Proceedings of the
44th IEEE Conference on Decision and Control, and
the European Control Conference 2005
Seville, Spain, December 12-15, 2005

Asymptotic Robust Adaptive Tracking of Parametric Strict-Feedback Systems with Additive Disturbance

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Abstract: This paper deals with the tracking control of multi-input/multi-output nonlinear parametric strict-feedback systems in the presence of additive disturbances and parametric uncertainties. For such systems, robust adaptive controllers usually cannot ensure asymptotic tracking or even regulation. In this work, under the assumption the disturbances are $\mathcal{C}^2$ with bounded time derivatives, we present a $\mathcal{C}^0$ robust adaptive control construction that guarantees the tracking error is asymptotically driven to zero.

1 Introduction

Robust adaptive control laws in the presence of additive disturbances can generally ensure closed-loop signal boundedness and convergence of the tracking error (state) to a residual bounded set with size of the order of the disturbance magnitude, *but not asymptotic tracking* (*or regulation*). See [3, 4, 5, 6, 7, 8, 10] for examples of such results. In this paper, we consider multi-input/multi-output (MIMO) nonlinear parametric strict-feedback systems subjected to bounded additive disturbances that are twice continuously differentiable and have bounded time derivatives. For these systems, we present a continuous robust adaptive control construction that guarantees *asymptotic tracking*. The proposed construction is based on the nonlinear robust control technique of [9], which was originally used to compensate for unstructured uncertainties. Here, we use it as a robustifying mechanism for adaptive controllers. That is, adaptation is used to compensate for structured (parametric) uncertainties while the robust mechanism compensates for disturbances, hence recovering the disturbance-free, asymptotic tracking property of the adaptive controller. The standard adaptive backstepping design is judiciously modified to allow the use of the robust control technique of [9]. Also instrumental to our new construction is the use of the *sufficiently smooth* projection-based adaptation law recently introduced in [2]. This allows the adaptive stabilizing functions of the backstepping design to be differentiable as many times as necessary. The stability analysis shows the proposed robust adaptive controller yields semi-global asymptotic tracking.

2 Problem Statement

We consider a class of parametric strict-feedback systems of the form

$$\dot{x}_1 = \varphi_1^{\intercal}(x_1)\,\theta + x_2 \tag{1a}$$

$$\vdots$$

$$\dot{x}_i = \varphi_i^{\intercal}(x_1,...,x_i)\,\theta + x_{i+1} \tag{1b}$$

$$\vdots$$

$$\dot{x}_n = \varphi_n^{\intercal}(x_1,x_2,...,x_n)\,\theta + d + u \tag{1c}$$

where $x_i(t) \in \mathbb{R}^m$, $i = 1,...,n$ are the system states, $\varphi_i \in \mathbb{R}^{p\times m}$, $i = 1,...,n$ are known nonlinearities, $\theta \in \mathbb{R}^p$ is an uncertain constant parameter vector, $d(t) \in \mathbb{R}^m$ is an uncertain additive disturbance, $u(t) \in \mathbb{R}^m$ is the control input, and $y = x_1$ is the system output. We make the following assumptions regarding the system:

A1. $\varphi_i \in \mathcal{C}^{n+1-i}$, $i = 1,...,n$.

A2. $d \in \mathcal{C}^2$ and $\|d(t)\|_{\mathcal{L}_\infty} \le \bar{d}_0$, $\left\|\dot{d}(t)\right\|_{\mathcal{L}_\infty} \le \bar{d}_1$, and $\left\|\ddot{d}(t)\right\|_{\mathcal{L}_\infty} \le \bar{d}_2$ where $\bar{d}_0, \bar{d}_1, \bar{d}_2$ are known positive constants.

A3. The parameter vector θ belongs to a compact convex set $\Omega := \{\theta : \|\theta\| \le \theta_0\}$ where θ_0 is a known positive constant.

Let the output tracking error be defined as

$$e := y - y_r \tag{2}$$

where the $\mathcal{C}^{n+1}$ reference trajectory $y_r(t) \in \mathbb{R}^m$ is such that

$$y_r^{(i)}(t) \in \mathcal{L}_\infty,\ i = 0,...,n+1, \tag{3}$$

and $(\cdot)^{(i)}(t)$ denotes the ith derivative with respect to time. Our goal is to construct a state feedback control $u(x_1,x_2,...,x_n)$ that ensures $e(t) \to 0$ as $t \to \infty$ and the boundedness of all closed-loop signals. The following notation will be used throughout the paper: $\hat{\theta}_i(t) \in \mathbb{R}^p$, $i = 1,...,n-1$ are parameter estimates;

$$\tilde{\theta}_i(t) := \theta - \hat{\theta}_i(t),\ i = 1,...,n-1 \tag{4}$$

denote the corresponding parameter estimation errors; $\text{Proj}(\mu_i, \hat{\theta}_i) \in \mathbb{R}^p$, $i = 1,...,n-1$, $\forall \mu_i(t) \in \mathbb{R}^p$ denote $\mathcal{C}^{n-i-1}$ projection operators used to ensure $\hat{\theta}_i(t) \in \mathcal{L}_\infty$ independent of the stability analysis (see Appendix A for details); $\Gamma_i \in \mathbb{R}^{p\times p}$, $i = 1,...,n-1$ are constant, diagonal, positive-definite matrices; and c_i, $i = 1,...,n+1$ are positive constants. To reduce the notational complexity and facilitate the readability of the paper, the control construction that follows is presented for the case where $m = 1$. Note, however, that *the main result is readily applicable to the MIMO case*. The following inequalities will also be used throughout the paper:

$$\ln(\cosh(\|\xi\|)) \le \sum_{i=1}^{q} \ln(\cosh(\xi_i)) \le \|\xi\|^2,\ \forall \xi \in \mathbb{R} \tag{5}$$

$$\|\xi\| \le (\|\xi\| + 1)\,\|\tanh(\xi)\|,\ \forall \xi \in \mathbb{R}. \tag{6}$$

3 Construction of Control Law

Step 1

We begin by differentiating (2) and substituting from (1a) to obtain

$$\dot{e} = \varphi_1^\intercal(y)\theta + x_2 - \dot{y}_r. \tag{7}$$

After adding and subtracting the term $\varphi_1^\intercal(y_r)\theta$ to (7), the error system becomes

$$\dot{e} = \varphi_1^\intercal(y_r)\theta + x_2 - \dot{y}_r + \tilde{w}_1. \tag{8}$$

where

$$\tilde{w}_1 = \left(\varphi_1^\intercal(y) - \varphi_1^\intercal(y_r)\right)\theta \tag{9}$$

Remark 1 Due to assumption A1 and (6), we can use the Mean Value theorem to show

$$\|\tilde{w}_1\| \le \rho_{11}\left(\|e\|\right)\|\tanh(e)\| \tag{10}$$

where $\rho_{11}(\cdot) \in \mathbb{R}_{\ge 0}$ is some globally invertible, nondecreasing function.

Let

$$\eta_2 := \frac{1}{c_1}\left(x_2 - \alpha_1\right) \tag{11}$$

where α_1 is a stabilizing function yet to be designed. To facilitate the notation, let

$$\eta_1 := e, \tag{12}$$

and define

$$V_1 = \ln\left(\cosh(\eta_1)\right) + \frac{1}{2}\tilde{\theta}_1^\intercal\Gamma_1^{-1}\tilde{\theta}_1. \tag{13}$$

Differentiating (13) along (8) gives

$$\begin{aligned} \dot{V}_1 &= \tanh(\eta_1)\left(\varphi_1^\intercal(y_r)\theta - \dot{y}_r + \alpha_1 + \tilde{w}_1 + c_1\eta_2\right) \\ &\quad -\tilde{\theta}_1^\intercal\Gamma_1^{-1}\dot{\hat{\theta}}_1 . \end{aligned} \tag{14}$$

Based on (14), we design the stabilizing function and parameter update law as follows

$$\alpha_1 = -c_1\tanh(\eta_1) - \varphi_1^\intercal(y_r)\hat{\theta}_1 + \dot{y}_r \tag{15}$$

$$\dot{\hat{\theta}}_1 = \Gamma_1 \mathrm{Proj}(\mu_1, \hat{\theta}_1), \quad \mu_1 = \varphi_1(y_r)\tanh(\eta_1) \tag{16}$$

Substituting (15) and (16) into (14) gives

$$\dot{V}_1 \le -c_1\tanh^2(\eta_1) + \tanh(\eta_1)\left(c_1\eta_2 + \tilde{w}_1\right) \tag{17}$$

where property P2 of the projection operator was used (see Appendix A).

Remark 2 The final step of our design will require that $\dot{\hat{\theta}}_i\ (t) \in \mathcal{L}_\infty$, $i = 1, ..., n-1$ independently of the stability analysis. This motivates the use of the term $\ln(\cosh(\eta_i))$ in the Lyapunov function of the first $n-1$ steps of the backstepping procedure. In particular, due to property P3 of the projection operator (see Appendix A), we know $\dot{\hat{\theta}}_i\ (t) \in \mathcal{L}_\infty$ if $\mu_i(t) \in \mathcal{L}_\infty$. The boundedness of μ_i is facilitated by the fact that $\partial\ln(\cosh(\eta_i))/\partial\eta_i = \tanh(\eta_i)$.

Remark 3 Using (11) and (15), the state x_2 can be decomposed into

$$x_2 = \underbrace{c_1\eta_2 - c_1\tanh(\eta_1)}_{\zeta_{\eta 1}} \underbrace{-\varphi_1^\intercal(y_r)\hat{\theta}_1 + \dot{y}_r}_{\zeta_{b1}} \tag{18}$$

where the term $\zeta_{\eta 1}$ is a function of $\eta_1$and η_2, and the term ζ_{b1} is *bounded*. The usefulness of this decomposition will become apparent in the next step.

Step i $(2 \le\ i\ \le n-1)$

Let

$$\eta_{i+1} := \frac{1}{c_i}\left(x_{i+1} - \alpha_i\right) \tag{19}$$

where α_i is a stabilizing function, and differentiate $\eta_i := \frac{1}{c_{i-1}}\left(x_i - \alpha_{i-1}\right)$ to obtain

$$\dot{\eta}_i = \frac{1}{c_{i-1}}\left(\varphi_i^\intercal\theta + c_i\eta_{i+1} + \alpha_i - \dot{\alpha}_{i-1}\right) \tag{20}$$

where (1b) was used. The derivative of $\alpha_{i-1}(x_1, ..., x_{i-1}, \hat{\theta}_1, ..., \hat{\theta}_{i-1}, y_r, ..., y_r^{(i-1)})$ can be written as

$$\begin{aligned} \dot{\alpha}_{i-1} &= \Psi_{i-1}(x_1, ..., x_i, \hat{\theta}_1, ..., \hat{\theta}_{i-1}, y_d, ..., y_d^{(i)}) \\ &\quad + \sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_j}\varphi_j^T\theta \end{aligned} \tag{21}$$

where

$$\begin{aligned} \Psi_{i-1} &= \sum_{j=1}^{i-1}\left(\frac{\partial\alpha_{i-1}}{\partial x_j}x_{j+1} + \frac{\partial\alpha_{i-1}}{\partial\hat{\theta}_j}\dot{\hat{\theta}}_j\right) \\ &\quad + \sum_{j=1}^{i}\frac{\partial\alpha_{i-1}}{\partial y_d^{(j-1)}}y_d^{(j)} \end{aligned}$$

is known. Using (21), we can rewrite (20) as

$$\begin{aligned} \dot{\eta}_i &= \frac{1}{c_{i-1}}\Bigg[-\Psi_{i-1} + \alpha_i + c_i\eta_{i+1} \\ &\quad + \underbrace{\left(\varphi_i^T - \sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_j}\varphi_j^T\right)}_{w_i(x_1, ..., x_i, \hat{\theta}_1, ..., \hat{\theta}_{i-1})}\theta\Bigg]. \end{aligned} \tag{22}$$

Adding and subtracting the term $w_{bi}\theta/c_{i-1}$ to (22) yields

$$\dot{\eta}_i = \frac{1}{c_{i-1}}\left(w_{bi}\theta - \Psi_{i-1} + \alpha_i + c_i\eta_{i+1} + \tilde{w}_i\right) \tag{23}$$

where $w_{bi} := w_i(y_r, \zeta_{b1}, ..., \zeta_{b(i-1)}, \hat{\theta}_1, ..., \hat{\theta}_{i-1})$ and

$$\tilde{w}_i = \left(w_i - w_{bi}\right)\theta. \tag{24}$$

Remark 4 Using assumption A1 and (6), we can show

$$\|\tilde{w}_i\| \le c_{i-1} \sum_{j=1}^{i} \rho_{ij}\left(\|\bar{\eta}_i\|\right) \left\|\tanh\left(\eta_j\right)\right\| \quad (25)$$

where $\rho_{ij}(\cdot) \in \mathbb{R}_{\ge 0}$, $j = 1, ..., i$ are some globally invertible, nondecreasing functions and $\bar{\eta}_i := (\eta_1, ..., \eta_i)^\intercal$. Note that the calculation of (25) is facilitated by the fact that $x_i - \zeta_{b(i-1)} = \zeta_{\eta(i-1)}$ (see Remark 3).

Define

$$V_i = V_{i-1} + \ln\left(\cosh\left(\eta_i\right)\right) + \frac{1}{2}\tilde{\theta}_i^\intercal \Gamma_i^{-1} \tilde{\theta}_i \quad (26)$$

whose derivative is

$$\begin{aligned} \dot{V}_i \quad \le \quad & \sum_{j=1}^{i-1} \left[\frac{\tanh\left(\eta_j\right)}{c_{j-1}} \left(c_j \eta_{j+1} + \tilde{w}_j\right) - \frac{c_j \tanh^2(\eta_j)}{c_{j-1}} \right] \\ & + \frac{\tanh\left(\eta_i\right)}{c_{i-1}} \left(w_{bi}\theta - \Psi_{i-1} + \alpha_i + c_i \eta_{i+1} + \tilde{w}_i\right) \\ & - \tilde{\theta}_i^T \Gamma_i^{-1} \dot{\hat{\theta}}_i \end{aligned} \quad (27)$$

where $c_0 = 1$. Based on (27), we design

$$\alpha_i = -c_i \tanh\left(\eta_i\right) - w_{bi}\hat{\theta}_i + \Psi_{i-1} \quad (28)$$

$$\dot{\hat{\theta}}_i = \Gamma_i \mathrm{Proj}(\mu_i, \hat{\theta}_i), \quad \mu_i = \frac{1}{c_{i-1}} w_{bi}^\intercal \tanh\left(\eta_i\right) \quad (29)$$

Substituting (28) and (29) into (27) gives

$$\dot{V}_i \le \sum_{j=1}^{i} \left[\frac{\tanh\left(\eta_j\right)\left(c_j \eta_{j+1} + \tilde{w}_j\right)}{c_{j-1}} - \frac{c_j \tanh^2(\eta_j)}{c_{j-1}} \right]. \quad (30)$$

Remark 5 Using (19) and (28), the state x_{i+1} can be written as

$$x_{i+1} = \zeta_{\eta i} + \zeta_{bi} \quad (31)$$

where

$$\zeta_{\eta i} = c_i \eta_{i+1} - c_i \tanh\left(\eta_i\right) + \Psi_{b(i-1)} - \Psi_{i-1}, \quad (32)$$

$$\zeta_{bi} = \Psi_{b(i-1)} - w_{bi}\hat{\theta}_i, \quad (33)$$

where $\Psi_{b(i-1)} := \Psi_{i-1}(y_r, \zeta_{b1}, ..., \zeta_{b(i-1)}, \hat{\theta}_1, ..., \hat{\theta}_{i-1}, y_r, ..., y_r^{(i)})$, $\zeta_{\eta i}$ is a function of $\eta_1, ..., \eta_{i+1}$, and ζ_{bi} is bounded.

Step n

In the last step, we modify the standard backstepping procedure in order to use the new robust control mechanism of [9] to deal with the additive disturbance in (1c). Let the variable $r(t)$ be defined as

$$r := \frac{1}{c_n}\dot{\eta}_n + \eta_n \quad (34)$$

where

$$\eta_n := \frac{1}{c_{n-1}}\left(x_n - \alpha_{n-1}\right). \quad (35)$$

After differentiating (34), we obtain

$$\dot{r} = \frac{1}{c_n}\ddot{\eta}_n + c_n r - c_n \eta_n. \quad (36)$$

Differentiating η_n twice produces

$$\ddot{\eta}_n = \frac{1}{c_{n-1}}\left(\dot{\varphi}_n^\intercal \theta + \dot{d} + \dot{u} - \ddot{\alpha}_{n-1}\right) \quad (37)$$

where (1c) was used.

We first concentrate on the calculation of the term $\ddot{\alpha}_{n-1}$ in (37). To that end, we have

$$\dot{\alpha}_{n-1} = \sum_{j=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial x_j} \varphi_j^T\left(x_1, \cdots, x_j\right)\theta + \Phi \quad (38)$$

where

$$\Phi = \sum_{j=1}^{n-1} \left(\frac{\partial \alpha_{n-1}}{\partial x_j} x_{j+1} + \frac{\partial \alpha_{n-1}}{\partial \hat{\theta}_j} \dot{\hat{\theta}}_j \right) + \sum_{j=1}^{n} \frac{\partial \alpha_{n-1}}{\partial y_d^{(j-1)}} y_d^{(j)} \quad (39)$$

is *known*. Differentiating (38) gives

$$\begin{aligned} \ddot{\alpha}_{n-1} \quad = \quad & \dot{\Phi} + f_1(x_1, ..., x_n, \hat{\theta}_1, ..., \hat{\theta}_{n-1}, y_d, ..., y_d^{(n-1)}) \\ & + g_1(x_1, ..., x_n, \hat{\theta}_1, ..., \hat{\theta}_{n-1}, y_d, ..., y_d^{(n)}) \end{aligned} \quad (40)$$

where

$$\begin{aligned} g_1(\cdot) \quad = \quad & \sum_{j=1}^{n-1} \left\{ \frac{\partial \alpha_{n-1}}{\partial x_j} \left[\sum_{k=1}^{j} \left(\frac{\partial \varphi_j^T}{\partial x_k} \left(x_{k+1} + \varphi_k^T \theta\right) \right) \right] \right. \\ & + \sum_{k=1}^{n} \frac{\partial^2 \alpha_{n-1}}{\partial x_j \partial y_d^{(k-1)}} y_d^{(k)} + \sum_{k=1}^{n-1} \left[\frac{\partial^2 \alpha_{n-1}}{\partial x_j \partial \hat{\theta}_k} \gamma_{bk} \right. \\ & \left. \left. \frac{\partial^2 \alpha_{n-1}}{\partial x_j \partial x_k} \left(x_{k+1} + \varphi_k^T \theta\right) \right] \right\} \varphi_j^T \theta, \end{aligned} \quad (41)$$

$$f_1(\cdot) = \sum_{j=1}^{n-1} \left(\sum_{k=1}^{n-1} \frac{\partial^2 \alpha_{n-1}}{\partial x_j \partial \hat{\theta}_k} \gamma_{\mu k} \right) \varphi_j^T \theta, \quad (42)$$

and property P3 of the projection operator was used.

We now turn our attention to the calculation of the term $\dot{\varphi}_n^T \theta$ in (37). To that end, we have

$$\begin{aligned} \dot{\varphi}_n^T \theta \quad = \quad & g_2(x_1, \cdots, x_n, \hat{\theta}_1, \cdots, \hat{\theta}_{n-1}, y_d, ..., y_d^{(n)}) \\ & + f_2(x_1, \cdots, x_n, \hat{\theta}_1, \cdots, \hat{\theta}_{n-1}, y_d, ..., y_d^{(n)}) \end{aligned} \quad (43)$$

where

$$\begin{aligned} g_2(\cdot) \quad = \quad & \frac{\partial \varphi_n^T}{\partial x_n} \sum_{j=1}^{n-1} \left(\frac{\partial \alpha_{n-1}}{\partial x_j} \varphi_j^T \theta + \frac{\partial \alpha_{n-1}}{\partial x_j} x_{j+1} \right. \\ & \left. + \frac{\partial \alpha_{n-1}}{\partial \hat{\theta}_j} \gamma_{bj} + \frac{\partial \alpha_{n-1}}{\partial y_d^{(j-1)}} y_d^{(j)} \right) \\ & + \sum_{j=1}^{n-1} \frac{\partial \varphi_n^T}{\partial x_j} \left(\varphi_j^T \theta + x_{j+1}\right) \theta \end{aligned} \quad (44)$$

and

$$f_2(\cdot) = \frac{\partial \varphi_n^T}{\partial x_n}\left(c_{n-1}c_n r - c_{n-1}c_n\eta_n + \sum_{j=1}^{n-1}\frac{\partial \alpha_{n-1}}{\partial \hat{\theta}_j}\gamma_{\mu j}\right). \quad (45)$$

Substituting (40), (43), and (37) into (36), we obtain

$$\dot{r} = \frac{1}{c_{n-1}c_n}\left(f_2 - f_1 + \dot{d} + \dot{u} - \dot{\Phi} + g_2 - g_1\right) + c_n r - c_n\eta_n. \quad (46)$$

We now add and subtract the terms $g_{b1} = g_1(y_d, \zeta_{b1}, ..., \zeta_{b(n-1)}, \hat{\theta}_1, ..., \hat{\theta}_{n-1}, y_d, ..., y_d^{(n)})$ and $g_{b2} = g_2(y_d, \zeta_{b1}, ..., \zeta_{b(n-1)}, \hat{\theta}_1, ..., \hat{\theta}_{n-1}, y_d, ..., y_d^{(n)})$ to the right-hand side of (46) to obtain

$$\dot{r} = c_n r - c_n\eta_n + \frac{1}{c_{n-1}c_n}\Big(f_2 - f_1 + \dot{d} + \dot{u} - \dot{\Phi} + \underbrace{g_2 - g_{b2} + g_{b1} - g_1}_{f_3} + \underbrace{g_{b2} - g_{b1}}_{h}\Big) \quad (47)$$

where $h(y_d, \zeta_{b1}, ..., \zeta_{b(n-1)}, \hat{\theta}_1, ..., \hat{\theta}_{n-1}, y_d, ..., y_d^{(n)})$ has the special property that

$$h(t), \dot{h}(t) \in \mathcal{L}_\infty \quad (48)$$

due to (3) and properties P1 and P3 of the projection operator. Finally, we rewrite (47) as

$$\dot{r} = \frac{1}{c_{n-1}c_n}\left(-\dot{\Phi} + h + \dot{d} + \dot{u}\right) + f + c_n r - c_n\eta_n \quad (49)$$

where $f := (f_2 - f_1 + f_3)/c_{n-1}c_n$.

Remark 6 Using (84) and (6), we can show f has the special property that

$$\|f\| \leq \rho_f(\|x\|)\|z\| \quad (50)$$

where $\rho_f(\cdot) \in \mathbb{R}_{\geq 0}$ is some globally invertible, non-decreasing function, and

$$x = (\eta_1, \eta_2, ..., \eta_n, r)^\intercal$$
$$z = \left(\tanh(\eta_1), \tanh(\eta_2), ..., \tanh(\eta_{n-1}), \eta_n, r\right)^\intercal. \quad (51)$$

Based on (49), we design $\dot{u}$ as [9]

$$\dot{u} = \dot{\Phi} - \beta\text{sgn}(\eta_n) - c_{n-1}c_n(c_n + c_{n+1} + 1)r \quad (52)$$

where $\beta > 0$. The actual $\mathcal{C}^0$ control input can be written from (52) as follows

$$u(t) = -\int_0^t [c_{n-1}c_n(c_n + c_{n+1} + 1)\eta_n(\tau) + \beta\text{sgn}(\eta_n(\tau))]\, d\tau + \Phi(t) - \Phi(0) - c_{n-1}c_n(c_n + c_{n+1} + 1)(\eta_n(t) - \eta_n(0)) \quad (53)$$

where $u(0) = 0$. After substituting (52) into (49), we obtain the closed-loop system

$$\dot{r} = -r - c_n\eta_n + f - c_{n+1}r + \frac{1}{c_{n-1}c_n}\left(h + \dot{d} - \beta\text{sgn}(\eta_n)\right). \quad (54)$$

4 Main Result

Before presenting the main result in Theorem 1, we state a technical lemma.

Lemma 1 (See [9] for proof.) Let the function $L(t) \in \mathbb{R}$ be defined as follows

$$L := \frac{r}{c_{n-1}c_n}\left(h + \dot{d} - \beta\text{sgn}(\eta_n)\right). \quad (55)$$

If the control gain β is selected to satisfy the following sufficient condition

$$\beta > \|h(t)\|_{\mathcal{L}_\infty} + \left\|\dot{d}(t)\right\|_{\mathcal{L}_\infty} + \frac{1}{c_n}\left(\left\|\dot{h}(t)\right\|_{\mathcal{L}_\infty} + \left\|\ddot{d}(t)\right\|_{\mathcal{L}_\infty}\right), \quad (56)$$

then

$$\int_0^t L(\tau)d\tau \leq \zeta_b \quad (57)$$

where the positive constant ζ_b is defined as

$$\zeta_b = \frac{1}{c_{n-1}c_n^2}\left[\beta|\eta_n(0)| + \eta_n(0)\left(h(0) + \dot{d}(0)\right)\right]. \quad (58)$$

Theorem 1 The control law (53) ensures that all system signals are bounded and $e(t) \to 0$ as $t \to \infty$, provided β is adjusted according to (56), $c_n > (c_{n-1}/2c_{n-2})^2$, and the control gains c_i, $i = 1, ..., n+1$ are selected sufficiently large relative to the system initial conditions.

Proof: Let the function $P(t) \in \mathbb{R}$ be defined as follows

$$P(t) := \zeta_b - \int_0^t L(\tau)d\tau \quad (59)$$

where ζ_b and $L(t)$ were defined in Lemma 1. If β is selected to according to (56), it follows from Lemma 1 that $P(t) \geq 0$.

We now define the following function V

$$V := V_{n-1} + \frac{1}{2}\eta_n^2 + \frac{1}{2}r^2 + P. \quad (60)$$

Using (5), we can bound (60) as follows

$$\lambda_1 \ln(\cosh(\|s\|)) \leq V \leq \lambda_2\|s\|^2 \quad (61)$$

where

$$s = [x^T\ \tilde{\theta}_1^T\ ...\ \tilde{\theta}_{n-1}^T\ \sqrt{P}]^T,$$
$$\lambda_1 = \frac{1}{2}\min\left\{1, \lambda_{\min}\left(\Gamma_i^{-1}\right)\right\},$$
$$\lambda_2 = \max\left\{\frac{1}{2}\lambda_{\max}\left(\Gamma_i^{-1}\right), 1\right\}, \quad (62)$$

and x was defined in (51).

Taking the time derivative of (60) and substituting from (34), (54), and (59), we obtain[1]

$$\dot{V} = \dot{V}_{n-1} - c_n\eta_n^2 - r^2 + rf - c_{n+1}r^2 \quad (63)$$

[1] If $\bar{f}(s,t)$ denotes the right-hand side of the closed-loop system on which the stability analysis is being performed, notice from (54) and (55) that $\bar{f}(s,t)$ has a discontinuity on the set of Lebesgue measure zero $\{(s,t) : \eta_n = 0\}$. Since Lemma ?? requires that a solution exist for $\dot{s} = \bar{f}(s,t)$, see [9] for a discussion on the existence of Filippov's generalized solution.

upon use of (55). Substituting now from (30) for $j = n-1$ and (50) gives

$$\dot{V} \leq \sum_{j=1}^{n-1}\left[\frac{\tanh(\eta_j)}{c_{j-1}}(c_j\eta_{j+1}+\tilde{w}_j) - \frac{c_j\tanh^2(\eta_j)}{c_{j-1}}\right] - c_n\eta_n^2 - r^2 + \left[\rho_f(\|x\|)\|z\|\|r\| - c_{n+1}r^2\right] \quad (64)$$

We can upper bound (64) by using (25) as follows

$$\begin{aligned}\dot{V} \leq & -\sum_{j=1}^{n-1}\frac{c_j}{c_{j-1}}\tanh^2(\eta_j) - c_n\eta_n^2 - r^2 - c_{n+1}r^2 \\ & +\rho_f(\|x\|)\|z\|\|r\| + \sum_{j=1}^{n-1}\left[\frac{c_j}{c_{j-1}}\tanh(\eta_j)\eta_{j+1}\right. \\ & \left. +\|\tanh(\eta_j)\|\sum_{k=1}^{j}\left[\rho_{jk}(\|\bar{\eta}_j\|)\|\tanh(\eta_k)\|\right]\right]. \end{aligned} \quad (65)$$

Using (6), we can show that for $k = 1, ..., n-2$

$$\begin{aligned} & \|\tanh(\eta_{k+1})\|\rho_{k+1,k}(\|\bar{\eta}_{k+1}\|)\|\tanh(\eta_k)\| \\ & +\frac{c_k}{c_{k-1}}\tanh(\eta_k)\eta_{k+1} \\ & \leq \|\tanh(\eta_k)\|\|\tanh(\eta_{k+1})\| \quad (66) \\ & \times \underbrace{\left[\frac{c_k}{c_{k-1}}(\|\eta_{k+1}\|+1) + \rho_{k+1,k}(\|\bar{\eta}_{k+1}\|)\right]}_{\bar{\rho}_{k+1,k}(\|\bar{\eta}_{k+1}\|)}. \end{aligned}$$

Now, let $c_j = c_{j-1}(k_j + n - j)$, $j = 1, ..., n-2$ and $c_{n-1} = c_{n-2}(k_{n-1}+2)$ where $k_j > 0$, $j = 1, ..., n-1$, and rewrite (65) as

$$\begin{aligned}\dot{V} \leq & -\sum_{j=1}^{n-1}(k_j+1)\tanh^2(\eta_j) - c_n\eta_n^2 - r^2 \\ & +\sum_{j=1}^{n-1}\sum_{k=1}^{j-1}\left[\rho_{jk}(\|\bar{\eta}_j\|)\|\tanh(\eta_k)\|\|\tanh(\eta_j)\|\right. \\ & \left. -\tanh^2(\eta_j)\right] + \sum_{j=1}^{n-1}\rho_{jj}(\|\bar{\eta}_j\|)\tanh^2(\eta_j) \\ & +\left[\frac{c_{n-1}}{c_{n-2}}\tanh(\eta_{n-1})\eta_n - \tanh^2(\eta_{n-1})\right] \\ & +\left[\rho_f(\|x\|)\|z\|\|r\| - c_{n+1}r^2\right] \end{aligned} \quad (67)$$

where $\rho_{j+1,j} = \bar{\rho}_{j+1,j}$ and $\rho_{10} = 0$. Completing squares on the above bracketed terms yields

$$\begin{aligned}\dot{V} \leq & -\sum_{j=1}^{n-1}\tanh^2(\eta_j) - r^2 - \left[c_n - \left(\frac{c_{n-1}}{2c_{n-2}}\right)^2\right]\eta_n^2 \\ & +\frac{\rho_f^2(\|x\|)}{4c_{n+1}}\|z\|^2 - \sum_{j=1}^{n-1}\left[k_j - \rho_{jj}(\|\bar{\eta}_j\|)\right. \\ & \left. -\frac{1}{4}\sum_{k=1}^{j-1}\rho_{jk}^2(\|\bar{\eta}_j\|)\right]\tanh^2(\eta_j). \end{aligned} \quad (68)$$

Let

$$\rho_j(\|\bar{\eta}_j\|) = \rho_{jj}(\|\bar{\eta}_j\|) - \frac{1}{4}\sum_{k=1}^{j-1}\rho_{jk}^2(\|\bar{\eta}_j\|) \quad (69)$$

$$c_n > (c_{n-1}/2c_{n-2})^2, \quad (70)$$

where $j = 1, ..., n-1$, and let $\sigma = \min\left\{1, c_n - \left(\frac{c_{n-1}}{2c_{n-2}}\right)^2\right\}$, then (68) can be written as

$$\begin{aligned}\dot{V} \leq & -\sum_{j=1}^{n-1}\left(k_j - \rho_j(\|\bar{\eta}_j\|)\right)\tanh^2(\eta_j) \\ & -\left(\sigma - \frac{\rho_f^2(\|x\|)}{4c_{n+1}}\right)\|z\|^2. \end{aligned} \quad (71)$$

It follows from (71) that

$$\dot{V} \leq -\gamma\|z\|^2 \quad (72)$$

for

$$\begin{aligned} k_j &> \rho_j(\|\bar{\eta}_j\|),\ j = 1, ..., n-1 \quad \text{and} \\ c_{n+1} &> \frac{\rho_f^2(\|x\|)}{4\sigma} \end{aligned} \quad (73)$$

where the constant γ is such that $0 < \gamma < 1$.where the constant γ satisfies $0 < \gamma < 1$.

From (61) and (72), let

$$\begin{aligned} W_1(s) &:= \lambda_1\ln(\cosh(\|s\|)),\ W_2(s) := \lambda_2\|s\|^2, \\ W(s) &:= \gamma\|z\|^2. \end{aligned} \quad (74)$$

From (73), we define the sets $\mathcal{D}$ and $\mathcal{S}$ as follows

$$\mathcal{D} := \{s : \|s\| < s_{\min}\} \quad (75)$$

$$\mathcal{S} := \{s \in \mathcal{D} : W_2(s) < \lambda_1\ln(\cosh(s_{\min}))\} \quad (76)$$

where $s_{\min} = \min\left\{\rho_j^{-1}(k_j), \rho_f^{-1}(2\sqrt{\sigma c_{n+1}})\right\}$. We can now invoke Lemma 2 of [1] to state that $s(t) \in \mathcal{L}_\infty$. From (3), we then know $x_1(t) \in \mathcal{L}_\infty$. From (15), (16), and property P3 of the projection operator, we know $\alpha_1(t), \dot{\hat{\theta}}_1(t) \in \mathcal{L}_\infty$. We can now use (11) to show $x_2(t) \in \mathcal{L}_\infty$. From (1a), we know $\dot{x}_1(t) \in \mathcal{L}_\infty$. Continuing with this procedure, we can show $\alpha_i(t), x_{i+1}(t), \in \mathcal{L}_\infty$, $i = 2, ..., n-1$. We can then state $\dot{\eta}_i(t), \dot{\alpha}_{i-1}(t) \in \mathcal{L}_\infty$, $i = 1, ..., n-1$ by using (22). Using (54) and assumption A2, we can show $\dot{r}(t) \in \mathcal{L}_\infty$. From (34) and (38), we know $\dot{\eta}_n(t), \dot{\alpha}_{n-1}(t) \in \mathcal{L}_\infty$; hence, from the time derivative of (35), we have $\dot{x}_n(t) \in \mathcal{L}_\infty$. Finally, we can use (1c) to show that $u(t) \in \mathcal{L}_\infty$.

Now, it is clear from (74) that $\dot{W}(s(t)) \in \mathcal{L}_\infty$, which is a sufficient condition for $W(s)$ being uniformly continuous. It then follows from Lemma 2 of [1] that $\gamma\|z(t)\|^2 \to 0$ as $t \to \infty$ $\forall s(0) \in \mathcal{S}$, which implies from (51) that $e(t) \to 0$ as $t \to \infty$ $\forall s(0) \in \mathcal{S}$.

Note that the region of stability in (76) can be made arbitrarily large to include any initial conditions by increasing the control gains c_i, $i = 1, ..., n+1$ (i.e., a semi-global stability result). Specifically, we can use the second equation in (74) and (76) to calculate the region of stability as follows

$$
\begin{aligned}
& k_i > \rho_i \left(\cosh^{-1} \left(\exp \left(\frac{\lambda_2}{\lambda_1} \|s(0)\|^2 \right) \right) \right), \\
& i = 1, ..., n-1, \quad \text{and} \\
& c_{n+1} > \frac{1}{4\sigma} \rho_f^2 \left(\cosh^{-1} \left(\exp \left(\frac{\lambda_2}{\lambda_1} \|s(0)\|^2 \right) \right) \right)
\end{aligned} \tag{77}
$$

where

$$
\|s(0)\| = \sqrt{\sum_{i=1}^{n} \eta_i^2(0) + r^2(0) + \sum_{i=1}^{n-1} \tilde{\theta}_i^{\mathsf{T}}(0) \tilde{\theta}_i(0) + P(0)} \tag{78}
$$

Note that the inequalities in (70) and (77) can be satisfied for large enough gains c_i, $i = 1, ..., n+1$ because: i) $\rho_i(\cdot)$ is not a function of c_i, ii) $\eta_i(0)$, $r(0)$, and $P(0)$ are only a function of $1/c_i$, and iii) $r(0)$ is not a function of c_{n+1} since $u(0) = 0$. ■

5 Conclusion

This work considered the tracking problem for MIMO nonlinear parametric strict-feedback systems in the presence of additive disturbances and parametric uncertainties. The continuous robust adaptive controller, whose construction is founded on the fusion of an adaptation law and a dynamic robust control mechanism, exploits the two times continuous differentiability of the disturbances. The robust adaptive control law guarantees the semi-global asymptotic convergence of the tracking error to zero and boundedness of all signals.

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A Projection Operator

We use the following projection operator, which was recently proposed in [2]:

$$
\text{Proj}(\mu_i, \hat{\theta}_i) = \mu_i - \frac{\pi_1 \pi_2 \nabla p(\hat{\theta}_i)}{4\left(\varepsilon^2 + 2\varepsilon\theta_0\right)^{n-i} \theta_0^2}, \; i = 1, ..., n-1 \tag{79}
$$

where

$$
p(\hat{\theta}_i) = \hat{\theta}_i^T \hat{\theta}_i - \theta_0^2, \tag{80}
$$

$$
\pi_1 = \begin{cases} p^{n-i}(\hat{\theta}_i) & \text{if } p(\hat{\theta}_i) > 0 \\ 0 & \text{otherwise,} \end{cases} \tag{81}
$$

$$
\pi_2 = \frac{1}{2} \nabla p(\hat{\theta}_i) \mu_i + \sqrt{\left(\frac{1}{2} \nabla p(\hat{\theta}_i) \mu_i \right)^2 + \delta^2}, \tag{82}
$$

μ_i was defined in (29), ∇ is the gradient operator, ε, δ are arbitrary positive constants, and θ_0 was defined in assumption A3. It can be proven that the above projection operator has the following properties [2]: If $\hat{\theta}_i(0) \in \Omega$, then

P1. $\left\|\hat{\theta}_i(t)\right\| \leq \theta_0 + \varepsilon \; \forall t \geq 0$;

P2. $\tilde{\theta}_i^T \text{Proj}(\mu_i, \hat{\theta}_i) \geq \tilde{\theta}_i^T \mu_i$;

P3. $\text{Proj}(\mu_i, \hat{\theta}_i) = \gamma_{\mu i} + \gamma_{bi}$ and $\left\|\text{Proj}(\mu_i, \hat{\theta}_i)\right\| \leq \|\mu_i\| \left[1 + \left(\frac{\theta_0 + \varepsilon}{\theta_0} \right)^2 \right] + \frac{\theta_0 + \varepsilon}{2\theta_0^2} \delta$, where

$$
\begin{aligned}
\gamma_{\mu i} &= \mu_i - \frac{\pi_1 \left(\pi_2 - \delta\right) \nabla p(\hat{\theta}_i)}{4\left(\varepsilon^2 + 2\varepsilon\theta_0\right)^{n-i} \theta_0^2}, \\
\gamma_{bi} &= -\frac{\pi_1 \delta \nabla p(\hat{\theta}_i)}{4\left(\varepsilon^2 + 2\varepsilon\theta_0\right)^{n-i} \theta_0^2},
\end{aligned} \tag{83}
$$

and

$$
\begin{aligned}
\|\gamma_{\mu i}\| &\leq \|\mu_i\| \left[1 + \left(\frac{\theta_0 + \varepsilon}{\theta_0} \right)^2 \right], \\
\|\gamma_{bi}\| &\leq \frac{\theta_0 + \varepsilon}{2\theta_0^2} \delta;
\end{aligned} \tag{84}
$$

P4. $\text{Proj}(\mu_i, \hat{\theta}_i) \in \mathcal{C}^{n-i-1}$.