Proceedings of the
44th IEEE Conference on Decision and Control, and
the European Control Conference 2005
Seville, Spain, December 12-15, 2005



# Identification of Continuous-Time Power Spectra by Generalized Fourier Series

Hüseyin Akçay and Semiha Türkay

*Abstract*— **In this paper, we present a generalized Fourier series based algorithm for the identification of continuous-time, linear-time invariant systems from power spectrum measurements. The algorithm is strongly consistent and it is used in the design of a linear shaping filter for a single-track road power spectrum.**

## I. Introduction

Identification of linear systems from a measured power spectrum is a problem arising in certain applications. An example is the design of *linear shaping filter* for noise processes; and a practical application is the modeling of stochastic disturbances experienced by a vehicle moving forward. Indeed, this problem inspired us to undertake the current work. The goal here is to model the road spectrum by a rational transfer function of reasonably low order and to use this approximation for a design of a linear shaping filter with a white noise input. Once such an approximation is made, the vehicle control problem can be formulated in standard form.

There are many identification algorithms available for solving this problem. A parametric approach consists of minimizing a quadratic loss function. Then, the optimized parameter values are found by an iterative search. Although quite successful in practice, there is no guarantee that the iterations will terminate in a finite-time and the global minumum will be attained. Discussion of parametric as well as nonparametric methods that mostly use time-domain data can be found in the books [13], [16], [18]. More recently [22], [4], parametric, but non-iterative subspace-based identification algorithm have been proposed. The algorithm in [22] identifies discrete-time spectra and requires the frequencies be uniformly spaced while the algorithm in [4] does not require the discrete-frequencies be uniformly spaced. A non-parametric approach would typically be based on the Fourier series development of the power spectrum [17] following a transformation of the estimation problem from the continuous-time to the discrete-time.

The present paper deals with a frequency-domain *spectral factorization* problem from noisy values of the power spectrum of a continuous-time system at a given set of frequencies. Spectral factorization is an important problem in system theory. For example, it plays an important role in linear-quadratic-optimal control [7]. In this paper, we propose an algorithm based on the generalized Fourier series, and show that over a large class of continuous-time spectra this algorithm is *strongly consistent* when the corruptions are stochastic noise. This result is an extension of the well-known consistency result for the finite-impulse response models, which correspond to the Laguerre models in the continuous-time, to model structures parameterized in terms of the fixed pole rational basis functions. The latter allow more flexibility in coding prior knowledge of the spectrum.

Now, we briefly describe the contents of the paper. In Section II, we formulate the spectral estimation problem for a particular class of continuous-time power spectra. Power spectra in this class are analytic and bounded in a strip containing the imaginary axis and also bounded away from zero on the imaginary axis. In addition, they satisfy a smoothness condition at infinity. In Section III, we show that this class of spectra is Dini-Lipschitz continuous. This result enables us to develop in Section IV the generalized Fourier series with respect to arbitrary uniformly bounded rational orthonormal bases with prescribed poles which converge uniformly on the imaginary axis with probability one to the true spectrum. The choice for the class of continuous-time spectra is not conservative. In fact, if the smoothness assumption at infinity is dropped, then there is no guarantee for the uniform convergence of the Fourier series even in the case of trigonometric basis functions; and the mapping that associates a spectral factor to given spectral density may not be continuous, hence sensitive to errors in spectra. In Section V, we use the proposed algorithm in the design of a linear shaping filter for a single-track road power spectrum [10].

In this paper, we consider single-input/single-output systems. The results extend to multi-input/multi-output systems with no modifications.

### A. Notation

The notation $y_k = O(x_k)$ as $k \to \infty$ will mean $y_k/x_k$ remains bounded and $z_k = o(x_k)$ will mean $z_k/x_k$ vanishes as $k \to \infty$. Let $\mathbf{R}$ and $\mathbf{C}$ denote respectively, the sets of the real and the complex numbers. Let $\bar{\mathbf{R}}$ denote the set of extended reals $\mathbf{R} \cup \{\pm\infty\}$. For a scalar complex measurable function $S(z)$, we define its supremum norm by

$$\|S\|_\infty \triangleq \sup_{\omega \in \mathbf{R}} |S(j\omega))|.$$

Let $\mathcal{H}(\alpha, M)$ denote the set of complex functions which are analytic and bounded by $M$ on the vertical strip $D_\alpha \triangleq \{s \in \mathbf{C} : |\mathrm{Re}\, s| < \alpha\}$ and continuous on $j\bar{\mathbf{R}}$. Here, $\mathrm{Re}\, x$

Department of Electrical and Electronics Engineering, Anadolu University, 26470 Eskişehir, Turkey.
E-mail: huakcay@anadolu.edu.tr; Tel: + (90) 222 335 0580 -X 6459; Fax: +(90) 222 323 9501.

and $\operatorname{Im} x$ are the real and the imaginary parts of $x$. Recall that a continuous-time stable system has a transfer function which is analytic and bounded on the open right half plane. For each $\alpha, M > 0$, we define a set of spectral densities by

$$\mathcal{S}(\alpha, M) \triangleq \{S \in \mathcal{H}(\alpha, M) : \; S(-s) = S(s), \\ \forall s \in D_\alpha; \; S(j\omega) > 0, \; \forall \omega \in \bar{\mathbf{R}}\}.$$

When $f(\theta)$ is a continuous function on $[0, 2\pi]$, we write

$$\zeta_f(\delta) = \sup_{|x-y|\le\delta} |f(x) - f(y)| \tag{1}$$

for the *modulus of continuity* of $f(\theta)$. A continuous function $f(\theta)$ is said *Dini-Lipschitz continuous* if

$$\zeta_f(\delta)\, \ln(\delta^{-1}) \to 0 \qquad (\delta \to 0). \tag{2}$$

## II. Problem Formulation

Consider a scalar linear-time invariant continuous-time stable system represented by the input-output relation

$$y(t) = \int_0^t g(t-\tau)u(\tau)\mathrm{d}\tau, \qquad t \ge 0 \tag{3}$$

where $g(t)$ is the impulse response, $u(t)$ and $y(t)$ are, respectively, the input and the output of the system. The transfer function of the system (3) is defined as

$$G(s) \triangleq \int_0^\infty g(t)e^{-st}\mathrm{d}t. \tag{4}$$

Assuming that $u(t)$ is unit intensity white noise process, the power spectrum of $y(t)$ denoted by $S(s)$ is defined as

$$S(s) \triangleq G(s)G(-s). \tag{5}$$

The transfer function $G(s)$ is called the spectral factor of $S(s)$.

We will assume that for each $N$, the noise $\eta$ corrupting the spectrum samples is a zero mean complex white noise process with a covariance function satisfying

$$\mathbf{E}\begin{bmatrix} \operatorname{Re}\eta_k^{(N)} \\ \operatorname{Im}\eta_k^{(N)} \end{bmatrix} \begin{bmatrix} \operatorname{Re}\eta_s^{(N)} \; \operatorname{Im}\eta_s^{(N)} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} R_k & 0 \\ 0 & R_k \end{bmatrix}\delta_{ks}.$$

Here $\mathbf{E}(x)$ denotes the expected value of random variable $x$ and $\delta_{ks}$ is the Kronecker delta. Furthermore, we assume that the fourth-order moments are bounded by some $K < \infty$ as

$$\mathbf{E}|\eta_k^{(N)}|^4 < K, \qquad \text{for all } k \text{ and } N. \tag{6}$$

The problem studied in this paper can be stated as follows:

*Given:* $N$ noisy samples $S_k^{(N)} \in \mathbf{C}$ of the power spectrum $S \in \mathcal{S}(\alpha, M)$ for some $\alpha, M > 0$ evaluated at $N$ frequencies:

$$S_k^{(N)} = S(j\omega_k^{(N)}) + \eta_k^{(N)}, \qquad k = 1, 2, \cdots, N. \tag{7}$$

*Find:* A stable, minimum phase transfer function $\widehat{G}_N(s)$ such that the estimated power spectrum

$$\widehat{S}_N(s) = \widehat{G}_N(s)\,\widehat{G}_N(-s) \tag{8}$$

is strongly consistent, *i.e.,*

$$\lim_{N\to\infty} \|\widehat{S}_N - S\|_\infty = 0, \qquad \text{w. p. 1.} \tag{9}$$

We will propose an algorithm to estimate the spectral factor of $S$ by a method based on the generalized Fourier series. The above identification problem can be thought as the design of a linear shaping filter from corrupted power spectrum measurements.

## III. A Convergence Result

The continuous-time estimation problem formulated in § II can be transformed to a discrete-time one via to the bilinear map:

$$s = \psi(z) = \lambda\frac{z-1}{z+1} \tag{10}$$

where $\lambda > 0$ is chosen in the order of the bandwidth of the spectral data. Let

$$S^{\mathrm{d}}(\theta) \triangleq S(j\lambda\tan(\theta/2)). \tag{11}$$

The following result will be instrumental in the convergence analysis of our algorithm.

*Lemma 1:* Let $S \in \mathcal{S}(\alpha, M)$ for some $\alpha, M > 0$ and let $S^{\mathrm{d}}$ be as in (11). Suppose

$$\sup_{|\omega|\ge\mu} |S(j\omega) - S(\infty)|\,\ln\mu \to 0 \qquad (\mu \to \infty). \tag{12}$$

Then, $S^{\mathrm{d}}$ is Dini-Lipschitz continuous.

**Proof.** Let $\widetilde{\alpha}$ be a positive number less than $\alpha$. Since $S$ is analytic and bounded by $M$ on $D_\alpha$, from the Cauchy's integral formula its derivative has the following representation:

$$S'(j\omega) = \frac{1}{2\pi j}\oint_{|s-j\omega|=\widetilde{\alpha}} \frac{S(s)}{(s-j\omega)^2}\,\mathrm{d}s \tag{13}$$

Thus, letting $\widetilde{\alpha} \to \alpha$ in the following inequality obtained from (13):

$$|S'(j\omega)| \le M/\widetilde{\alpha},$$

we get $|S'(j\omega)| \le M/\alpha$. Then, from an application of the chain rule of differentiation:

$$\frac{\mathrm{d}S^{\mathrm{d}}}{\mathrm{d}\theta} = S'(j\omega)\frac{\mathrm{d}\omega}{\mathrm{d}\theta} = \frac{\lambda S'(j\omega)}{2\cos^2(\theta/2)}. \tag{14}$$

Put $\theta \triangleq \pi - \nu \;\; (\nu > 0)$. Since

$$\frac{\sin x}{x} \ge \frac{2}{\pi}, \qquad 0 \le x \le \frac{\pi}{2},$$

we get from (14)

$$\left|\frac{\mathrm{d}S^{\mathrm{d}}}{\mathrm{d}\theta}\right| \le \frac{\pi^2\lambda}{2\nu^2}\,|S'(j\omega)| \le \frac{\pi^2\lambda M}{2\alpha\nu^2}. \tag{15}$$

It follows for all $\delta < \nu_0$ that

$$\sup_{|\widetilde{\theta}|\le\delta,\,\nu\ge\nu_0} |S^{\mathrm{d}}(\theta + \widetilde{\theta}) - S^{\mathrm{d}}(\theta)| \le \frac{\pi^2\lambda M}{2\alpha(\nu_0 - \delta)^2}\delta. \tag{16}$$

Next, by an application of the triangle inequality

$$\sup_{|\widetilde{\theta}|\le\delta,\,\nu<\nu_0} |S^{\mathrm{d}}(\theta+\widetilde{\theta})-S^{\mathrm{d}}(\theta)| \le$$

$$\sup_{|\widetilde{\theta}|\le\delta,\,\nu<\nu_0} |S^{\mathrm{d}}(\theta+\widetilde{\theta})-S^{\mathrm{d}}(\pi)| + \sup_{\nu<\nu_0} |S^{\mathrm{d}}(\theta)-S^{\mathrm{d}}(\pi)|$$

$$\le 2 \sup_{|\widetilde{\theta}|\le\delta+\nu_0} |S^{\mathrm{d}}(\pi+\widetilde{\theta})-S^{\mathrm{d}}(\pi)|. \quad (17)$$

Combining inequalities (16) and (17), we obtain

$$\sup_{|\widetilde{\theta}|\le\delta} |S^{\mathrm{d}}(\theta+\widetilde{\theta})-S^{\mathrm{d}}(\theta)| \le$$

$$\max\left\{2 \sup_{|\widetilde{\theta}|\le\delta+\nu_0} |S^{\mathrm{d}}(\pi+\widetilde{\theta})-S^{\mathrm{d}}(\pi)|,\ \frac{\pi^2\lambda M}{2\alpha(\nu_0-\delta)^2}\delta\right\}$$

provided that $\delta<\nu_0$. Thus, if $\delta<\nu_0$

$$\zeta_{S^{\mathrm{d}}}(\delta)\ln(1/\delta) \le \max\left\{\frac{\pi^2\lambda M}{2\alpha(\nu_0-\delta)^2}\delta\ln(1/\delta),\right.$$

$$\left. 2\ln(1/\delta) \sup_{|\widetilde{\theta}|\le\delta+\nu_0} |S^{\mathrm{d}}(\pi+\widetilde{\theta})-S^{\mathrm{d}}(\pi)|\right\}.$$

Set $\nu_0=\delta^{1/3}$ in the above expression. Then for all $\delta<1/2$,

$$\zeta_{S^{\mathrm{d}}}(\delta)\ln(1/\delta) \le \max\left\{\frac{8\pi^2\lambda M}{9\alpha}\delta^{1/3}\ln(1/\delta),\right.$$

$$\left. 2\ln(1/\delta) \sup_{|\widetilde{\theta}|\le 2\delta^{1/3}} |S^{\mathrm{d}}(\pi+\widetilde{\theta})-S^{\mathrm{d}}(\pi)|\right\}.$$

Since

$$\delta^{1/3}\ln(1/\delta)\to 0 \qquad (\delta\to 0),$$

the left hand side of the above inequality converges to zero if

$$\ln(1/\delta)\sup_{|\widetilde{\theta}|\le\delta} |S^{\mathrm{d}}(\pi+\widetilde{\theta})-S^{\mathrm{d}}(\pi)| \to 0 \qquad (\delta\to 0) \quad (18)$$

Since $\omega=\lambda\tan(\theta/2)$ and $\tan x = x+O(x^3)$ for all small $x$, (18) is equivalent to

$$\sup_{|w|\ge\mu} |S(j\omega)-S(\infty)|\ln\mu \to 0 \qquad (\mu\to\infty).$$

∎

The spectral factorization problem is a well-known subject in statistics and engineering. Formally, a discrete-time stochastic process has a factorable spectral density function $S^{\mathrm{d}}(\theta)$ if and only if it is regular:

$$\int_0^{2\pi} \ln S^{\mathrm{d}}(\theta)\,\mathrm{d}\theta > -\infty. \quad (19)$$

Then, it is possible to write the spectral factors of $S^{\mathrm{d}}(\theta)$ in terms of the Fourier coefficients of the function $\ln(S^{\mathrm{d}}(\theta))^{1/2}$ [17]. For a continuous-time stochastic process, the regularity is guaranteed by the familiar Paley-Wiener condition:

$$\int_0^{\infty} \frac{\ln S(j\omega)}{1+\omega^2}\,\mathrm{d}\omega > -\infty \quad (20)$$

which is obtained from (19) by an application of (10).

It is desired to have the mapping which associates a spectral factor to a spectral density be continuous since any irrational spectral factor will typically be approximated by a rational one and calculation by a computer will always introduce small errors. It is important to know that the calculated spectral factor is still within a certain allowed error region around the exact solution. However, this does not hold if the spectral densities are bounded strictly positive functions on the imaginary axis [5]. This does not hold either for the class of spectral densities that are analytic and bounded away from zero in some strip including the imaginary axis [12]. Therefore, the choice $S(\alpha, M)$ for the class of continuous-time spectra is not conservative.

The condition (12) combined with the analyticity of $S$ in a strip containing the imaginary axis implies the Dini-Lipschitz continuity of $S^{\mathrm{d}}(\theta)$ by Lemma 1. It is a well-known fact that the Fouries series of a Dini-Lipschitz continuous function converges uniformly to that function [9]. Recently [6], [1], this fact was extended from the trigonometric basis to arbitrary but uniformly bounded orthonormal bases, which is the basis of our algorithm developed in the next section.

## IV. A Generalized Fourier Series Based Algoritrhm

Let $\{z_k\}$ be a given sequence of complex numbers satisfying $z_0=0$ and $|z_k|<1$ for all $k$. We define a set of rational functions by

$$B_0(z)\triangleq 1; \quad B_k(z)\triangleq \frac{(1-|z_k|^2)^{1/2}}{1-\overline{z_k}z}\prod_{i=0}^{k-1}\frac{z-z_i}{1-\overline{z}_i z} \qquad (k\ge 1). \quad (21)$$

The rational functions defined by (21) are *complete* in the space of complex functions which are analytic inside and square integrable on the unit circle if and only if $\sum_{k=0}^{\infty}(1-|z_k|)=\infty$ [15]. Furthermore, the orthogonal complement of this space in the space of square integrable complex functions is spanned by the functions [2]

$$B_{-k}(z)\triangleq \frac{(1-|z_k|^2)^{1/2}}{z-z_k}\prod_{i=1}^{k-1}\frac{1-\overline{z_i}z}{z-z_i} \qquad (k\ge 1). \quad (22)$$

The rational functions in (21) and (22) are *orthonormal* with respect to the inner product:

$$\langle f,g\rangle \triangleq \frac{1}{2\pi}\int_0^{2\pi} f(e^{j\theta})\,\overline{g(e^{j\theta})}\,\mathrm{d}\theta.$$

The basis functions (21) generalize the well-known finite-pulse response, the Laguerre and the Kautz, and the generalized orthonormal basis functions. In contrast to the Laguerre and the Kautz bases, the basis defined by (21) enjoys increased flexibility of pole location.

In any application involving the modeling of a physical system it is necessary that the underlying modeled impulse response is real-valued. A requirement is that the set $\{z_k\}$ used to define basis via (21) and (22) always contains the complex conjugates. Then, the constraint of realness is easily accommodated by taking suitable linear combinations of the basis functions (21) and (22) [2]. Let $\widetilde{B}_k$ denote the new basis functions with real-valued impulse responses.

The Fourier series of $S^{\rm d}$ with respect to (21) and (22) is defined by

$$\begin{aligned}\mathcal{F}_M(z) &\triangleq \sum_{k=-M}^{M} \langle S^{\rm d}, B_k\rangle\, B_k(z) \\ &= \sum_{k=-M}^{M} \langle S^{\rm d}, \widetilde{B}_k\rangle\, \widetilde{B}_k(z) \qquad (23)\end{aligned}$$

where $M$ is such that $\{z_1, \cdots, z_M\}$ contains the complex conjugates. The proof of the second equality in (23) is given in [2]. Since $S^{\rm d}(z) = S^{\rm d}(z^{-1})$, the new basis functions constructed in [2] satisy the following equalities for all $k$

$$\langle S^{\rm d}, \widetilde{B}_{-k}\rangle = \langle S^{\rm d}, \widetilde{B}_k\rangle \in \mathbf{R}; \qquad \widetilde{B}_k(z^{-1}) = \widetilde{B}_{-k}(z).$$

which can be directly verified from (21) and (22) when $z_k \in \mathbf{R}$ for all $k$. Therefore, (23) can be written as

$$\mathcal{F}_M(z) = \langle S^{\rm d}, 1\rangle + \sum_{k=1}^{M} \langle S^{\rm d}, \widetilde{B}_k\rangle\, [\widetilde{B}_k(z) + \widetilde{B}_k(z^{-1})].$$

Since $S^{\rm d}$ is Dini-Lipschitz continuous, $\mathcal{F}_M$ converges uniformly to $S^{\rm d}$ on the unit circle [1] provided that $\sup_k |z_k| < \infty$. Moreover, in the above equation, we may replace the Fourier coefficients of $S^{\rm d}$ with their least-squares estimates denoted by $c_k^{(N)}$ without changing convergence properties if $M$ does not grow too fast with $N$. Thus, we propose the following as an estimator of $S^{\rm d}$:

$$\widehat{S}_N^{\rm d}(z) \triangleq \widehat{H}_N(z) + \widehat{H}_N(z^{-1}) \qquad (24)$$

where

$$\begin{aligned}\widehat{H}_N(z) &\triangleq \frac{c_0^{(N)}}{2} + \sum_{k=1}^{M} c_k^{(N)} \widetilde{B}_k(z); &(25)\\ c^{(N)} &\triangleq (\Theta_N^T \Theta_N)^{-1} \Theta_N^T S^{(N)}; &(26)\\ \theta_k^{(N)} &= 2\tan^{-1}(\omega_k^{(N)}/\lambda), \qquad k \le N; &(27)\\ \Theta_N &\triangleq \mathrm{Re}\begin{bmatrix} 1 & \cdots & 2\widetilde{B}_M(e^{j\theta_1^{(N)}}) \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 2\widetilde{B}_M(e^{j\theta_N^{(N)}}) \end{bmatrix}; &(28)\\ S^{(N)} &\triangleq \begin{bmatrix} S_1^{(N)} \\ \vdots \\ S_N^{(N)} \end{bmatrix}. &(29)\end{aligned}$$

The consistency result is the following.

*Lemma 2:* Let $S_k^{(N)}$ be as given by (7) where $\eta_k^{(N)}$ has bounded fourth order moments as in (6). Consider the spectral estimator in (24). Let $S$ be as in Lemma 1. Assume that $\sup_k |z_k| < \infty$ and the frequencies in (27) satisfy $\pi(k-1)/N \le \theta_k^{(N)} < \pi k/N$ for all $k = 1, 2, \cdots, N$. Choose $M$ such that $M\,\zeta_{S^{\rm d}}(1/N) \to 0$ as $N \to \infty$ and $M = O(N^{1/4}(\ln N)^{-\gamma})$ for some $\gamma > 1/4$. Then,

$$\lim_{N\to\infty} \|\widehat{S}_N^{\rm d} - S^{\rm d}\|_\infty = 0 \qquad \text{w.p.1.}$$

**Proof.** Since $S^{\rm d}$ is Dini-Lipschitz continuous, all the conditions in Theorem 4.4 in [3] are satisfied. Hence, the conclusion follows. ∎

The condition on the frequencies can be fulfilled by sampling for each $N$ the frequencies $\omega_k^{(N)}$ and the corresponding spectral data $S_k^{(N)}$. Put $\mu = N$ in (12). Then, $\sup_{|\omega|\ge N} |S(j\omega) - S(\infty)| = o(1/\ln N)$; and thus from the condition $M\,\zeta_{S^{\rm d}}(1/N) \to 0$ $(N \to \infty)$ we obtain $M = O(\ln N)$, which is a severe restriction on the number of basis functions. We say that $f$ is of *Lipschitz* class denoted by $\Lambda_\beta$ $(0 < \beta \le 1)$ if $\zeta_f(\delta) = O(\delta^\beta)$. The only consistency requirement for the class $\Lambda_\beta$ $(\beta > 1/4)$ turns out to be $M = O(N^{1/4}(\ln N)^{-\gamma})$, $\gamma > 1/4$. Note that any time-delayed finite-dimensional spectral factor has a spectrum in the class $\Lambda_1$.

We are left with the problem of extracting the spectral factor $\widehat{G}_N^{\rm d}(z)$ from the *spectral summand* $\widehat{H}_N(z)$. To this end, we first obtain a state-space realization of $\widehat{H}_N(z)$ denoted by $(\widehat{A}_N, \widehat{F}_N, \widehat{C}_N, \frac{1}{2}\widehat{E}_N)$. A *minimal balanced realization* of $\widehat{H}_N(z)$ could readily be computed by using the recursive algorithm in [21] developed for the realization of a product of successive rational *inner* functions. Then, we solve the *Riccati* equation for $\widehat{P}_N$:

$$\begin{aligned}\widehat{P}_N &= \widehat{A}_N \widehat{P}_N \widehat{A}_N^T + (\widehat{F}_N - \widehat{A}_N \widehat{P}_N \widehat{C}_N^T) \\ &\quad \cdot (\widehat{E}_N - \widehat{C}_N \widehat{P}_N \widehat{C}_N^T)^{-1} (\widehat{F}_N - \widehat{A}_N \widehat{P}_N \widehat{C}_N^T)^T \qquad (30)\end{aligned}$$

and compute $\widehat{B}_N$ and $\widehat{D}$ as follows:

$$\begin{aligned}\widehat{B}_N &\triangleq (\widehat{F}_N - \widehat{A}_N \widehat{P}_N \widehat{C}_N^T)(\widehat{E}_N - \widehat{C}_N \widehat{P}_N \widehat{C}_N^T)^{-\frac{1}{2}} &(31)\\ \widehat{D}_N &\triangleq (\widehat{E}_N - \widehat{C}_N \widehat{P}_N \widehat{C}_N^T)^{1/2}. &(32)\end{aligned}$$

The quadruplet $(\widehat{A}_N, \widehat{B}_N, \widehat{C}_N, \widehat{D}_N)$ is a state-space realization of $\widehat{G}_N^{\rm d}(z)$.

It is a well-known fact that $\widehat{P}_N$ is positive definite if and only if $\widehat{S}_N^{\rm d}$ is positive. Since $S^{\rm d}$ is positive, from Lemma 2 we have $\widehat{S}_N^{\rm d}$ positive w.p.1 for all sufficiently large $N$. Due to undermodeling and noise, $\widehat{S}_N^{\rm d}$ may happen to be non-positive. In this case, the spectral factor can not be computed. But, this problem can be fixed by an adjustment of the zeros of $\widehat{S}_N^{\rm d}$. Several methods to modify rational transfer matrices which are not positive real so that they become positive real after the modification have been suggested in the works [20], [22], [14], [11].

Let us summarize the final algorithm in the following.

*Algorithm 3 (Generalized Fourier series based algorithm):*

1) Given the data $\omega_k^{(N)}$, $S_k^{(N)}$ compute $\Theta_N$ in (28) with the real-valued basis functions $\widetilde{B}_k(z)$ evaluated at $\theta_k^{(N)}$ defined by (27) with a scaling factor $\lambda > 0$.
2) Calculate the basis coefficient vector $c^{(N)}$ in (26).
3) Obtain a minimal balanced realization of $\widehat{H}_N$ and truncate it if necessary.
4) If $\widehat{H}_N$ is strictly positive real, calculate $\widehat{B}_N$ and $\widehat{D}_N$ from (30)–(32); otherwise, use one of the schemes in [20], [22], [14], [11] to calculate $\widehat{B}_N$ and $\widehat{D}_N$.
5) From the state-space parameters $\widehat{A}_N, \widehat{B}_N, \widehat{C}_N, \widehat{D}_N$ of $\widehat{G}_N^{\mathrm{d}}(z)$ and $\lambda$, compute the continuous-time spectral factor $\widehat{G}_N(s)$ by the inverse bilinear transform. ■

The following is the main result of this paper.

*Theorem 4:* Let $S_k^{(N)}$ be as given by (7) where $\eta_k^{(N)}$ satisfies (6). Consider Algorithm 3. Let $S$ be as in Lemma 1 and suppose that $z_k$, $\omega_k^{(N)}$, and $M$ satisfy the conditions in Lemma 2. Then, Algorithm 3 is strongly consistent.

In Algorithm 3, $c^{(N)}$ can effectively be computed using the discrete Fourier transform when $B_k$ equals $z^k$ for all $k$ and $\theta_k^{(N)}$ are uniformly spaced. Moreover, if the spectrum of the true system does not contain sharp peaks, the uniformly spaced spectral data may be recovered from the non-uniformly spaced data by interpolation; and the resulting algorithm is still strongly consistent. The details will be provided in the following section.

## V. Stochastic Road Modeling Example

In this section, we illustrate Algorithm 3 in the modeling of the road spectrum [10] by a low order rational spectrum. In Figure 1, the spectral data are shown with the split power law approximation [10] and the integrated white noise approximation [8]. The split power approximation is defined by

$$\widehat{S}_{\mathrm{sp}}(j2\pi\widetilde{n}) \triangleq \begin{cases} \kappa|\widetilde{n}/\widetilde{n}_0|^{-2\delta_1}, & 0 < |\widetilde{n}| < \widetilde{n}_0 \\ \kappa|\widetilde{n}/\widetilde{n}_0|^{-2\delta_2}, & \widetilde{n}_0 \le |n| < \infty \end{cases}$$

and the integrated white noise approximation by

$$\widehat{S}_{\mathrm{iw}}(j2\pi\widetilde{n}) \triangleq \kappa(\widetilde{n}_0/\widetilde{n})^2. \tag{33}$$

Both the approximations are made to match the spectral data at $\widetilde{n}_0 = 0.15708$ cycles/m. Thus, $\kappa = 0.76 \times 10^{-5}$. The values of $\delta_1$ and $\delta_2$ obtained by trial and error are respectively, 1.6 and 1.1. It is clear that the fit by the integrated white-noise modeling is rather poor; in particular at the frequencies below $\widetilde{n_0}$.

The number of data is $N = 63$. We picked $\lambda = 0.2$ (rather arbitrarily) and piece-wise linearly interpolated $S_k^{(N)}$ to get 128-point uniformly spaced spectral data on $[0, \pi]$:

$$\widetilde{S}_l \triangleq S_k^{(63)} + \frac{S_{k+1}^{(63)} - S_k^{(63)}}{\theta_{k+1}^{(63)} - \theta_k^{(63)}} \left( \frac{\pi l}{128} - \theta_k^{(63)} \right)$$

for $l = 0, 1, \cdots, 128$; $\theta_k^{(63)} \le \frac{\pi l}{128} \le \theta_{k+1}^{(63)}$ where we let $S_{64}^{(63)} = 0$ for $\theta_{64}^{(63)} = \pi$ and extrapolate $S_1^{(63)}$ and $S_2^{(63)}$ to get the spectral datum at $\theta_0^{(63)} = 0$ if $\theta_1^{(63)} > 0$.

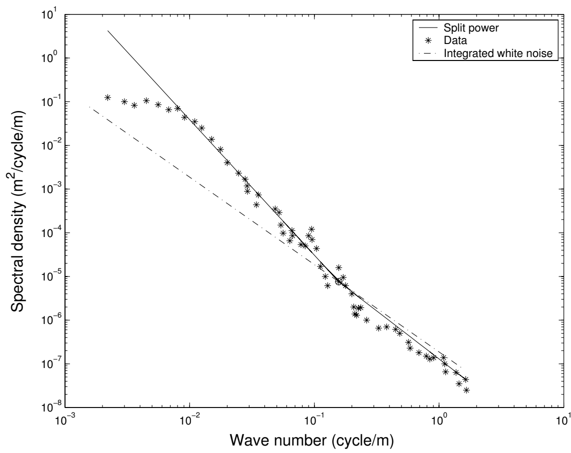


Fig. 1. The spectral data and its approximate modeling by the split power law and the integrated white noise.

We expanded the 129-point spectral data $\widetilde{S}_l$ to 256-point uniformly spaced data according to:

$$\widetilde{S}_{128+l} = \widetilde{S}_{128-l}, \qquad l = 1, \cdots, 127.$$

The use of piece-wise linear splines is not essential; for example, cubic splines may also be used. We chose $M = 126$ and computed $c^{(N)}$ defined in (26) by taking 256-point inverse discrete Fourier transform of $\widetilde{S}$:

$$c_k^{(63)} = \frac{1}{256} \sum_{l=0}^{257} \widetilde{S}_l \, e^{j2\pi kl/256}, \qquad k = 0, 1, \cdots, 126. \tag{34}$$

In Step 3 of Algorithm 3, we truncated balanced realization of $\widehat{H}_N$ to orders one and eight. The truncated transfer functions were still strictly positive real. In step 4, we obtained $\widehat{B}_N$ and $\widehat{D}_N$ simply from polynomial factoring.

In Figures 2 and 3, the estimation results are plotted for the modified spectral factors $\chi\widehat{G}_N$ denoted by $\widetilde{G}_N$ with orders $n = 8$ and $n = 2$, respectively. Here, $\chi(s) = 0.3/(s + 0.3)$ is a convergence factor rolling the frequency response to zero at high frequencies. The modified spectral factors are used in [19] with a quarter-car model to study the response of the vehicle to profile imposed excitation with randomly varying traverse velocity and variable vehicle forward velocity.

Figure 2 demonstrates that the eighth order spectral factor is capable of capturing the road dynamics up to 0.1 cycles/m. Beyond this frequency, the road dynamics is negligible since the power spectral density drops below $10^{-4}$ m$^2$/cycle/m. On the other hand, Fig 3 tells us that the second order spectral factor is accurate up to only 0.01 cycles/m. Henceforth, it is not suitable for modeling road profiles. Based on these limited observations, we suggest using high-order shape filters in road modeling.

The modified spectral factors estimated for $n = 8$ and

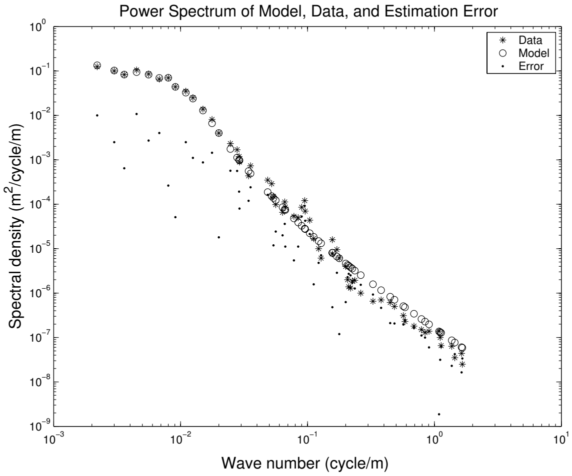


Fig. 2. The spectral data and its estimate $|\widetilde{G}_{63}(j2\pi\widetilde{n})|^2$ for $n = 8$.

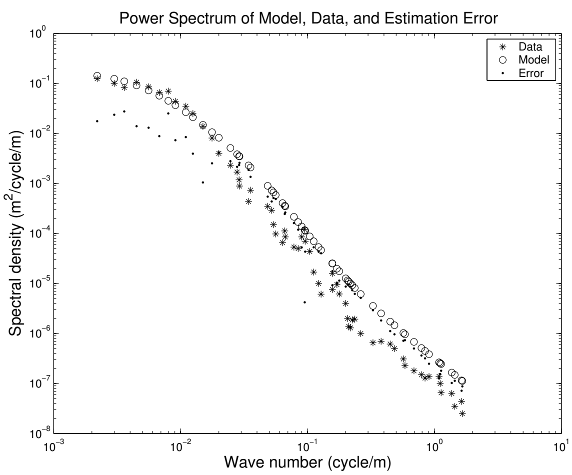


Fig. 3. The spectral data and its estimate $|\widetilde{G}_{63}(j2\pi\widetilde{n})|^2$ for $n = 2$.

$n = 2$ are given by

$$\widetilde{G}_{63}(s) = 0.0084\,\frac{0.3}{s+0.3}\,\frac{(s+0.0024)(s+0.5587)}{(s+0.0023)(s+0.0214)} \times\frac{(s+0.0042\pm j0.0254)(s+0.0032\pm j0.0516)}{(s+0.0041\pm j0.0263)(s+0.0028\pm j0.0514)} \times\frac{(s+0.0980\pm j0.0491)}{(s+0.0383\pm j0.0675)} \qquad (35)$$

and

$$\widetilde{G}_{63}(s) = 0.0116\,\frac{0.3}{s+0.3}\,\frac{s+1.0860}{s+0.0304}. \qquad (36)$$

The near by pole-zero cancellations in (35) suggests using a third order modified spectral factor. We omit the results for this case for the sake of brevity.

## VI. Conclusions

In this paper, we presented a generalized Fourier series based identification algorithm which fits rational models to given noisy power spectrum measurements. A detailed convergence analysis of the algorithm was carried out. A successful application of the algorithm was the design of a linear shaping filter with output spectrum matching the measured road spectrum.

## References


[1] H. Akçay. On the uniform approximation of discrete-time systems by generalized Fourier series. *IEEE Transactions on Signal Processing* **49**:1461–1467, 2001.

[2] H. Akçay. Discrete-time system modeling in $L_p$ with orthonormal basis functions. *Systems and Control Letters* **39**:365–376, 2000.

[3] H. Akçay. A stochastic analysis of robust estimation algorithms in $H_\infty$ with rational basis functions. *International Journal of Robust and Nonlinear Control* **12**:71–86, 2002.

[4] H. Akçay and S. Türkay Frequency domain subspace-based identification of discrete-time power spectra from nonuniformly spaced measurements. *Automatica* **40**:1333–1347, 2004.

[5] B. D. O. Anderson. Continuity of the matrix spectral factorization operation. *Math. Appl. Comput.* **4**:139–156, 1985.

[6] A. Bultheel and P. Van gucht. Boundary asymptotics for orthogonal rational functions on the unit circle. *Acta Appl. Math.* **62**:333–349, 2000.

[7] F. M. Callier and J. Winkin. Spectral factorization and LQ-optimal regulation for multivariable distributed systems. *International Journal of Control*, **52**:55–75, 1990.

[8] R. M. Chalasani. Ride performance potential of active suspension systems -Part I: Simplified anaysis based on a quarter-car model. *ASME-AMD*, **80**:187–204, 1986.

[9] E. W. Cheney. *Introduction to Approximation Theory*. Mc-Graw Hill, New York, 1966.

[10] C. J. Dodds and J. D. Robson . The description of road surface roughness. *Journal of Sound and Vibration*, **31**:175–183, 1973.

[11] I. Goethals, T. Van Gestel, J. Suykens, P. Van Dooren, and B. De Moor. Identification of positive real models in subspace identification by using regularization. *IEEE Transactions on Automatic Control*, **48**:1843–1847, 2003.

[12] B. Jacob, J. Winkin, and H. Zwart. Does the spectral factor depend continuously on the spectral density? In *Proceedings of the Mathematical Theory of Networks and Systems*, Beghi A, Finesso I, Picci G (eds). Il Poligrafo, Padova, Italy, 217-220, 1998.

[13] S. Kay. *Modern Spectral Estimation, Theory & Application*. Prentice-Hall, Englewood Cliffs, NJ, 1988.

[14] J. Mari. Modifications of rational transfer matrices to achieve positive reals. *Signal Processing*, **80**:615–635, 2000.

[15] B. Ninness and F. Gustafsson. A unifying construction of orthonormal bases for system identification. *IEEE Transactions of Automatic Control*, **42**:515–521, 1997.

[16] M. B. Priestley. *Spectral Analysis and Time Series*. Academic Press, London, England, 1989.

[17] C. L. Rino. Factorization of spectra by discrete Fourier transforms. *IEEE Transactions on Information Theory*, **16**:484–485, 1970.

[18] P. Stoica and R. Moses. *Introduction to Spectral Analysis*. Prentice-Hall, Upper Saddle River, NJ, 1997.

[19] S. Türkay and H. Akçay. A Study of random vibration charecteristics of the quarter-car model. In *Proceedings of the 1st IFAC Symposium on Advances in Automotive Control*. Salerno, Italy. April, 757–762, 2004.

[20] R.J. Vaccaro and T. Vukina. A solution to the positivity problem in the space-space approach to modeling vector-valued time series. *Journal of Economic Dynamics and Control*, **17**:401–421, 1993.

[21] P. Van gucht and A. Bultheel. State space representation for arbitrary orthogonal rational functions. *Systems & Control Letters*, **49**: 91-98, 2003.

[22] P. Van Overschee, B. De Moor, W. Dehandschutter, and J. Swevers. A subspace algorithm for the identification discrete-time frequency domain power spectra. *Automatica*, **33**:2147–2157, 1997.