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On Event Triggered Trajectory Tracking for Control Affine Nonlinear Systems

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***Abstract*— In this paper we study an event based control algorithm for trajectory tracking in control affine nonlinear systems. The desired trajectory is modelled as the solution of a reference system with an exogenous input. It is assumed that the desired trajectory and the exogenous input to the reference system are uniformly bounded. Given a continuous-time controller that guarantees global uniform asymptotic tracking of the desired trajectory our algorithm provides an event based controller that not only guarantees semiglobal uniform ultimate boundedness of the tracking error, but also ensures non-accumulation of inter-execution times. In the special case that the derivative of the exogenous input to the reference system is also uniformly bounded, the proposed control algorithm can be used to design an ultimate bound that is arbitrarily small. The main ideas in the paper are illustrated through simulations of trajectory tracking by a nonlinear system.**

I. Introduction

Traditional computer based control systems rely on periodic sampling of the sensors and computation/execution of the control. The reason for the popularity of this paradigm is a well developed theory and the ease of analysis of such systems. However, such control algorithms may be very inefficient from a computational perspective as the period for sampling and control execution is determined by a worst case analysis and the rate of control execution is independent of the state of the system. On the other hand, in event based control systems, timing of control execution is not necessarily periodic and can be state dependent. Thus, event based control is useful in systematically designing controllers that make better use of computational and communication resources in a wide variety of applications such as embedded control systems and decentralized systems (a representative list of references includes [1], [2], [3], [4]).

While there have been some efforts in the past to study event based control systems [5], [6], [7], their systematic design for tasks such as stabilization has been undertaken only recently [8], [1], [9], [10], [11]. Of these, [1] has significantly influenced the the proposed controller in this paper. In [1], an event-triggering algorithm was proposed that ensures global asymptotic stability as well as a lower bound on the inter-execution times of the control law for general nonlinear systems that are rendered Input-to-State Stable (ISS) with respect to measurement errors by a continuous time controller.

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In this paper, we investigate an event triggered control algorithm for trajectory tracking. Tracking a time varying trajectory or even a set-point is of tremendous practical importance in many control applications. In these applications, the goal is to make the state of the system follow a reference or desired trajectory, which is usually specified as an exogenous input to the system. In this paper, the reference trajectory is modelled as a solution of a reference system. To the best of our knowledge, all the previous works in the event-triggered control literature assumed a state feedback control strategy with no exogenous input signals, notable exceptions being [8], [9], [10], [11], [12], [13], where unknown disturbances appear as exogenous inputs. However, in this paper, we consider exogenous inputs that are available to the controller through measurements, namely the reference trajectory and the input to the reference system.

The main **contribution** of this paper is the design of event-triggered controllers for trajectory tracking in control affine nonlinear systems, which is a special case of nonlinear systems with exogenous inputs. It is assumed that the reference trajectory and the exogenous input to the reference system are uniformly bounded. Given a nonlinear system and a continuous-time controller that ensures global uniform asymptotic tracking of the desired trajectory, the proposed algorithm provides an event based controller that guarantees semiglobal uniform ultimate boundedness of the tracking error and ensures that the inter-execution times of the control are bounded away from zero. In the special case that the derivative of the exogenous input to the reference system is also uniformly bounded, an arbitrarily small ultimate bound for the tracking error can be designed. It is to be noted that our assumption regarding the closed loop system with continuous-time control is weaker than the ISS like property assumed in [1], and hence is a minor contribution in itself.

The rest of the paper is organized as follows. In Section II we set up the problem and introduce the notation used in the paper. Subsequently, in Section III, the major assumptions are stated and the event triggering condition is introduced. The main analytical results are presented in Section IV. The theoretical results in the paper are illustrated through numerical simulations of a second order nonlinear system in Section V. Finally, the results are summarized in Section VI.

II. Problem statement and notation

Consider a nonlinear system of the form

$$\dot{x} = f(x) + g(x)u, \tag{1}$$

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where $x \in \mathbb{R}^n$, $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $g : \mathbb{R}^n \longrightarrow \mathbb{R}^n \times \mathbb{R}^m$. Let the reference or the desired trajectory that has to be tracked be defined implicitly by the dynamical system

$$\dot{x}_d = f_r(x_d, v) \tag{2}$$

where $x_d \in \mathbb{R}^n$, $v \in \mathbb{R}^q$ and $f_r : \mathbb{R}^n \times \mathbb{R}^q \longrightarrow \mathbb{R}^n$. The external signal v and the initial condition of the signal x_d determine the reference trajectory. The tracking error is defined as

$$\tilde{x} \triangleq x - x_d. \tag{3}$$

In general, a control for tracking a reference trajectory depends on both the tracking error as well as the reference trajectory. Hence, we assume that the control signal is of the form

$$u = \gamma(\xi), \quad \text{where} \quad \xi \triangleq [\tilde{x}, x_d, v]^T \tag{4}$$

where the notation $[a_1, a_2, a_3]^T$ denotes the concatenation of the vectors a_1, a_2 and a_3. Consequently, the closed loop system that describes the tracking error is given as

$$\dot{\tilde{x}} = f(\tilde{x} + x_d) + g(\tilde{x} + x_d)\gamma(\xi) - \dot{x}_d. \tag{5}$$

Now, consider a controller that updates the control only intermittently and not continuously in time. Let t_i for $i = 0, 1, 2, \ldots$ be the time instances at which the control is computed and updated. Then the tracking error evolves as

$$\begin{aligned} &\dot{\tilde{x}} = f(\tilde{x} + x_d) + g(\tilde{x} + x_d)\gamma(\xi(t_i)) - \dot{x}_d, \\ &\text{for } t \in [t_i, t_{i+1}),\ i \in \{0, 1, 2, ...\}. \end{aligned} \tag{6}$$

The above dynamical system can also be viewed as a continuously updated control system, albeit with an error in the measurement of the state and the exogenous input. Let

$$e^i \triangleq \xi(t_i) - \xi. \tag{7}$$

Now, by defining the measurement error as

$$\begin{aligned} &\bar{e} \triangleq \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} \triangleq e^i \triangleq \begin{bmatrix} \tilde{x}(t_i) - \tilde{x} \\ x_d(t_i) - x_d \\ v(t_i) - v \end{bmatrix}, \\ &\text{for } t \in [t_i, t_{i+1}),\ i \in \{0, 1, 2, ...\} \end{aligned} \tag{8}$$

the system in (6) can be rewritten as

$$\begin{aligned} \dot{\tilde{x}} = &\left[f(\tilde{x} + x_d) + g(\tilde{x} + x_d)\gamma(\xi) - \dot{x}_d\right] \\ &+ g(\tilde{x} + x_d)\left[\gamma(\xi + \bar{e}) - \gamma(\xi)\right] \end{aligned} \tag{9}$$

where we have expressed the above system as a perturbed version of the dynamical system described in (5). Note that $\bar{e}$ is discontinuous at $t = t_i$, for each i, because $\bar{e}(t_i) = e^i(t_i) = 0$ while $\bar{e}(t_i^-) \triangleq \lim_{t \uparrow t_i} \bar{e}(t) = \lim_{t \uparrow t_i} e^{i-1}(t)$.

In time-triggered or periodic control systems, $t_{i+1} - t_i = T_s$ for all $i \in \{0, 1, 2, \ldots\}$, where $T_s > 0$ is a constant sampling time. On the other hand, in an event-triggered system the time instants t_i in general are not uniformly separated, and are determined dynamically by an event-triggering condition.

The objective of this paper is to develop an event based control for tracking a trajectory within a desired ultimate bound. To this end, we assume that when the control is updated continuously in time, the state x tracks the desired trajectory asymptotically, that is, for system (5) $\tilde{x} \to 0$ as $t \to \infty$. In the next section we investigate the conditions under which it is possible to track the desired trajectory with the same control function, but updated intermittently based on event-triggering, as in (6), rather than continuously in time.

III. EVENT-TRIGGERING CONDITION FOR EMULATION BASED TRAJECTORY TRACKING CONTROL

There are two main requirements for an event based controller for trajectory tracking. It needs to (i) guarantee that the tracking error is at least uniformly ultimately bounded, and (ii) ensure that there are no accumulation of execution times. In this section, an event-triggering condition that satisfies both these requirements is developed. We begin by formally stating the **main assumptions** in this paper regarding the system in (5).

- **(A1)** There exists a C^1 Lyapunov function for the dynamical system in (5), $V(\tilde{x}) : \mathbb{R}^n \to \mathbb{R}$, such that for all admissible x_d and v,

$$\begin{aligned} &\alpha_1(\|\tilde{x}\|) \le V(\tilde{x}) \le \alpha_2(\|\tilde{x}\|) \\ &\frac{\partial V}{\partial \tilde{x}}\left[f(\tilde{x} + x_d) + g(\tilde{x} + x_d)\gamma(\xi) - \dot{x}_d\right] \le -\alpha_3(\|\tilde{x}\|) \end{aligned}$$

  where $\alpha_1(.)$, $\alpha_2(.)$, and $\alpha_3(.)$ are class $\mathcal{K}_\infty$ [1] functions.
- **(A2)** $f(.)$, $g(.)$, $f_r(.,.)$, and $\gamma(.,.,.)$ are Lipschitz on compact sets.
- **(A3)** For all time $t \ge 0$, $\|[x_d, v]^T\| \le d$ for some $d \ge 0$ and v is piecewise continuous.
- **(A4)** For all time $t \ge 0$, $\|\dot{v}\| \le c$ for some $c \ge 0$.
- **(A5)** The initial tracking error is bounded by a known constant, that is, $\|\tilde{x}(0)\| \le R$, and $R > 0$ is a known constant.

Let L be the Lipschitz constant for the function $\gamma(.,.,.)$ on the compact set

$$B = \{\xi : \|\tilde{x}\| \le \mu,\ \|[x_d, v]^T\| \le d\},\ \mu = \alpha_1^{-1}(\alpha_2(R)). \tag{10}$$

Note that the set B includes all the admissible reference signals. Next, by assumption (A2)

$$\|\gamma(\xi + \bar{e}) - \gamma(\xi)\| \le L\|\bar{e}\|, \quad \forall\, \xi,\ (\xi + \bar{e}) \in B \tag{11}$$

We also define

$$\beta(\|\tilde{x}\|) \triangleq \max_{\|w\| \le \|\tilde{x}\|,\ \|x_d\| \le d} \left\|\frac{\partial V(w)}{\partial w} g(w + x_d)\right\|. \tag{12}$$

We now derive the triggering condition that determines the time instants t_i at which the control is updated.

Consider the Lyapunov function, $V(.)$, in assumption (A1) as a candidate Lyapunov function for the system defined

[1] A continuous function $\alpha : [0, \infty) \to [0, \infty)$ is said to belong to the class $\mathcal{K}_\infty$ if it is strictly increasing, $\alpha(0) = 0$ and $\alpha(r) \to \infty$ as $r \to \infty$ [14].

by (6). The time derivative of $V(\tilde{x})$, along the flow of the tracking error system, $\dot{V} = (\partial V/\partial \tilde{x})\dot{\tilde{x}}$, may be obtained through the measurement error interpretation, (9).

$$\begin{aligned}\dot{V} =& \frac{\partial V}{\partial \tilde{x}}\big[f(\tilde{x}+x_d) + g(\tilde{x}+x_d)\gamma(\xi) - \dot{x}_d\big] \\ &+ \frac{\partial V}{\partial \tilde{x}} g(\tilde{x}+x_d)\big[\gamma(\xi+\bar{e}) - \gamma(\xi)\big] \\ \leq& -\alpha_3(\|\tilde{x}\|) + \frac{\partial V}{\partial \tilde{x}} g(\tilde{x}+x_d)[\gamma(\xi+\bar{e}) - \gamma(\xi)] \qquad (13) \\ \leq& -\alpha_3(\|\tilde{x}\|) + \beta(\|\tilde{x}\|)L\|\bar{e}\|, \quad \forall \xi,\, (\xi+\bar{e}) \in B \qquad (14)\end{aligned}$$

where (13) is obtained from assumption (A1), and (14) is then obtained from (10)-(12). From (14) it appears that a triggering condition that ensures $\xi, (\xi+\bar{e}) \in B$ for all $t \geq 0$ also ensures ultimate boundedness of the tracking error, $\tilde{x}$. In the sequel, this statement is formally shown to be true. But first, we define a triggering condition based on this idea.

Consider the following **triggering condition** (for the sake of clarity, the complete system description including the state equation and the triggering condition are given).

$$\begin{aligned}&\dot{\tilde{x}} = f(\tilde{x}+x_d) + g(\tilde{x}+x_d)\gamma(\xi(t_i)) - \dot{x}_d, \\ &\text{for } t \in [t_i, t_{i+1}),\ i \in \{0,1,2,...\} \qquad (15) \\ &\text{where} \\ &t_0 = \min\{t \geq 0 : \|\tilde{x}\| \geq r > 0\},\ \text{and} \\ &t_{i+1} = \min\{t \geq t_i : \|e^i\| \geq W(\|\tilde{x}\|),\ \text{and } \|\tilde{x}\| \geq r > 0\} \\ &W(\|\tilde{x}\|) = \frac{\sigma\alpha_3(\|\tilde{x}\|)}{L\beta(\|\tilde{x}\|)},\ 0 < \sigma < 1, \quad \text{for } \tilde{x} \neq 0 \qquad (16)\end{aligned}$$

where the parameter r is a design choice that determines the ultimate bound of the tracking error. It is necessary to update the control only when $\|\tilde{x}\| \geq r$, for some $r > 0$, else it may result in the accumulation of control update times. Further, without loss of generality, it is assumed that $r \leq R$, where R is the bound on the initial condition (assumption (A5)). Notice that each update instant t_{i+1} is defined implicitly with respect to t_i. Hence, the initial update instant t_0 has been specified separately. As the proposed triggering condition does not allow the control to be updated whenever $\|\tilde{x}\| < r$, the first update instant, t_0, need not be at $t = 0$. Therefore, it is assumed that $u = 0$ for $0 \leq t < t_0$. In the next section it is shown, for two different classes of reference trajectories, that the triggering condition (16) ensures uniform semiglobal ultimate boundedness of the tracking error.

IV. Uniform Ultimate Boundedness of the Trajectory Tracking Error

The following lemma demonstrates that the event-triggering condition (16) ensures ultimate boundedness of the tracking error, provided the sequence of control execution times does not exhibit *Zeno behavior* (accumulation of inter-execution times), that is either the sequence of control execution times is finite or $\lim_{i\to\infty} t_i = \infty$.

Lemma 1: Consider the system (5). Suppose that assumptions (A1), (A2), (A3) and (A5) are satisfied, and let r be any constant such that $0 < r \leq R$. In the event-triggered system (15)-(16), if the sequence of control execution times does not exhibit Zeno behavior, then the tracking error, $\tilde{x}$, is uniformly ultimately bounded by a ball of radius $r_1 = \alpha_1^{-1}(\alpha_2(r))$.

Proof: **Case I:** Suppose that the sequence of control execution times is infinite and $\lim_{i\to\infty} t_i = \infty$.

From the definition of t_0, it is clear that $\|\tilde{x}\| \leq r \leq R$ for $t \in [0, t_0)$, and $\|\tilde{x}(t_0)\| \leq R$. Thus, by assumption (A1), $V(\tilde{x}(t_0)) \leq \alpha_2(R)$. Now, we *show by induction* that for all $t \in [t_0, t_i]$, and for each $i \in \{0,1,2,\ldots\}$, $V(\tilde{x}(t)) \leq \alpha_2(R)$. Clearly, the statement is true for $i = 0$. Now, assume that for all $t \in [t_0, t_i]$, $V(\tilde{x}(t)) \leq \alpha_2(R)$, and hence $\xi \in B$ (or, equivalently $\|\tilde{x}\| \leq \mu$). Then, we need to show that the induction statement is true for $t \in [t_0, t_{i+1}]$. Observe that for each $k \in \{0,1,\ldots,i\}$, and for $t \in [t_k, t_{k+1})$, $\xi + \bar{e} = \xi(t_k) \in B$. Hence, $(\xi+\bar{e}) \in B$ for all $t \in [t_0, t_{i+1})$. Then by (14) we have that for all $t \in [t_0, t_{i+1})$

$$\begin{aligned}\dot{V} &\leq -\alpha_3(\|\tilde{x}\|) + \beta(\|\tilde{x}\|)L\|\bar{e}\|,\ \text{for all } \tilde{x} \text{ s.t. } \|\tilde{x}\| \leq \mu \\ &\leq -(1-\sigma)\alpha_3(\|\tilde{x}\|),\ \text{for all } \tilde{x} \text{ s.t. } r \leq \|\tilde{x}\| \leq \mu \qquad (17)\end{aligned}$$

where the second relation follows from the triggering condition (16). Note that $\alpha(\|\tilde{x}\|) \triangleq (1-\sigma)\alpha_3(\|\tilde{x}\|)$ is a class $\mathcal{K}_\infty$ function. By the induction hypothesis $V(\tilde{x}(t_i)) \leq \alpha_2(R)$. Thus, (17) and continuity of the tracking error imply that $V(\tilde{x}(t)) \leq \alpha_2(R)$ for all $t \in [t_0, t_{i+1}]$. Therefore, by induction we see that $V(\tilde{x}(t)) \leq \alpha_2(R)$ for all $t \in [t_0, t_i]$, for each $i \in \{0,1,2,\ldots\}$.

Now, consider the sets $\Omega \triangleq \{\tilde{x} : r \leq \|\tilde{x}\| \leq \mu\}$ and $E \triangleq \{\tilde{x} : V(\tilde{x}) \leq \alpha_2(r)\}$. The assumption that $\lim_{i\to\infty} t_i = \infty$, together with (17) implies that $\dot{V} \leq -\alpha(r) < 0$ for all $\tilde{x} \in \Omega$ and $t \geq t_0$. The set $\Omega \cap E$ is non-empty, and moreover the level set $\delta E \triangleq \{\tilde{x} : V(\tilde{x}) = \alpha_2(r)\} \subset \Omega$. Consequently, all trajectories satisfying assumption (A5) eventually enter the set E in finite time and stay there, as E is positively invariant.

Finally, α_1 is a class $\mathcal{K}_\infty$ function, and there exists an r_1 such that $r_1 = \alpha_1^{-1}(\alpha_2(r))$ and $E \subseteq \{\tilde{x} : \|\tilde{x}\| \leq r_1\}$. Therefore, the tracking error, $\tilde{x}$, is uniformly ultimately bounded by the closed ball of radius r_1.

Case II: Suppose that the sequence of control execution times, $\{t_0, t_1, \ldots, t_N\}$, is finite.

The induction hypothesis in Case I holds in this case for each $i \in \{0,1,\ldots,N\}$ with $t_{N+1} = \infty$. The rest of the proof is similar to that of Case I. ■

Henceforth, Case II is not considered explicitly as it is included in Case I. Next, we show that the system (15-16) does not exhibit Zeno behavior by demonstrating that the inter-execution times are uniformly bounded away from zero. We consider two different classes of reference trajectories in Lemmas 2 and 3, respectively.

Lemma 2: Consider the system (5). Suppose that assumptions (A1), (A2), (A3), (A4) and (A5) are satisfied, and let r be any constant such that $0 < r \leq R$. Then, in the event-triggered system (15)-(16), the inter-update times $(t_{i+1} - t_i)$ for $i \in \{0,1,2,\ldots\}$ are uniformly bounded away from zero.

Proof: It follows from the proof of Lemma 1 that $\|\tilde{x}\| \leq \mu = \alpha_1^{-1}(\alpha_2(R))$ and $r \leq \|\tilde{x}(t_i)\| \leq \mu$, for each i. Additionally, we note that by definition $\|\bar{e}(t_i)\| = 0$, for each i. Hence $t_{i+1} - t_i \geq T$, where T is the time it takes $\|\bar{e}\|$ to grow from 0 to $\epsilon = \sigma\alpha_3(r)/L\beta(\mu)$, which is a lower bound for the minimum value of $\sigma\alpha_3(\|\tilde{x}\|)/L\beta(\|\tilde{x}\|)$ on the compact set $\{\tilde{x} : r \leq \|\tilde{x}\| \leq \mu\}$. If we show that $T > 0$, then the proof is complete.

From (9), and the triangle inequality property, we observe that

$$\begin{aligned}\|\dot{\tilde{x}}\| \leq& \|f(\tilde{x} + x_d) + g(\tilde{x} + x_d)\gamma(\xi) - \dot{x}_d\| \\ &+ \|g(\tilde{x} + x_d)[\gamma(\xi + \bar{e}) - \gamma(\xi)]\|. \end{aligned} \tag{18}$$

By assumptions (A2) and (A3), there exists a Lipschitz constant M_1 for the function on the right hand side of (5) in the set B. Recall that for all $t \in [t_0, t_i)$, $\xi \in B$ and $(\xi + \bar{e}) \in B$, thus (11) holds. Also, by virtue of assumption (A1), $f(0) + g(0)\gamma(0, 0, 0) - 0 = 0$. Finally, $g(.)$ is Lipschitz on compact sets, and hence $\|g(\tilde{x} + x_d)\|$ attains a maximum value, M_2, on the compact set $\{\tilde{x}+x_d : \|\tilde{x}\| \leq \mu, \|x_d\| \leq d\}$. Using these facts we see that

$$\begin{aligned}\|\dot{\tilde{x}}\| &\leq M_1(\|\tilde{x}\| + \|[x_d, v]^T\|) + L\|\bar{e}\|\|g(\tilde{x} + x_d)\| \\ &\leq M_1(\|\tilde{x}\| + \|[x_d, v]^T\|) + M_2L\|\bar{e}\|. \end{aligned} \tag{19}$$

Therefore, $\|\dot{\tilde{x}}\| \leq P_1(\|\tilde{x}\| + \|\bar{e}\| + d)$ for some $P_1 > 0$, where d is the uniform bound on $\|[x_d, v]^T\|$ in assumption (A3). Since $\|\tilde{x}\| \leq \mu$ for all $t \in [t_0, t_i)$, for each i, $\|\dot{\tilde{x}}\| \leq P_1(\mu + \|\bar{e}\| + d)$. Hence, by the definition $\dot{\bar{e}} = -[\dot{\tilde{x}}, \dot{x}_d, \dot{v}]^T$ there exists a finite $P > 0$ such that

$$\frac{\mathrm{d}\|\bar{e}\|}{\mathrm{d}t} \leq \|\dot{\bar{e}}\| \leq P(\mu + \|\bar{e}\| + d + c) \tag{20}$$

where c is the uniform bound on $\|\dot{v}\|$ in assumption (A4). Note that for $\|\bar{e}\| = 0$, the first inequality holds for all the directional derivatives of $\|\bar{e}\|$. Then, according to the Comparison Lemma [14]

$$\|\bar{e}\| \leq (\mu + d + c)(\mathrm{e}^{P(t-t_i)} - 1), \quad \text{for} \quad t \geq t_i. \tag{21}$$

Thus, the inter-execution times are uniformly lower bounded by T, which satisfies

$$T \geq \frac{1}{P} \log\left(1 + \frac{\sigma\alpha_3(r)}{L\beta(\mu)(\mu + d + c)}\right). \tag{22}$$

As P and L are finite, we conclude that the inter-execution times have a uniform lower bound, T, that is greater than zero. ■

This leads to the first main result of this paper, which is presented below.

Theorem 1: Consider the system (5). Suppose that assumptions (A1), (A2), (A3), (A4) and (A5) are satisfied, and let r be any constant such that $0 < r \leq R$. Then, for the event-triggered system (15)-(16), the tracking error, $\tilde{x}$, is uniformly ultimately bounded by a ball of radius $r_1 = \alpha_1^{-1}(\alpha_2(r))$, and the inter-update times $(t_{i+1} - t_i)$ for $i \in \{0, 1, 2, \ldots\}$ are uniformly bounded away from zero.

Proof: The proof follows from Lemma 1 and Lemma 2. ■

In the next result, we relax the conditions on the reference trajectory by no longer requiring it to satisfy assumption (A4). Instead, certain conditions on the function $W(\|\tilde{x}\|) = \sigma\alpha_3(\|\tilde{x}\|)/L\beta(\|\tilde{x}\|)$ are assumed to demonstrate the absence of Zeno behavior. The new assumption implies that there is a certain region in the state space where the numerator, $\sigma\alpha_3(\|\tilde{x}\|)$, dominates the denominator, $L\beta(\|\tilde{x}\|)$, by a factor determined by the bound on $[x_d, v]^T$ in assumption (A3).

In this case also we demonstrate that the inter-execution times are uniformly bounded away from zero. However, as compared to Lemma 2, where the choice of r was completely arbitrary, the new assumptions lead to a constraint on the choice of the radius r in the triggering condition.

Lemma 3: Consider the system defined by (5). Suppose that the following assumptions are satisfied
(1) assumptions (A1), (A2), (A3) and (A5) hold.
(2) r is any constant such that $0 < r \leq R$.
(3) There exist constants $s_0 > 0$ and $\delta > 0$ such that $(2d + \delta) \leq W(\|\tilde{x}\|)$, for all $s_0 \leq \|\tilde{x}\| \leq \mu$.
If $r > s_0$, then in the event-triggered system (15)-(16), the inter-update times $(t_{i+1} - t_i)$ for $i \in \{0, 1, 2, \ldots\}$ are uniformly bounded away from zero.

Proof: The proof is very similar to that of Lemma 2, and hence only the essential steps are described here. We note that for each i, $s_0 < r \leq \|\tilde{x}(t_i)\| \leq \mu$. Furthermore, due to assumption (3), $(2d + \delta) \leq W(\|\tilde{x}\|)$ for all $\tilde{x}$ s.t. $s_0 < r \leq \|\tilde{x}\| \leq \mu$. The triggering condition in (16) implies that $\|\bar{e}(t_i^-)\| \geq W(\|\tilde{x}(t_i)\|)$, and therefore $\|\bar{e}(t_i^-)\| \geq (2d + \delta)$, for each i. We know by Assumption (A3) that $\|\bar{e}\| \leq \|e_1\| + \|[e_2, e_3]^T\| \leq \|e_1\| + 2d$. Hence, the inter-execution times $t_{i+1} - t_i \geq T$, where T is the time it takes $\|e_1\|$ to grow from 0 to δ. If we show that $T > 0$, then the proof is complete.

From the proof of Lemma 2, it is known that $\|\dot{\tilde{x}}\| \leq P_1(\mu + \|\bar{e}\| + d)$ for some $P_1 > 0$. Since by definition $\dot{e}_1 = -\dot{\tilde{x}}$, we see that

$$\begin{aligned}\|\dot{e}_1\| &\leq P_1(\mu + \|\bar{e}\| + d) \\ &\leq P_1(\mu + \|e_1\| + 3d), \text{ as } \|\bar{e}\| \leq \|e_1\| + 2d. \end{aligned}$$

Therefore, there exists a finite $P > 0$ such that

$$\frac{\mathrm{d}\|e_1\|}{\mathrm{d}t} \leq \|\dot{e}_1\| \leq P(\mu + \|e_1\| + d). \tag{23}$$

Note that for $\|e_1\| = 0$, the first inequality holds for all the directional derivatives of $\|e_1\|$. Then, according to the Comparison Lemma [14]

$$\|e_q\| \leq (\mu + d)(\mathrm{e}^{P(t-t_i)} - 1). \tag{24}$$

Hence, the inter-execution times are uniformly lower bounded by T, which satisfies

$$T \geq \frac{1}{P} \log\left(1 + \frac{\delta}{(\mu + d)}\right). \tag{25}$$

As P is finite, we conclude that the inter-execution times have a lower bound, T, that is greater than zero. ■

This leads to the second main result of this paper, which is presented below.

Theorem 2: Consider the system defined by (5). Suppose that the following assumptions are satisfied

(1) assumptions (A1), (A2), (A3) and (A5) hold.
(2) r is any constant such that $0 < r \leq R$.
(3) There exist constants $s_0 > 0$ and $\delta > 0$ such that $(2d + \delta) \leq W(\|\tilde{x}\|)$, for all $s_0 \leq \|\tilde{x}\| \leq \mu$.
If $r > s_0$, then for the event-triggered system (15)-(16), the tracking error, $\tilde{x}$, is uniformly ultimately bounded by a ball of radius $r_1 = \alpha_1^{-1}(\alpha_2(r))$, and the inter-update times $(t_{i+1} - t_i)$ for $i \in \{0, 1, 2, \ldots\}$ are uniformly bounded away from zero.

Proof: The proof follows from Lemma 1 and Lemma 3. ■

Remark 1: In Theorem 1, the uniform ultimate bound of the tracking error can be made arbitrarily small by choosing an arbitrarily small value for r. Although Theorem 2 holds for a wider class of reference trajectories, the ultimate bound cannot be made arbitrarily small.

Remark 2: Equations (22) and (25) provide very conservative lower bounds on the inter-execution times. It may appear that smaller the r (and r_1), the higher the average triggering frequency. However, it is not necessarily true. In fact, this is the advantage of event-triggered control over time-triggered control. In time-triggered control the period of control execution has to be uniformly less than a worst-case bound such as (22), which depends on the desired ultimate bound of the tracking error.

Remark 3: It is not necessary for L to be constant for all $t \geq t_0$. At each t_i we can choose $L = L_i$, the Lipschitz constant for γ on the compact set $B_i = \{\xi : \|\tilde{x}\| \leq \alpha_1^{-1}(V(\tilde{x}(t_i))),\ \|[x_d, v]^T\| \leq d\}$. As $\dot{V} < 0$ for $r \leq \|\tilde{x}\| \leq \mu$, and α_1^{-1} is a monotonously increasing function, the sequence of sets B_i and the sequence of Lipschitz constants L_i are decreasing as long as $\|\tilde{x}\| \geq r$. Subsequently, a constant value of L may be used. These constants L_i may be pre-computed, and by appropriately partitioning the $\tilde{x}$ space only a finitely many of them are needed.

In the next section our theoretical results are illustrated through simulations of a second order nonlinear system.

V. Examples and simulation results

The theoretical results developed in the previous sections are illustrated through simulations of the following second order nonlinear system.

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \frac{1}{l}(-g\cos(x_1) + u). \end{aligned} \tag{26}$$

In the simulations the parameters g and l were chosen as 10 and 0.2, respectively. The desired trajectory is a solution of the system

$$\begin{aligned} \dot{x}_{d,1} &= x_{d,2} \\ \dot{x}_{d,2} &= v \end{aligned} \tag{27}$$

where v is an exogenous input, which along with the initial conditions determines the specific trajectory.

The evolution of the tracking error, $\tilde{x} = [\tilde{x}_1, \tilde{x}_2]^T$, can be written as

$$\begin{aligned} \dot{\tilde{x}}_1 &= \tilde{x}_2 \\ \dot{\tilde{x}}_2 &= \frac{1}{l}(-g\cos(\tilde{x}_1 + x_{d,1}) + u) - v. \end{aligned} \tag{28}$$

Let the control input be given as

$$\begin{aligned} u &= \gamma(\tilde{x}, x_d, v) \\ &= l(v - \lambda\tilde{x}_2) + g\cos(\tilde{x}_1 + x_{d,1}) - K(\tilde{x}_2 + \lambda\tilde{x}_1). \end{aligned} \tag{29}$$

where $K > 0$ and $\lambda > 0$. Now consider the Lyapunov function

$$V(\tilde{x}) = \frac{l}{2}(\tilde{x}_2 + \lambda\tilde{x}_1)^2 + \lambda K\tilde{x}_1^2. \tag{30}$$

Then along the flow defined by (28) and (29)

$$\dot{V} = -K\tilde{x}_2^2 - K\lambda^2\tilde{x}_1^2 \leq -\min(K, K\lambda^2)\|\tilde{x}\|^2 \tag{31}$$

As the candidate Lyapunov function is a radially unbounded positive definite function, the origin of system (28) is globally asymptotically stable, that is the nonlinear system tracks the desired trajectory asymptotically.

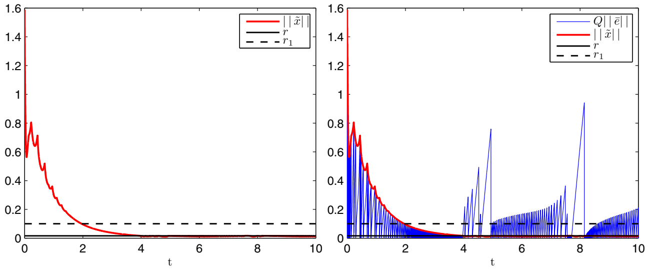
If instead, the control input is updated using event triggers, then

$$\dot{V}(\tilde{x}) \leq -\min(K, K\lambda^2)\|\tilde{x}\|^2 + L\sqrt{\lambda^2 + 1}\|\tilde{x}\|\|\bar{e}\| \tag{32}$$

where $\tilde{x}$ and $\bar{e}$ are defined as in Section II and $L = \sqrt{(\lambda K + g)^2 + (K + \lambda l)^2 + g^2 + l^2}$. By comparison with the results, we see that in this case $\alpha_3(\|\tilde{x}\|) = \min(K, K\lambda^2)\|\tilde{x}\|^2$ and $\beta(\|\tilde{x}\|) = \sqrt{\lambda^2 + 1}\|\tilde{x}\|$. Consequently, given a desired ultimate bound for the trajectory tracking error, we can design a r in the triggering condition. In this system, L is a global Lipschitz constant. Therefore, $W(\|\tilde{x}\|) = \|\tilde{x}\|/Q$, where Q is a constant (σ was chosen as 0.99). Next, we present simulation results for two cases corresponding to the two classes of reference trajectories considered in this paper.

Case I: The signals $x_{d,1}$, $x_{d,2}$, and v were chosen as sinusoidal signals with amplitude $1/2$. We selected $K = 7$, $\lambda = 1$, and following the conditions in Theorem 1, we chose $r = 0.0164$ in the triggering condition to achieve an ultimate bound of $r_1 = 0.1$ in the tracking error. The simulation results are shown in Figure 1. Figure 1a shows the norm of the tracking error, the radius r in the triggering condition, and the desired ultimate bound r_1. The figure demonstrates that the tracking error is ultimately bounded, and well below the desired bound. Figure 1b shows the scaled measurement error, $Q\|\bar{e}\|$, in addition to the data in the first figure. We recall that according to the triggering condition (16), the control is not updated when $\|\tilde{x}\| < r$. Hence, as long as $\|\tilde{x}\| \geq r$, the scaled measurement error, $Q\|\bar{e}\|$, is bounded above by the norm of the tracking error, $\|\tilde{x}\|$, and an event is triggered (control is sampled) each time $Q\|\bar{e}\| \geq \|\tilde{x}\|$. However, when $\|\tilde{x}\| < r$, $Q\|\bar{e}\|$ may exceed $\|\tilde{x}\|$.

The number of control executions in the simulated time duration was 319, and the minimum inter-execution time was observed to be 0.0043s. Therefore, the observed average

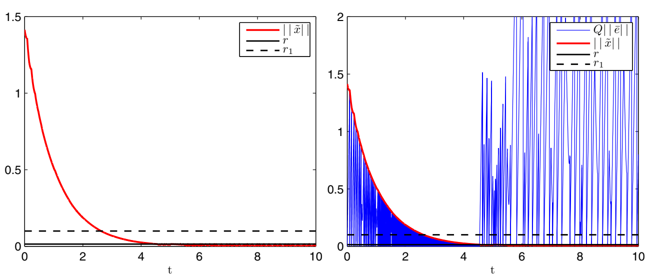


(a) (b)

Fig. 1: Case I

frequency of control updates is around 32Hz. Since most of the updates occur before $\tilde{x}$ first enters the ball of radius r, it is important to also consider the average frequency for this time period, and in this simulation it was found to be around 51Hz.

Case II: In this case the input signal v is continuous but not differentiable ($\dot{v}$ was chosen as a piecewise constant function), and $x_{d,1}$, $x_{d,2}$ were chosen as sinusoidal signals. In this case $K = 9$ was chosen, and other parameters were kept the same as in the earlier simulation. This, however, changed the parameter r to 0.0145. Figures 2a and 2b show the results. The number of control updates were observed to be higher in this case at 1047, with the minimum execution time at 9×10^{-4}s. The observed average frequencies of control updates were found to be around 105Hz and 221Hz for the simulated time duration and the time duration that $\tilde{x}$ takes to first enter the ball of radius r, respectively.



(a) (b)

Fig. 2: Case II

VI. Conclusions

In this paper, we developed an event based control algorithm for trajectory tracking in control affine nonlinear systems. Using two main results, it was demonstrated that given a nonlinear dynamical system, and a continuous-time control that ensures uniform asymptotic tracking of the desired trajectory, an event based controller can be designed that not only guarantees uniform ultimate boundedness of the tracking error, but also ensures that the inter-execution times for the control algorithm are uniformly bounded away from zero. The first result demonstrated that an arbitrary ultimate bound for the tracking error can be designed, provided the reference trajectory, the exogenous input to the reference system, and its derivative are all uniformly bounded. However, the choice of an arbitrary ultimate bound is constrained by the minimum *guaranteed* inter-execution time, which decreases along with the ultimate bound. In the second result, we relaxed the assumption on the second derivative of the input to the reference system, and demonstrated that the tracking error is uniformly ultimately bounded. In this case, the analytical result demonstrated that it may not be feasible to reduce the ultimate bound below a certain threshold.

The theoretical results were demonstrated through simulations of a second order nonlinear system. Numerical simulations indicated that the ultimate bound of the tracking error is much lower than the desired value. Therefore, this is one area for improvement of the theoretical predictions. Another area of future research is finding better estimates of the lower bounds on the inter-execution times.

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