the roots of Δ_k can be chosen (at Step 4) independently inside $\mathbb{D}$. If $\ell < p - 1$, these roots should be selected in a special way, because not all possible polynomials Δ_k satisfy A2.

Remark 2. If we choose $\delta(\Delta_k) < \ell + p$ at Step 4, dead-beat modes will appear. Namely, the characteristic polynomial in z-plane will have $\ell + p - \delta(\Delta_k)$ roots at the origin.

Remark 3. The optimization technique at the lower level (Step 6) is determined by the properties of the functional $J(\xi, \Delta)$. In the next section we investigate a class of functionals appearing in optimization problems for discrete and sampled-data systems.

Remark 4. If the optimal controller C_{opt} minimizing the cost function (without additional restrictions on the stability domain) can be computed analytically, it is reasonable to take the characteristic polynomial of the optimal system as a starting point for the search procedure. Of course, all its roots must be projected onto $\mathbb{D}$.

VI. Lower level optimization for quadratic functionals

Let the superscript * denote the adjoint $F^*(\zeta) = F(\zeta^{-1})$. A polynomial (real rational function) in ζ will be called *stable* if it is free of roots (respectively, poles) inside the closed unit disk.

As was shown in [2], the $\mathcal{H}_2$- and $\mathcal{L}_2$-optimization problems for SISO sampled-data systems reduce to minimizing functionals of the form

$$J_2 = \frac{1}{2\pi j}\oint_\Gamma \left[M^*AM + BM + M^*B^* + E\right]\frac{d\zeta}{\zeta} \tag{8}$$

over the set of stabilizing controllers. Here $A(\zeta) = A^*(\zeta)$, $B(\zeta)$, and $E(\zeta) = E^*(\zeta)$ are known real rational functions in ζ, and

$$M(\zeta) = \frac{C(\zeta)}{1 - D_{22}(\zeta)C(\zeta)}$$

contains the controller $C(\zeta)$ to be determined. The integration path Γ in (8) is the unit circle passed in an anticlockwise fashion. Using (2), (4), and (6), we obtain

$$M(\zeta) = -\frac{(a_0 + d\xi)d}{\Delta}\,.$$

Using this formula in (8) yields

$$J_2 = \frac{1}{2\pi j}\oint_\Gamma \left[\xi^*\widetilde{A}\xi - \widetilde{B}\xi - \xi^*\widetilde{B}^* + \widetilde{E}\right]\frac{d\zeta}{\zeta} \tag{9}$$

where

$$\begin{aligned}
\widetilde{A}(\zeta) &= \frac{(dd^*)^2}{\Delta\Delta^*}A\\
\widetilde{B}(\zeta) &= \frac{d^2}{\Delta}B - \frac{d^2a_0^*d^*}{\Delta\Delta^*}A\\
\widetilde{E}(\zeta) &= E + \frac{a_0a_0^*dd^*}{\Delta\Delta^*}A - \frac{a_0d}{\Delta}B - \frac{a_0^*d^*}{\Delta^*}B^*\,.
\end{aligned}$$

Assume that the function $\widetilde{A}(\zeta)$ is free of poles on the unit circle. Then, it admits the factorization

$$\widetilde{A}(\zeta) = K(\zeta)K^*(\zeta)$$

where all poles of $K(\zeta)$ are outside the unit disk. Completing the squares in the integrand in (9), we obtain

$$J_2 = \frac{1}{2\pi j}\oint_\Gamma (\xi^*K^* - L^*)(K\xi - L)\frac{d\zeta}{\zeta} + J_0$$

where J_0 is independent of ξ, and

$$L(\zeta) = \frac{\widetilde{B}^*}{K^*}\,.$$

Assuming that L has no poles at the unit circle, we can perform the following separation:

$$L(\zeta) = L_+(\zeta) + L_-(\zeta)$$

where L_+ is a stable function, while L_- is strictly proper and strictly antistable. Then, repeating the arguments of [2], it can be shown that

$$J_2 = \frac{1}{2\pi j}\oint_\Gamma (\xi^*K^* - L_+^*)(K\xi - L_+)\frac{d\zeta}{\zeta} + \widetilde{J}_0 \tag{10}$$

where $\widetilde{J}_0$ is independent of ξ. This results in a parametric optimization problem over the set of coefficients of the polynomial ξ. For the functional (10) one can use, for example, the method described in [19]. Nevertheless, below we present a new polynomial solution to this problem, which is superior to the latter as regards computational complexity.

Consider the optimization problem

$$J_\xi = \frac{1}{2\pi j}\oint_\Gamma (\xi^*W^* - V^*)(W\xi - V)\frac{d\zeta}{\zeta} \;\to\; \min\,, \tag{11}$$

where $W(\zeta)$ and $V(\zeta)$ are stable rational functions such that

$$W(\zeta) = \frac{n_w(\zeta)}{d_w(\zeta)}\,,\; W^*(\zeta) = \frac{\tilde{n}_w(\zeta)}{\tilde{d}_w(\zeta)\zeta^\nu}\,,\; V(\zeta) = \frac{n_v(\zeta)}{d_v(\zeta)}\,,$$

where ν is a nonnegative integer, and $\{n_w(\zeta), d_w(\zeta)\}$, $\{\tilde{n}_w(\zeta), \tilde{d}_w(\zeta)\}$, and $\{n_v(\zeta), d_v(\zeta)\}$ are pairs of coprime polynomials such that d_w, d_v, and $\tilde{d}_w$ are free of roots at $\zeta = 0$.

Theorem 2: The following statements are equivalent:

i) Polynomial $\xi(\zeta)$ is the solution of (11) over the set of polynomials of order $\le \delta_\xi$;
ii) There exists a solution $\{\xi, \pi, \theta\}$ of the following polynomial equation:

$$\tilde{n}_w n_w d_v \xi - d_w d_v \pi - \tilde{d}_w \zeta^{\nu+\delta_\xi+1}\theta = \tilde{n}_w n_v d_w\,, \tag{12}$$

such that $\delta(\xi) \le \delta_\xi$ and $\delta(\pi) < \delta(\tilde{d}_w) + \nu$.

Proof: is given in the Appendix. ■

By construction, the functions $K(\zeta)$ and $L_+(\zeta)$ in (10) are stable, so that Theorem 3 is directly applicable.

Remark 1. It should be noted that equations similar to (12) were introduced, for some specific cases, in [20] as sufficient conditions of optimality. Theorem 3 considers the general case and proves necessity as well.

Remark 2. Using the method of [19], one has to compute $2(\delta_\xi + 1)$ inner products of rational functions, after that a linear system of equations must be solved. Each inner product is a contour integral, so this is a time-consuming operation. The solution proposed in Theorem 3 uses a single Diophantine equation, which is equivalent to a linear system of equations. Computational experiments (with randomly generated second-order W and V) demonstrated that for $\delta_\xi = 0$ new algorithm is almost 3 times faster than that of [19], while for $\delta_\xi = 10$ this ratio exceeds 17. Nevertheless, a detailed analysis of computational aspects are beyond the scope of the paper and will be considered elsewhere.

VII. Numerical example

Consider the $\mathcal{L}_2$-optimization problem for a sampled-data tracking system shown in Fig. 2. The reference signal $r(t)$ has the Laplace transform $R(s)$. The system performance is evaluated by the $\mathcal{L}_2$-norm of the error $e(t)$ between the actual output $y(t)$ and the ideal signal $\hat{y}(t)$ formed by a linear block with transfer function $Q(s)$:

$$J = \int_0^\infty e^2(t)\,dt = \int_0^\infty \left[y(t) - \hat{y}(t)\right]^2\,dt\,. \tag{13}$$

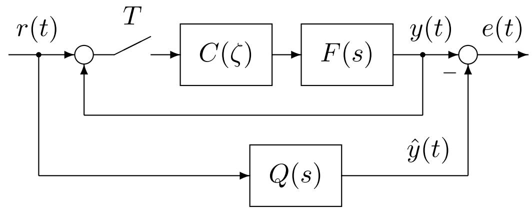

Fig. 2. Block-diagram of the sampled-data tracking system

Let

$$F(s)=\frac{1}{s^2}\,,\quad Q(s)=\frac{1}{T_0 s+1}\,,\quad R(s)=\frac{1}{s}\,,$$

$$H(s)=\frac{1-\mathrm{e}^{-sT}}{s}\,,\quad T_0=2\,\mathrm{sec}\,,\quad T=0.5\,\mathrm{sec}\,.$$

It is required to find a digital controller C minimizing (13) such that $\mathrm{ord}\, C\le 3$, the degree of stability is greater than $\alpha=0.2$, and the damping ratio is restricted by a sector with $\tan\theta=\beta=2$ [8]. This means that

$$\Re\, s_i \le -\alpha\,,\quad \left|\frac{\Im\, s_i}{\Re\, s_i}\right|\le\beta \tag{14}$$

holds for all $s_i=-\frac{1}{T}\log p_i$, where p_i are the roots of the characteristic polynomial in the ζ-plane.

The analytical solution to the $\mathcal{L}_2$-problem without additional restrictions on pole locations is given for SISO systems in [2]. A formal application of the procedure described in [2] gives

$$C_{opt}(\zeta)=\frac{1.4986(1-\zeta)(1-0.119\zeta)}{(1+0.5225\zeta)(1-0.015\zeta)}\,.$$

Fig. 3 indicates that the optimal controller seems to ensure a nearly ideal tracking behavior to step inputs (with $J=0.00186$). However, the closed-loop poles p_i in the ζ-plane are $\{1, 1.284, -2.323, -23.204\}$, i.e., the system is marginally stable and will not work in practice. It is easy to check that the first restriction in (14) is violated for $p_1=1$, while the second one is false for $p_3=-2.323$. Then, we compare different controller design methods that provide for the prescribed degree of stability. Applying the idea of Anderson and Moore [3], [4] to sampled-data systems for $\alpha=0.2$, we obtain

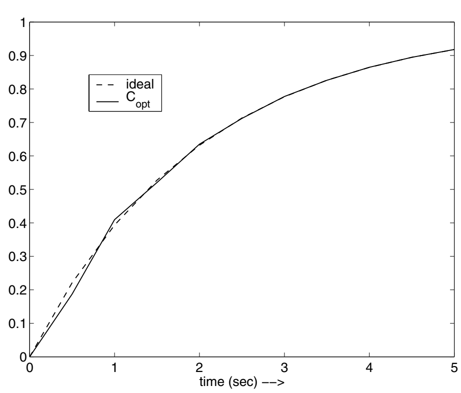

Fig. 3. Transient of the system with $\mathcal{L}_2$-optimal controller

$$C_{AM}(\zeta)=\frac{6.2354(1-0.8708\zeta)(1-0.7127\zeta)}{(1+0.7146\zeta)(1-0.6633\zeta)}\,.$$

Now the closed-loop poles are located at $\{1.221, 1.284, -2.569, -25.685\}$, and the actual degree of stability is 0.4. The controller yields $J=1.021$. This method does not take into account the required damping ratio, and the second condition in (14) is violated for $p_3=-2.569$.

Calculations show that the method of [12] is hardly applicable to this problem, because an approximation of Youla parameter with required precision leads to controllers of very high order (more than 5).

Using the search procedure developed in the paper we found the optimal modal controllers satisfying (14). The first-order controller

$$C_1(\zeta)=\frac{1.0039(1-0.95\zeta)}{1+0.2867\zeta}$$

ensures $J=0.289$, and the poles are located at the points $\{-4.812, 1.123, 1.105\}$.

For the optimal controller of order 2

$$C_2(\zeta)=\frac{2.0823(1-0.9501\zeta)(1-0.5568\zeta)}{(1+0.4664\zeta)(1-0.2759\zeta)}$$

we have $J=0.218$, and the closed-loop poles are $\{-18.868, -4.819, 1.105, 1.105\}$.

The controller of order 3:

$$C_3(\zeta)=\frac{1.0407(1-0.9659\zeta)(1-1.878\zeta+0.8872\zeta^2)}{(1+0.1254\zeta)(1-1.875\zeta+0.8886\zeta^2)}$$

places all roots of the characteristic polynomial in the neighborhood of the point $\zeta=1.105$ and gives $J=0.137$. This cost value is almost 7.5 times less than that for C_{AM}.

The transients for the systems with the above controllers shown in Fig. 4 demonstrate effectiveness of the optimal modal controllers that ensure much better performance as compared with C_{AM}.

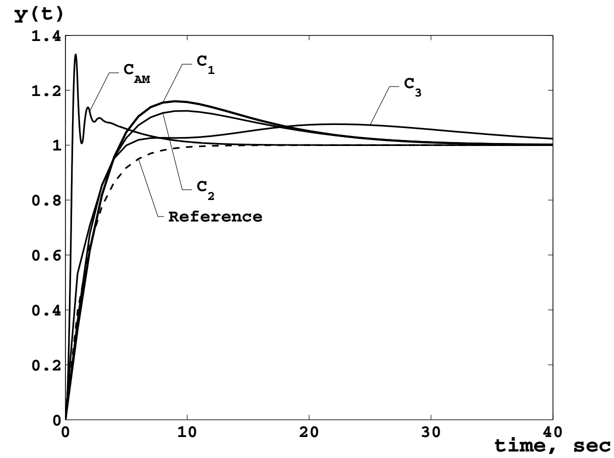

Fig. 4. Transients for systems with controllers C_{AM}, C_3, C_2, and C_1

VIII. Conclusions

The paper investigates the problem of optimal modal design of digital controllers, which minimize a specified cost function and place all closed-loop poles in a prescribed region $\mathbb{D}$ of the complex plane.

A two-level optimization procedure is developed that uses, at the upper level, variation of desired characteristic polynomial so that its roots remain inside $\mathbb{D}$, and, at the lower level, design of a suboptimal controller for the system with a fixed characteristic polynomial. For a quadratic cost function, a new effective polynomial-based method was proposed for analytical lower-level optimization.

Compared with the simplest parametric optimization of controller transfer function, the proposed method guarantees that the closed-loop poles remain in $\mathbb{D}$ at all stages of the algorithm.

It is important that the controller is optimized with respect to the original (rather than modified!) criterion, and this fact makes it possible to overcome many drawbacks of existing methods. Moreover, the algorithm is suitable for an arbitrary region $\mathbb{D}$ and can be used for the design of reduced-order controllers. The idea developed here is also applicable to continuous-time and discrete-time LTI systems.

Appendix

The proof of Theorem 3 presented below is based on the following lemma.

Lemma 1: Let $G(\zeta)$ be a real rational function free of poles on the unit circle:

$$G(\zeta) = \frac{n_g}{d_g^+\, d_g^-}\,, \tag{15}$$

where $n_g(\zeta)$, $d_g^+(\zeta)$, and $d_g^-(\zeta)$ are polynomials such that d_g^+ is stable and d_g^- is strictly antistable. Then, the following statements are equivalent:

i) For all $i = 0, \ldots, n$

$$\frac{1}{2\pi j}\oint_\Gamma \frac{G(\zeta)}{\zeta^i}\frac{d\zeta}{\zeta} = 0\,. \tag{16}$$

ii) The polynomial Diophantine equation

$$d_g^- \zeta^{n+1}\theta + d_g^+\pi = n_g \tag{17}$$

has a solution $\{\theta, \pi\}$ such that $\delta(\pi) < \delta(d_g^-)$.

Proof: is omitted for space limitation. ∎

Proof of Theorem 3: i) → ii). Let a polynomial ξ of degree δ_ξ solve (11). Then, by the projection theorem [21], it is equivalent to

$$\frac{1}{2\pi j}\oint_\Gamma \frac{G(\zeta)}{\zeta^i}\frac{d\zeta}{\zeta} = 0\,, \qquad i = 0, \ldots, n\,, \tag{18}$$

where

$$\begin{aligned} G(\zeta) &= W^*\left[W\xi - V\right] = \frac{\tilde n_w}{\tilde d_w \zeta^\nu}\left[\frac{n_w}{d_w}\xi - \frac{n_v}{d_v}\right] \\ &= \frac{\tilde n_w\left[n_w d_v \xi - n_v d_w\right]}{\tilde d_w \zeta^\nu d_w d_v}\,. \end{aligned}$$

Since W and V are stable, the polynomials d_w and d_v are stable, while $\tilde d_w$ is strictly antistable. Therefore, $G(\zeta)$ can be written in the form (15) with

$$\begin{aligned} n_g(\zeta) &= \tilde n_w\left[n_w d_v \xi - n_v d_w\right]\,, \\ d_g^-(\zeta) &= \tilde d_w \zeta^\nu\,, \quad d_g^+(\zeta) = d_w d_v\,. \end{aligned}$$

Since (18) holds, by Lemma 1 there exists a solution $\{\theta, \pi\}$ of

$$\tilde d_w \zeta^{\nu+n+1}\theta + d_w d_v \pi = \tilde n_w\left[n_w d_v \xi - n_v d_w\right]\,,$$

such that $\delta(\pi) < \delta(d_g^-)$. This equation is easily transformed into (12).

ii) → i) is proved by inverting the above argument. ∎

References

[1] K. Åström and B. Wittenmark, *Computer Controlled Systems: Theory and Design*, 3rd ed. Englewood Cliffs, NJ: Prentice-Hall, 1997.

[2] E. Rosenwasser and B. Lampe, *Computer Controlled Systems - Analysis and Design with Process-orientated Models*. London Berlin Heidelberg: Springer-Verlag, 2000.

[3] B. D. O. Anderson and J. B. Moore, "Linear system optimization with prescribed degree of stability," *IEE Proc.*, vol. 116, pp. 2083–2087, 1969.

[4] ——, *Optimal Control (Linear Quadratc Methods)*. Englewood Cliffs, NJ: Prentice-Hall, 1989.

[5] B. Wittenmark, R. J. Evans, and Y. C. Soh, "Constrained pole-placement using transformation and LQ-design," *Automatica*, vol. 23, no. 6, pp. 763–769, 1987.

[6] K. Furuta and S. B. Kim, "Pole assignment in a specified disk," *IEEE Trans. Automat. Contr.*, vol. AC-32, no. 5, pp. 423–427, 1987.

[7] S. B. Kim and K. Furuta, "Regulator design with poles in a specified region," *Int. J. Control*, vol. 47, no. 1, pp. 143–160, 1988.

[8] H. Zhang and S. G. Shu, "Analytic approach to quadratic control with prescribed relative stability," *Int. J. Control*, vol. 48, pp. 1843–1850, 1988.

[9] J. Medanic, H. S. Tharp, and W. R. Perkins, "Pole placement by performance criterion modification," *IEEE Trans. Automat. Contr.*, vol. AC-33, no. 5, pp. 469–472, 1988.

[10] W. M. Haddad and D. S. Bernstein, "Controller design with regional pole constraints," *IEEE Trans. Automat. Contr.*, vol. AC-37, no. 1, pp. 54–69, 1994.

[11] N. Sivashankar, I. Kaminer, and P. P. Khargonekar, "Optimal controller synthesis with D-stability," *Automatica*, vol. 30, no. 6, pp. 1003–1008, 1994.

[12] F. De Bruyne, B. D. O. Anderson, M. Gevers, and J. Leblond, "LQG control with pole constraints using the Youla parameterization," in *Proc. European Control Conf.*, Roma, Italy, 1995, pp. 2494–2499.

[13] T. Chen and B. A. Francis, *Optimal Sampled-Data Control Systems*. Berlin Heidelberg New York: Springer-Verlag, 1995.

[14] V. Kučera, *Discrete Linear Control. The Polynomial Approach*. Prague: Academia, 1979.

[15] G. Fishman, *Monte Carlo: Concepts, Algorithms, and Applications*. New York: Springer, 1996.

[16] C. Robert and G. Casella, *Monte Carlo Statistical Methods*. New York: Springer, 1999.

[17] K. Horspool, R. S. Erwin, C. T. Abdallah, and P. Dorato, "A randomized approach to the $\mathcal{H}_2/\mathcal{H}_\infty$ control problem via Q-parametrization," in *Proc. American Control Conf.*, Chicago, Illinois, June 2000, pp. 1400–1404.

[18] K. Polyakov, E. Rosenwasser, and B. Lampe, "DirectSD - a toolbox for direct design of sampled-data systems," in *Proc. IEEE Intern. Symp. CACSD'99*, Kohala Coast, Island of Hawai'i, Hawai'i, USA, 1999, pp. 357–362.

[19] D. N. Hoover, R. Longchamp, and J. Rosenthal, "Two-degree-of-freedom ℓ_2-optimal tracking with preview," *Automatica*, vol. 40, no. 1, pp. 155–162, 2004.

[20] L. Volgin, *Optimal discrete control of dynamic systems*. Moscow: Nauka, 1986, (in Russian).

[21] A. Balakrishnan, *Applied Functional Analysis*, 2nd ed. New York: Springer, 1981.