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Geometry of the stability domain in the parameter space: D-decomposition technique

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***Abstract*— The challenging problem in linear control theory is to describe the total set of parameters (controller coefficients or plant characteristics) which provide stability of a system. For the case of one complex or two real parameters and SISO system (with a characteristic polynomial depending linearly on these parameters) the problem can be solved graphically by the use of so called D-decomposition. Our goal is to extend the technique and to link it with general $M-\Delta$ framework. On this way we investigate the geometry of D-decomposition for polynomials and estimate the number of root invariant regions. Several examples verify that these estimates are tight. We also extend D-decomposition for the matrix case. For instance, we partition the real axis or the complex plane of the parameter k into regions with invariant number of stable eigenvalues of the matrix $A + kB$. Similar technique can be applied to double-input double-output systems with two parameters.**

Key words: **Stability analysis, stability domain, linear systems, D-decomposition, $M-\Delta$ framework.**

I. INTRODUCTION

Consider a linear system depending on a vector parameter k with a characteristic polynomial $p(s, k)$. The boundary of a stability domain (in the space k) is given by the equation

$$p(j\omega, k) = 0, \quad -\infty < \omega < \infty. \tag{1}$$

If $k \in \mathbb{R}^2$ (or $k \in \mathbb{C}$) then we have two equations (real and imaginary part of (1)) in two variables and (in general) can define the parametric curve $k(\omega), -\infty < \omega < \infty$ defining the boundary of the stability domain. Moreover, the curve $k(\omega)$ divides the plane into root invariant regions (i.e. regions with a fixed number of stable and unstable roots of $p(s, k)$). This is the basic idea of D-decomposition approach. The idea can be traced back to Vishnegradsky [22] who reduced a cubic polynomial to the form $p(s, k) = s^3 + k_1s^2 + k_2s + 1$ and treated the coefficients $k_1, \ k_2$ as parameters. Then equation (1) yields $k_1\omega^2 = 1$, $\omega(k_2 - \omega^2) = 0$. Eliminating ω we get that D-decomposition is given by the hyperbola $k_1k_2 = 1$. The stability domain is the set $k_1k_2 > 1$.

For the general case similar ideas were exploited by Frazer and Duncan [4]. Moreover, Nyquist plot can be considered as the realization of the same idea. But it was Yu. Neimark [12], [13] who developed the rigorous algorithm (and coined the name "D-decomposition").

In the Western literature the technique is described first by Mitrovic [11]; he also proposed the mapping of contours other that imaginary axis. This line of research was significantly developed by Siljak [18], [19], [20]. In his works D-decomposition (which he calls the parameter plane method) was broadened to become a useful tool for design purposes. D-decomposition is also described in the books [10], [20] and [1] and is often exploited for low-order controller design (e.g., [3], [2], [5], [21]).

In this paper we extend the approach to systems presented at the state space form. More specifically, given a class $\mathcal{K}$ of $r \times m$ matrices K, find all matrices $K \in \mathcal{K}$ such that $A + BKC$ is stable:

$$D = \{K \in \mathcal{K} : \ A + BKC \text{ is stable}\}. \tag{2}$$

Here A, B, C are given real matrices of dimensions $n \times n, \ n \times r, \ m \times n$ respectively; a stability is understood either in a continuous-time sense (all eigenvalues are in the open LHP) or a discrete-time sense (all eigenvalues are in the open unit disc). A class $\mathcal{K}$ may be different; below we analyze in detail the simplest cases:

$$K = k \in \mathbb{R}^n \text{ or } K = k^T, \ k \in \mathbb{R}^n;$$
$$K = kI, \ k \in \mathbb{R} \text{ or } k \in \mathbb{C}, \ m = r;$$
$$K \in \mathbb{R}^{2\times 2};$$

where all calculations can be performed explicitly in the graphical form. The first case ($m = 1$ or $r = 1$) is equivalent to the polynomial framework, two others are essentially matrix ones.

Nevertheless we present general description of D-decomposition. It is closely related to the standard $M-\Delta$ setting.

Problem (2) arises in design or robustness studies. For instance, to find all stabilizing static output controllers for the system $\dot{x} = Ax + Bu, \ y = Cx$ one can construct the set D (2) with $\mathcal{K} = \mathbb{R}^{r\times m}$; here K plays a role of the feedback gain. On the other hand, if A is a nominal stable matrix and it is perturbed as $A + BKC$, where K is a constant $r \times m$ matrix, then (2) provides all admissible perturbations which preserve stability. Of course, if we know a boundary of a stability domain ∂D, then we can find the distance to it:

$$\rho = \min_{K\in\partial D} \|K\|. \tag{3}$$

The quantity ρ^{-1} is closely related to μ (structured singular value) [23]. If $\mathcal{K}$ is a set of all $\mathbb{C}^{r\times m}$ ($\mathbb{R}^{r\times m}$) matrices, then ρ is a complex (real) stability radius [16]. Of course, the knowledge of the entire set D provides much more information than the value of ρ. For instance, for design purposes a designer can solve performance or specification problems on the set of all stabilizing controllers D. See examples in [9].

The paper is organized as follows. In Section 2 we present the general equation of D-decomposition and exhibit its links with $M-\Delta$ approach to robustness. The rank-one case (i.e. single-input or single-output systems) will be addressed in Section 3. The main contribution of this Section is the study of a D-decomposition geometry. In particular, we estimate the number of all root invariant regions as well as the number of simply connected stability regions and verify that the estimates are not conservative. The related results can be found in our papers [6]-[7]. The situation with $K = kI$, k being real or complex scalar, is analyzed in Section 4. Section 5 is devoted to double-input double-output systems with two parameters in a gain matrix.

II. EQUATION OF D-DECOMPOSITION

Let $A \in \mathbb{R}^{n\times n},\ B \in \mathbb{R}^{n\times r},\ C \in \mathbb{R}^{m\times n}$ be fixed real matrices while $\mathcal{K}$ is a class of real or complex $r\times m$ matrices. The class will be specified later. The only property required at the moment is: $\mathcal{K}$ is a connected set, i.e. $K_0 \in \mathcal{K}$, $K_1 \in \mathcal{K}$ imply the existence of a parametric family $K(t) \in \mathcal{K},\ 0 \le t \le 1$, $K(0) = K_0$, $K(1) = K_1$ with $K(t)$ continuously depending on t. We also assume that A has no imaginary eigenvalues in the continuous-time case and no eigenvalues on the unit circumference in the discrete-time case.

Define a transfer function $M(s) = C(A - sI)^{-1}B$ for the continuous-time case and $M(z) = C(A - zI)^{-1}B$ for the discrete-time case, where variables s and z are used to distinguish continuous-time and discrete-time settings elsewhere.

Definition. The set $D(l) = \{K \in \mathcal{K} : A + BKC$ has l stable eigenvalues$\}$, $l = 0, ..., n$ is called *eigenvalue invariant domain* (thus $D(n)$ is the set of stabilizing matrices). The equation for the boundaries of $D(l)$, $l = 0, ..., n$, is called *D-decomposition* of the parameter space. The simply connected components of $D(l)$ are *eigenvalue invariant regions*.

Theorem 1: The equation

$$\det(I + M(j\omega)K) = 0, \qquad -\infty < \omega < \infty \tag{4.a}$$

or

$$\det(I + M(e^{j\omega})K) = 0, \qquad 0 \le \omega < 2\pi \tag{4.b}$$

defines the D-decomposition of the class $\mathcal{K}$, i.e. if $Q \subset \mathcal{K}$ is a connected set and $\det(I + M(j\omega)K) \ne 0$, $-\infty < \omega < \infty$, $\forall K \in Q$ (continuous time) or $\det(I + M(e^{j\omega})K) \ne 0,\ 0 < \omega < 2\pi,\ \forall K \in Q$ (discrete time), then $A + BKC$ has the same number of stable and unstable eigenvalues for all $K \in Q$.

Equations (4.a-4.b) define D-decomposition in the implicit form. Our main goal below is to point out some particular cases where the boundaries can be constructed explicitly. That is in contrast with μ -analysis, where the problem $\min\limits_{K\in\mathcal{K},\ \det(I+M(j\omega)K)=0} \|K\|$ is under consideration (i.e. one is seeking for the largest ball contained in D).

Of course the complete description of D-decomposition is possible for exceptional cases only. They will be addressed in the following sections.

III. SINGLE-INPUT OR SINGLE-OUTPUT SYSTEMS

Suppose we deal with a single-input system, i.e. $r = 1$. Then K is a row vector: $K = [k_1, ..., k_m]$ while M is a column vector $M = [M_1, ..., M_m]^T \in \mathbb{C}^m$. For $a, b \in \mathbb{C}^m$ one has $\det(I + ab^T) = 1 + \sum\limits_{i=1}^{m} a_i b_i$. Thus (4.a) is reduced to

$$1 + \sum_{i=1}^{m} k_i M_i(j\omega) = 0 \tag{5}$$

We conclude that in this case equation (4.a) is linear in K. Similarly, for single-output systems $m = 1$, $K = [k_1, ..., k_r]^T$ is a column vector and we obtain the same equation. We will focus on the simplest cases when equation (5) provides graphical tools to describe D-decomposition in the space of parameters k.

A. ONE REAL PARAMETER

For a single-input single-output system with a real scalar gain k (4.a) reads as

$$1 + kM(j\omega) = 0,\ M(s) = \frac{b(s)}{a(s)}, \tag{6}$$

with a scalar transfer function $M(s) = \frac{b(s)}{a(s)}$, where $a(s)$, $b(s)$ are polynomials of degree n. We avoid the situations when $a(s)$, $b(s)$ have a common imaginary (or zero) root. Thus (4.a) is equivalent to $-1/k = M(j\omega)$ or to the standard Nyquist diagram: the critical values of the gain k (such that correspond to a change of the stable roots number for the polynomial $p(s, k) = a(s) + kb(s)$) are defined by intersections of the Nyquist plot $M(j\omega)$ with the real axis. Non-graphical tools to find critical gains are presented in [15].

Theorem 2: The real axis can be divided into $m \le n + 2$ root invariant intervals $(-\infty, k_1), (k_1, k_2), ..., (k_m, \infty)$ with $-\infty < k_1 < k_2 < ... < k_m < k_{m+1} < \infty$ such that for $k_i < k < k_{i+1}$ the polynomial $p(s, k)$ has the invariant number ν_i of stable roots. Moreover, the number of stability intervals (i.e. intervals (k_i, k_{i+1}) with $\nu_i = n$) is no more than $\lfloor \frac{n}{2} \rfloor + 1$ ($\lfloor \alpha \rfloor$ is the biggest integer smaller or equal α).

The examples below verify that the estimates of the number of root invariant intervals and stability intervals provided by Theorem 2 are not conservative. But we start with an example, where the D-decomposition is lacking – for any k the polynomial $p(s, k)$ has the same number of stable and unstable roots.

Example 1: Let for $n = 4m$, $p(s, k) = s^n + ks + 1$. Then $p(j\omega, k) = \omega^n + kj\omega + 1$, and Re $p(j\omega, k) \ne 0$ for all k. Thus there are no critical values of ω, and the entire real axis is the single root invariant region for the polynomial $p(s, k)$ (indeed it has $2m$ stable and $2m$ unstable roots for any k). A minor variation of the example: $p(s, k) = k(s^n + 1) + s$ provides real axis with an exception of the origin as the root invariant region: for any $k \ne 0$, $p(s, k)$ has $2m$ stable and $2m$ unstable roots.

Example 2: This is the modification of a $2D$ example in [14]. The polynomial $p(z, k) = z^n + kz^{n-1} + \alpha z^{n-2} + \beta$ with $1 < \alpha < 1 + \frac{2}{(n-2)^2}$, $\beta = 1 - \alpha - \frac{1}{n^2}$ has $\lfloor \frac{n}{2} \rfloor$ stability intervals in k.

Indeed, D-decomposition is given by $k = -e^{j\omega} - \alpha e^{-j\omega} - \beta e^{-(n-1)j\omega} = \psi(\omega)$. The equation Im $\psi(\omega) = 0$ reads $(\alpha - 1)\sin\omega + \beta\sin(n-1)\omega = 0$, it has n solutions on $[0, \pi]$ because $|\beta| > |\alpha - 1|$. The values $0 = \omega_1 < \omega_2 < ... < \omega_n \leq \pi$ increase monotonically and critical values $k_i =$ Re $\psi(\omega_i) = (\alpha+1)\cos\omega + \beta\cos(n-1)\omega$ also increase monotonically. The derivatives of Im $\psi(\omega)$ change sign at the points ω_i. For large k the polynomial $p(z, k)$ has $n-1$ stable roots (close to zero) and one unstable root ($z \approx -k$). Thus $\nu_0 = n-1$ and there are $\lfloor \frac{n}{2} \rfloor$ stability intervals for k varying from $-\infty$ to $+\infty$.

B. ONE COMPLEX PARAMETER

We are in the same setting as above ($r = m = 1$) but now $k \in \mathbb{C}$. Equation (6) gives the formula

$$k(\omega) = -M(j\omega)^{-1} = -\frac{a(j\omega)}{b(j\omega)}, \tag{7}$$

where $b(j\omega)$ has no roots with zero real part. The same result follows from the direct analysis of the characteristic polynomial

$$p(s, k) = a(s) + kb(s). \tag{8}$$

It has imaginary roots $j\omega$ for k defined by (7). Curve (7) for $-\infty < \omega < \infty$ decomposes the complex plane into root invariant regions. Their number is estimated below.

Theorem 3: The number N of root invariant regions for polynomial (8) on the complex plane k is $N \leq (n-1)^2 + 2$.

This result is valid for both continuous-time and discrete-time polynomials. The proof of Theorem 3 exploits some tools of algebraic geometry (e.g. Bezout theorem on the number of real roots for two polynomials in two variables and Euler formula).

Example 3: The polynomial $p(z, k) = z^n + kz^{n-1} + \alpha$, where $k \in \mathbb{C}$, has $(n-1)^2 + 1$ root invariant regions for $\alpha > 1$ and two root invariant regions for $\alpha < 1/(n-1)$.

D-decomposition is given by the parametric curve $k(\omega) = -e^{j\omega} - \alpha e^{-j\omega(n-1)}$, $0 \leq \omega < 2\pi$, which describes a hypotrohoid. The quantity of stable roots in each region

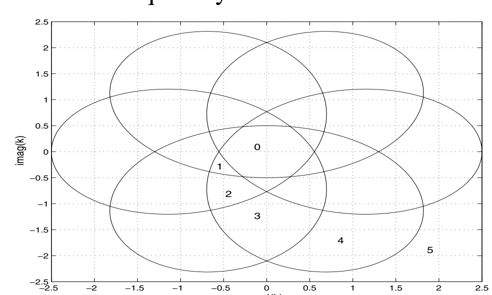

Fig. 1. Maximal number of root invariant regions in Example 3

is marked by digits, the same notation is used elsewhere. For $n = 6$, $\alpha = 1.5$ the decomposition is shown in Fig. 1. It is interesting to note that there are no stability regions in this case. Note that the minimal number of root invariant regions is one.

Example 4: D-decomposition for the polynomial $s^n + k$, where $n = 2m$, $k \in \mathbb{C}$, consists of one ray $(-\infty, 0]$ for m even and $[0, \infty)$ for m odd and there are m stable roots for any k except this ray.

Using the mapping $s = \frac{z+1}{z-1}$, we can proceed from the continuous-time case to the discrete one. Thus the discrete analog of this example is $(z+1)^n + k(z-1)^n$ and it also has one root invariant region.

C. TWO REAL PARAMETERS

This is the case of single-input double-output ($r = 1$, $m = 2$) or double-input single-output ($r = 2$, $m = 1$) systems. The parameters k_1, k_2 are assumed to be real and the equation of D-decomposition (5) reads $1 + k_1 M_1(j\omega) + k_2 M_2(j\omega) = 0$. For transfer functions $M_1(s) = \frac{b(s)}{a(s)}$, $M_2(s) = \frac{c(s)}{a(s)}$ the characteristic polynomial is

$$p(s, k) = a(s) + k_1 b(s) + k_2 c(s) \tag{9}$$

and the above equation is reduced to $p(j\omega, k) = a(j\omega) + k_1 b(j\omega) + k_2 c(j\omega)$. This is the classical setting of D-decomposition [12]-[13]. In general, D-decomposition consists of a parametric curve and singular lines. The curve separates regions with ± 2 difference in the number of stable roots and singular lines separate regions with ± 1 difference in the number of stable roots.

Theorem 4: The number N of root invariant regions for the polynomial (9) on the $\{k_1, k_2\}$ plane has the following upper bound: $N \leq 2n(n-1) + 3$.

The smallest number of root invariant regions is one, see the example below.

Example 5: Let $p(s, k) = s^n + k_1 s^3 + k_2 s + 1, n = 4m$. Then equation $p(j\omega, k) = 0$ has no solutions for arbitrary ω (because Re $p(j\omega, k) \neq 0$) and $\mathbb{R}^2$ plane is the only root invariant region: for any k the polynomial $p(s, k)$ has $2m$ stable and $2m$ unstable roots.

Example 6: The following example demonstrates that the number of root invariant regions N can achieve $O(n^2)$. Let $p(s, k) = a(s^2) + s(k_1 b(s^2) + k_2 c(s^2) + \alpha)$, where $a(t), b(t), c(t)$ are polynomials of degree $m, m-1, m-1$ correspondingly (thus $p(s, k)$ has degree $n = 2m$), $a(t)$ has m negative real roots $-t_i^2$, $i = 1, \ldots, m$. Then D-decomposition equation is $p(j\omega, k) = U(\omega^2) + j\omega V(\omega^2) = 0$ and we get two equations $U(\omega^2) = a(-\omega^2) = 0$, $\omega V(\omega^2) = \omega(k_1 b(-\omega^2) + k_2 c(-\omega^2) + \alpha) = 0$. The first equation does not depend on k, it has n real roots $\omega_i = \pm t_i$. Hence D-decomposition is generated by singular straight lines $k_1 b(\omega_i^2) + k_2 c(\omega_i^2) + \alpha = 0$, their total number equals m. The plane is divided into $(m^2 + m)/2 + 1$ regions by m straight lines of generic position, thus $N = n^2/8 + o(n^2)$.

Consider the characteristic polynomial with the structure $p(s, k_I, k_P, k_D) = a(s)(k_I + k_P s + k_D s^2) + b(s)$, which correspond to a system with PID controller. For any fixed k_P D-decomposition consists of straight lines. These lines divide $(k_I,\ k_D)$-plane into a finite number of convex polygons. An approach for the calculation of root invariant regions in the $(k_I,\ k_P,\ k_D)$-space is to grid k_P and use a tomographic representation of the result. This idea is developed in [3], [2], [5], [21].

What is the largest number of stability regions is an open problem. The following example (originated in [14]) demonstrates that this number can achieve $n-1$.

Example 7: Suppose $p(z,k) = z^n + k_1 z^{n-1} + \alpha z^{n-2} + k_2,\ 1 < \alpha < \frac{n}{n-2}$. Then there are $n-1$ simply connected stability regions in $\{k_1,\ k_2\}$-plane. The structure of the regions for $n = 5$ $\alpha = 1.05$ can be seen in Fig. 2; $n-3$ regions are the loops of the D-decomposition curve while two other regions are generated by the intersection of the curve with two singular lines.

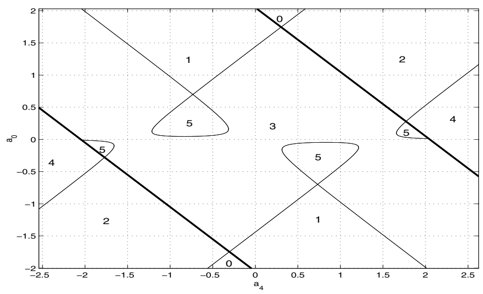

Fig. 2. Root invariant regions in Example 7

IV. SCALAR GAIN

In this section we address systems with a scalar gain, i.e. $m = r$ and $K = kI,\ k \in \mathbb{R}$ or $k \in \mathbb{C}$. By the terminology of μ-analysis this is the class $\mathcal{K}$ with one scalar block. Then the matrix $A + BKC$ is equal to $A + kBC$ and the problem is reduced to the simplest one: given $n \times n$ real matrices A and F, find $D(l) = \{k \in \mathbb{C}$ (or $k \in \mathbb{R}$) : $A + kF$ has l stable eigenvalues$\}$.

Equation (4.a) or (4.b) now reads

$$\det(I + kM(j\omega)) = 0, \qquad -\infty < \omega < \infty \tag{10.a}$$

or

$$\det(I + kM(e^{j\omega})) = 0, \qquad 0 \le \omega < 2\pi. \tag{10.b}$$

If we denote the eigenvalues of $M(j\omega)$ or $M(e^{j\omega})$ as $\lambda_i(\omega),\ i = 1,...,n$, equations (10.a)-(10.b) split into $1 + k\lambda_i(\omega) = 0,\ i = 1,...,n$ and a D-decomposition boundary consists of n branches

$$k_i(\omega) = -\frac{1}{\lambda_i(\omega)},\ i = 1,...,n. \tag{11}$$

Equation of D-decomposition (11) can be obtained in a different form with no use of the transfer function. If $A + kF$ ($F = BC$) has an imaginary eigenvalue then the matrix $A + kF - j\omega I$ is singular for some $\omega \in \mathbb{R}$, that is $(A + kF - j\omega I)x = 0$ for $x \in \mathbb{C}^n$ or $(A - j\omega I)x = -kFx$. Thus we conclude that k is a generalized eigenvalue for the matrix pair $A - j\omega I$ and $-F$:

$$k(\omega) = \mathrm{eig}(A - j\omega I, -F). \tag{12}$$

Similarly for the discrete-time case

$$k(\omega) = \mathrm{eig}(A - e^{j\omega} I, -F). \tag{13}$$

In contrast with (11), (12)–(13) can be used when $A - j\omega I$ (or $A - e^{j\omega} I$) is singular, however the total number of the generalized eigenvalues in this case can be less than n.

Note that eigenvalues are complex numbers, thus $k(\omega)$ provided by (11) or (12) are complex as well. For the case $k \in \mathbb{C}$ these equations generate the a boundary of eigenvalue invariant regions $D(l)$. There are some special cases, when eig(M) or eig$(A - j\omega I, -F)$ can be calculated explicitly. However in most situations we construct the boundary numerically as follows.

Algorithm.

a. Choose a grid $\omega \in \mathbb{R}$ (or $\omega \in [0, 2\pi]$ for discrete-time case).
b. Calculate $A - j\omega I$ (or $A - e^{j\omega} I$) for all ω in the grid.
c. Calculate $k(\omega) = \mathrm{eig}(A - j\omega I, -BC)$ (or $k(\omega) = \mathrm{eig}(A - e^{j\omega} I, -BC)$).
d. Plot $k(\omega)$ in the complex plane.

In general $k(\omega)$ consists of n branches, however the resulting curve can split into a smaller number of disconnected arcs. For instance, it may happen that $k(\omega)$ is the single Jordan curve (see Example 8 and Fig. 3).

Example 8: $A = \begin{bmatrix} 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 1 & 1 & \dots & 1 & 0 \end{bmatrix}, F = \beta I + \alpha A^T,$

where A, I are $n \times n$ matrices. D-decomposition for discrete-time case, $n = 4, \alpha = 0.01, \beta = 1$ is depicted in Fig. 3 and consists of one Jordan curve.

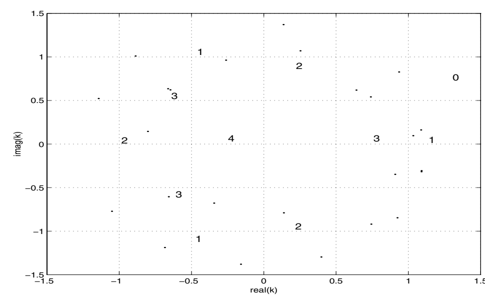

Fig. 3. D-decomposition in Example 8

It is not clear, what is the largest number N of eigenvalue invariant regions. We conjecture that $N = O(n^3)$ (compare with the estimates $O(n^2)$ from the previous section). Also the largest number of simply connected stability regions is not known yet. The smallest number of eigenvalue invariant regions is one. Indeed, set $\alpha = -1, \beta = 0$ in Example 8, then the characteristic polynomial of $A + kF$ can be calculated explicitly: $p(z,k) = (z+1)^n k + (z-1)^n$, and after a standard change of variables we are in the framework of Example 4.

For real k D-decomposition technique should be modified as follows. First, we draw the curves $k(\omega)$ as in the complex case. Second, we find the intersections k_i of $k(\omega)$ with the real axis. If we order these points such that $k_1 < k_2 < ... < k_N$, then intervals $(-\infty, k_1), (k_1, k_2), ...(k_N, \infty)$ are eigenvalue invariant regions. The number of such intervals can be estimated.

Theorem 5: For real k the number of intervals preserving the same number of stable eigenvalues of $A + kBC$ does not exceed $n(n+1) + 1$.

For $n = 2, 3$ the estimate is not conservative, as illustrated below.

Example 9: a. $n = 2, A = \begin{bmatrix} 0 & 0.9 \\ 0.9 & 0 \end{bmatrix},$

$B = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$, $C = I$. There are $N = 7$ eigenvalue invariant intervals, 3 of them are stability intervals.

b. $n = 3$, $A = \begin{bmatrix} 0.95 & 1 & 0 \\ 0 & 0 & 0.6 \\ 0 & 0 & -0.95 \end{bmatrix}$,
$B = \begin{bmatrix} 0 & 0 & -0.22 \\ 0 & -0.3 & 0 \\ 0.4 & 0 & 0 \end{bmatrix}$, $C = I$. Here (Fig. 4) there are $N = 13$ eigenvalue invariant intervals and 5 stability intervals.

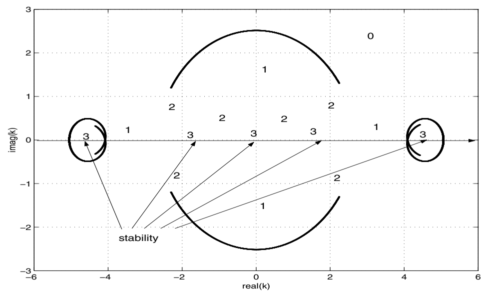

Fig. 4. Stability intervals in Example 9.b

V. DOUBLE-INPUT DOUBLE-OUTPUT SYSTEMS

We consider the case $r = m = 2$ and K real. Then for $M = \begin{bmatrix} m_1 & m_2 \\ m_3 & m_4 \end{bmatrix}$, $K = \begin{bmatrix} k_1 & k_3 \\ k_2 & k_4 \end{bmatrix}$, $m_i \in \mathbb{C}$, $k_i \in \mathbb{R}$, $i = 1, ..., 4$ equation (4.a) has the form

$$0 = \det(I + MK) = 1 + \sum_{i=1}^{4} k_i m_i + \det M \det K. \qquad (14)$$

This quadratic in K equation defines D-decomposition of the $4D$ space $K \in \mathbb{R}^{2\times 2}$. To take an opportunity of the graphical representation we restrict ourselves by situations with K depending on two parameters only.

A. CASE 1. $K = \mathrm{diag}(k_1,\ k_2)$

For $K = \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix}$ equation (14) becomes $0 = 1 + k_1 m_1 + k_2 m_4 + k_1 k_2 (m_1 m_4 - m_2 m_3)$. Substituting $m_i = u_i + j v_i,\ i = 1, ..., 4$ we get two quadratic equation in two variables $k_1,\ k_2$:

$$\begin{array}{c} 1 + k_1 u_1 + k_2 u_4 + \alpha k_1 k_2 = 0, \\ k_1 v_1 + k_2 v_4 + \beta k_1 k_2 = 0, \end{array} \qquad (15)$$

where $\alpha = u_1 u_4 - v_1 v_4 - u_2 u_3 + v_2 v_3$, $\beta = u_1 v_4 + v_1 u_4 - u_2 v_3 - v_2 u_3$. The quantities $u_i,\ v_i,\ \alpha,\ \beta$ depend on ω. For $\omega = 0$ the matrix $M(j\omega) = C(A - j\omega I)^{-1} B$ is real and $v_i(0) = 0,\ i = 1, ..., 4$ as well as $\beta(0) = 0$. Hence the second equation vanishes and the first equation $1 + k_1 u_1(0) + k_2 u_4(0) + \alpha(0) k_1 k_2 = 0$ defines the singular curve (a hyperbola).

For $\omega \neq 0$ we solve system (15). If for some ω the solution is complex we ignore it because it does not belong to D-decomposition.

Example 10: The discrete-time system $A + B \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix} C$, $A = \begin{bmatrix} -0.8848 & 0.4457 \\ -0.8733 & -0.9326 \end{bmatrix}$, $B = \begin{bmatrix} 0.3914 & 0.2508 \\ -0.5576 & 0.0266 \end{bmatrix}$, $C = \begin{bmatrix} 0.1514 & 0.7854 \\ -0.4255 & -0.8148 \end{bmatrix}$

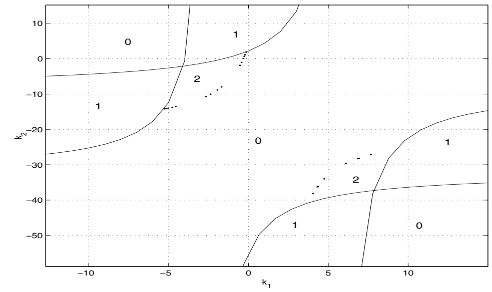

Fig. 5. D-decomposition in Example 10

has a typical D-decomposition structure. In Fig. 5 one can see two singular hyperbolas (subtle lines) and two branches of the nonsingular curve (solid lines).

5.2 CASE 2. $K = \begin{bmatrix} -k_1 & k_2 \\ k_2 & k_1 \end{bmatrix}$

The calculations are similar to the ones for Case 1 and (14) becomes $1 - k_1(u_1 - u_4) + k_2(u_2 - u_3) - \alpha(k_1^2 + k_2^2) = 0$, $-k_1(v_1 - v_4) + k_2(v_2 - v_3) - \beta(k_1^2 + k_2^2) = 0$. The D-decomposition curves are the parametric curve $k_1(\omega),\ k_2(\omega),\ \omega \neq 0$ and the singular curve – the circumference $1 + k_1(u_1 - u_4) + k_2(u_2 - u_3) - \alpha(k_1^2 + k_2^2) = 0$, $\omega = 0$.

Example 11: This continuous-time example is originated in [16] (p. 889, Example 2). Here $n = 4$, $m = r = 2$, $A = \begin{bmatrix} 79 & 20 & -30 & -20 \\ -41 & -12 & 17 & 13 \\ 167 & 40 & -60 & -38 \\ 33.5 & 9 & -14.5 & -11 \end{bmatrix}$, $B = \begin{bmatrix} .219 & .9346 \\ .047 & .3835 \\ .6789 & .5194 \\ .6793 & .831 \end{bmatrix}$, $C = \begin{bmatrix} .0346 & .5297 & .0077 & .0668 \\ .0535 & .6711 & .3834 & .4175 \end{bmatrix}$.

The smallest norm perturbation destroying the stability of $A + BKC$ is $K^* = \begin{bmatrix} -0.4996 & 0.1214 \\ 0.1214 & 0.4996 \end{bmatrix}$, that is it has the form considered in the present subsection. The D-decomposition of (k_1, k_2)-plane for matrices $K = \begin{bmatrix} -k_1 & k_2 \\ k_2 & k_1 \end{bmatrix}$ is shown in Fig.6. There are two disconnected

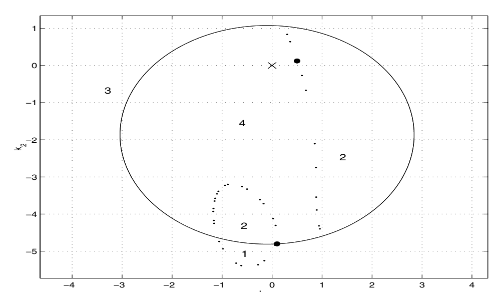

Fig. 6. D-decomposition in Example 11

components of the parametric curve (solid lines) and one singular curve — the circumference (subtle line). The nearest to the origin point on the boundary of the stability domain is $k_1^* = 0.4996$, $k_2^* = 0.1214$; it corresponds to the matrix K^* above. Other directions preserve stability for larger perturbations. For instance, $A + BKC$ is stable for

$K = \lambda \begin{bmatrix} -0.0211 & -0.9998 \\ -0.9998 & 0.0211 \end{bmatrix}$, $0 \leq \lambda \leq 4.8032$, in particular for $K_1 = \begin{bmatrix} -0.1013 & -4.802 \\ -4.8021 & 0.1013 \end{bmatrix}$, $||K_1|| = 4.8$ (0.5141 being the real stability radius).

5.3 CASE 3. $K = \begin{bmatrix} k_1 & k_2 \\ -k_2 & k_1 \end{bmatrix}$

This is a real 2×2 analog of a complex scalar (note that the eigenvalues of such K are $k_1 \pm jk_2$). For such K equation (14) reads $0 = 1 + k_1(m_1 + m_4) - k_2(m_2 - m_3) + (k_1^2 + k_2^2)(m_1 m_4 - m_2 m_3)$ and (15) is replaced with

$$\begin{aligned} &1 + k_1(u_1 + u_4) - k_2(u_2 - u_3) + \alpha(k_1^2 + k_2^2) = 0, \\ &k_1(v_1 + v_4) - k_2(v_2 - v_3) + \beta(k_1^2 + k_2^2) = 0, \end{aligned} \quad (16)$$

where $u_i(\omega)$, $v_i(\omega)$, $\alpha(\omega)$, $\beta(\omega)$ are the same as above. For $\omega = 0$ we get $v_i(0) = 0$, $\beta(0) = 0$ and the second equation vanishes while the first equation $1 + k_1(u_1(0) + u_4(0)) - k_2(u_2(0) - u_3(0)) + \alpha(0)(k_1^2 + k_2^2) = 0$ is the equation of a circumference. For $\omega \neq 0$ we can solve (16) and define $k_1(\omega)$, $k_2(\omega)$ (provided that k_1, k_2 are real and (16) is nonsingular). Thus D-decomposition consists of the components of this curve and singular circumference.

Example 12: This discrete-time example is again borrowed from [16] (p. 889, Example 3): ($n = 3$, $m = r = 2$). The optimal solution for (3) is supplied with $K^* = \begin{bmatrix} 0.8483 & 0.5971 \\ -0.5971 & 0.8483 \end{bmatrix}$; it has the form $\begin{bmatrix} k_1 & k_2 \\ -k_2 & k_1 \end{bmatrix}$. Thus we restrict ourselves with matrices K of this form and construct D-decomposition for these matrices. It has the same structure as in the previous subsection. It is generated by two singular circumferences (subtle lines) and one parametric curve (solid line) and is shown in Fig. 7. The distance from the origin to the boundary of the stability domain, in accordance with [16], equals 1.0374. However, other directions allow larger values of perturbations preserving stability. For instance, if $K = \lambda \begin{bmatrix} -0.9680 & -0.2508 \\ 0.2508 & -0.9680 \end{bmatrix}$, then $A + BKC$ remains stable for $0 \leq \lambda \leq 19.6932$, and an admissible perturbation has the norm 19.6932, that is 18.9832 times larger than the real stability radius.

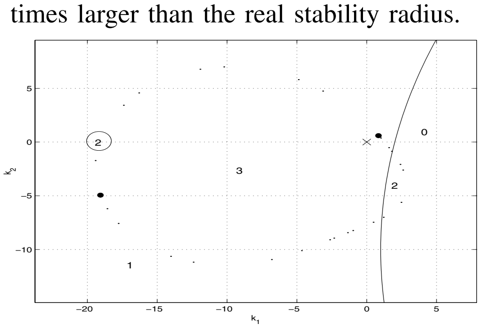

Fig. 7. D-decomposition in Example 12

VI. CONCLUSIONS

We provided the simple and effective techniques to construct the stability domain in the parameter space. This is an extension of D-decomposition method for polynomials. Simultaneously with the stability domain we construct all root invariant regions, i.e. simply connected regions of the domains with the invariant number of eigenvalues of the system matrix. This technique can be helpful for low-order controller design and for detailed robustness analysis. The main limitation of the proposed approach is the low dimensionality of the parameter space (one or two).

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