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Lower Bounds on the Performance of Analog to Digital Converters

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Abstract— **This paper deals with the task of finding certified lower bounds for the performance of Analog to Digital Converters (ADCs). A general ADC is modeled as a causal, discrete-time dynamical system with outputs taking values in a finite set. We define the performance of an ADC as the worst-case average intensity of the filtered input matching error. The input matching error is the difference between the input and output of the ADC. This error signal is filtered using a shaping filter, the passband of which determines the frequency region of interest for minimizing the error. The problem of finding a lower bound for the performance of an ADC is formulated as a dynamic game problem in which the input signal to the ADC plays against the output of the ADC. Furthermore, the performance measure must be optimized in the presence of quantized disturbances (output of the ADC) that can exceed the control variable (input of the ADC) in magnitude. We characterize the optimal solution in terms of a Bellman-type inequality. A numerical approach is presented to compute the value function in parallel with the feedback law for generating the worst case input signal. The specific structure of the problem is used to prove certain properties of the value function that allow for iterative computation of a certified solution to the Bellman inequality. The solution provides a certified lower bound on the performance of any ADC with respect to the selected performance criteria.**

I. INTRODUCTION AND MOTIVATION

Analog to Digital Converters (ADCs) act as the interface between the analog world and digital processors. They are present in almost all digital control and communication systems and modern high-speed data conversion and storage systems. Naturally, the design and analysis of ADCs have, for many years, attracted the attention and interest of researchers from various disciplines across academia and industry. Despite the progress that has been made in this field, the design of optimal ADCs remains an open challenging problem, and the fundamental limitations of their performance are not well understood. This paper is concerned with the latter problem.

A particular class of ADCs primarily used in high resolution applications is the Delta-Sigma Modulator (DSM). Fig. 1, illustrates the classical first-order DSM [1], where Q is a quantizer with uniform step size.

An extensive body of research on DSMs has appeared in the signal processing literature. One well known approach is based on linearized additive noise models and filter design for noise shaping [1]-[6]. The underlying assumption for validity of the linearized additive noise model is availability of a relatively high number of bits. Alternative approaches based on a formalism of the signal transformation performed by the quantizer have been exploited for deterministic analysis in [7]-[9]. Some other works that do not use linearized additive noise models are reported in [10]-[12].

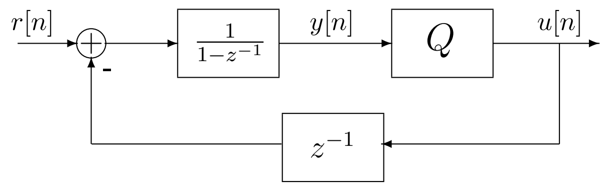

Fig. 1. Classical First-Order Sigma-Delta Modulator

In the control field, [13]-[15] find performance bounds and suboptimal policies for linear stochastic control problems using Bellman inequalities with quadratic value functions. The problem is relaxed and solved using linear matrix inequalities and semidefinite programming. Finally, some work on quantized control are reported in [16]-[18].

In [19] we provided a characterization of the solution to the optimal ADC design problem and presented a generic methodology for numerical computation of sub-optimal solutions along with computation of a certified upper bound on the performance. The performance of an ADC is evaluated with respect to a cost function which is a measure of the intensity of the error signal (the difference between the input signal and its quantized version) for the worst case input. The error signal is passed through a shaping filter which dictates the frequency region in which the error is to be minimized. Furthermore, we showed that the dynamical system within the optimal ADC is a copy of the shaping filter used to define the performance criteria. In [19] we also presented an exact analytical solution to the optimal ADC for first-order shaping filters, and showed that the classical first-order DSM (Figure 1) is identical to our optimal ADC. This result proved the optimality of the classical first-order DSM with respect to the adopted performance measure, and was a step towards understanding the limitations of performance.

In this paper, we present a framework for finding *certified lower bounds* for the performance of ADCs with shaping filters of arbitrary order. We use the same ADC model and performance measure adopted in [19]. The objective is to find a lower bound on the infimum of the cost function. The approach is to find a feedback law for generating the input of the ADC such that regardless of its output, the performance

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is bounded from below by a certain value. Thus, the input of the ADC is viewed as the control, and the problem is posed within a non-linear optimal feedback control/game framework. We show that the optimal control law can be characterized in terms of a *value function* satisfying an analog of the Bellman inequality. The value function in the Bellman inequality and the corresponding control law can be jointly computed via value iteration.

Since searching for the value function involves solving a sequence of infinite dimensional optimization problems, some approximations are needed for numerical computation. First, a finite-dimensional parameterization of the value function is selected. Second, the state space and the input space are discretized. Third, the computations are restricted to a finite subset of the space. The latter step deserves further elaboration. If the dynamical system inside the ADC is strictly stable, then a bounded control invariant set exists, thus it is possible to do the computations over a bounded region. The challenge arises when the filter has poles on the unit circle. In this case, there does not exist a bounded control invariant set, since the disturbances can exceed the control variable in magnitude. Under the condition that there is at most one pole on the unit circle, we present a theorem that states that the value function is zero outside a certain bounded space. Thus, we have an a priori knowledge of an analytic expression for the value function beyond a bounded region. As a result, the computations need to be carried out only over this bounded region. This is in dramatic contrast with the case of upper bound computations [19], something to be discussed in section III.

The organization is as follows. Section II provides a rigorous problem formulation. The main contributions are presented in Section III and IV. Section III describes our methodology for finding certified lower bounds for ADCs. Section IV provides our theoretical results. We provide an example in section V, and section VI concludes the paper.

A. Preliminaries

Notation 1: Function $f : \mathbb{R}^m \mapsto \mathbb{R}$ is called BIBO if condition

$$\sup_{x\in\Omega} |f(x)| < \infty \tag{1}$$

holds for every bounded set $\Omega \subset \mathbb{R}^m$.

Notation 2: Given a set P, $\ell_+(P)$ is the set of all sequences that map $\mathbb{Z}_+$ to P:

$$\ell_+(P) \overset{\text{def}}{=} \{x : \mathbb{Z}_+ \mapsto P\}. \tag{2}$$

II. PROBLEM FORMULATION

The problem setup in this section is taken from [19].

A. Analog to Digital Converters

In this paper, a general ADC is viewed as a causal, discrete-time, non-linear system Ψ, accepting arbitrary inputs in the $[-1, 1]$ range, and producing outputs in a fixed finite subset $U \subset \mathbb{R}$, as shown in Fig. 2. We assume that the smallest element in the set U is less than -1 and the largest element is greater than 1.

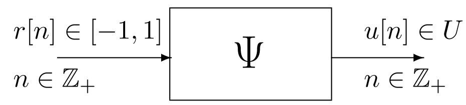

Fig. 2. Analog to Digital Converter as a Dynamical System

Equivalently, an ADC is defined by a sequence of functions $\Upsilon_n : [-1, 1]^{n+1} \mapsto U$ according to

$$\Psi : \; u[n] = \Upsilon_n (r[n], r[n-1], \cdots, r[0]), \quad n \in \mathbb{Z}_+. \tag{3}$$

The class of ADCs defined above is denoted by $\mathcal{Y}_U$.

B. Asymptotic Weighted Average Intensity (AWAI) of a Signal

The Asymptotic Weighted Average Intensity (AWAI) of a signal w is denoted by $\eta_{G,\phi}(w)$, which depends on the transfer function $G(z)$ of a strictly causal LTI dynamical system L_G and a non-negative function $\phi : \mathbb{R} \mapsto \mathbb{R}_+$:

$$\eta_{G,\phi}(w) = \limsup_{N\mapsto\infty} \frac{1}{N+1} \sum_{n=0}^{N} \phi(q[n]), \tag{4}$$

where the sequence q is the response to input w of the dynamical system L_G defined by:

$$x[n+1] = Ax[n] + Bw[n], \quad x[0] = 0, \quad \forall n \in \mathbb{Z}_+ \tag{5}$$

$$q[n] = Cx[n], \tag{6}$$

where A, B, C are given matrices of appropriate dimensions. Examples of functions that we consider for ϕ are: $\phi(\cdot) = |\cdot|$ and $\phi(\cdot) = |\cdot|^2$. The motivation for these selections for ϕ and for using the AWAI as a measure of the quality of analog to digital conversion is presented in [19].

C. ADC Performance Measure

The setup that we use to measure the performance of an ADC is illustrated in Fig. 3. The performance measure of $\Psi \in \mathcal{Y}_U$, denoted by $\mathcal{J}_{G,\phi}(\Psi)$, is the worst-case AWAI of the error signal for all input sequences $r \in \ell_+([-1, 1])$, that is:

$$\mathcal{J}_{G,\phi}(\Psi) = \sup_{r\in\ell_+([-1,1])} \eta_{G,\phi}(r - \Psi(r)). \tag{7}$$

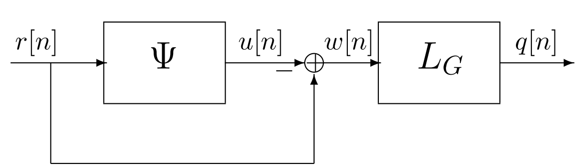

Fig. 3. Setup Used for Measuring the Performance of the ADC

D. ADC Optimization

Given L_G and ϕ, we consider $\Psi_o \in \mathcal{Y}_U$ an optimal ADC if $\mathcal{J}_{G,\phi}(\Psi_o) \leq \mathcal{J}_{G,\phi}(\Psi)$ for all $\Psi \in \mathcal{Y}_U$. The corresponding optimal performance measure $\gamma_{G,\phi}(U)$ is defined as

$$\gamma_{G,\phi}(U) = \inf_{\Psi \in \mathcal{Y}_U} \mathcal{J}_{G,\phi}(\Psi). \tag{8}$$

The objective is to find certified lower bounds for (8).

III. OUR APPROACH

We find the lower bound on the performance of any given ADC belonging to the class $\mathcal{Y}_U$ by associating the problem with a full-information feedback control problem. The objective is to find a feedback law for generating the input of the ADC, r, such that regardless of the output u, the performance is bounded from below by a certain value. Thus, in this setup, r is viewed as the control and u is viewed as the input of a strictly causal system with output r. The setup is depicted in Fig. 4, where the function $K_r : \mathbb{R}^m \mapsto [-1,1]$ is said to be an admissible controller if there exists $\gamma \in [0,\infty)$ such that every triplet of sequences (x,u,r) satisfying

$$x[n+1] = Ax[n] + Br[n] - Bu[n], \quad x[0] = 0, \tag{9}$$

$$r[n] = K_r(x[n]), \tag{10}$$

$$q[n] = Cx[n], \tag{11}$$

also satisfies the dissipation inequality

$$\inf_N \sum_{n=0}^{N} (\phi(q[n]) - \gamma) > -\infty. \tag{12}$$

Note that if (12) holds subject to (9)-(11), then $\gamma_{G,\phi}(U) \geq \gamma$. Let γ_o be the minimal upper bound of γ, for which an admissible controller exists. Then K_r is said to be an optimal controller if (12) is satisfied with $\gamma = \gamma_o$.

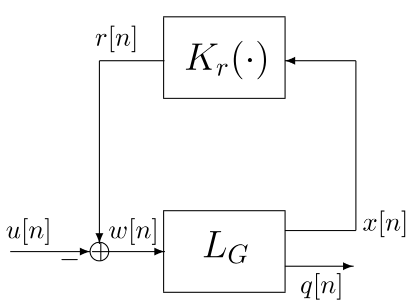

Fig. 4. Full State-Feedback Control Setup

A. The Bellman Inequality

The solution to a well-posed state-feedback optimal control problem can be characterized as the solution to the associated Bellman equation [20]-[23]. Herein, standard techniques are used to show that there exists a controller K_r such that (12) holds, if and only if the solution to an analog of the Bellman equation exists. The formulation will be made more precise as follows. Define function $\sigma_\gamma : \mathbb{R}^m \mapsto \mathbb{R}$ by

$$\sigma_\gamma(x) = \gamma - \phi(Cx). \tag{13}$$

The control sequence r satisfying (10) results in an output sequence q satisfying (12), if and only if there exists a function $V : \mathbb{R}^m \mapsto \mathbb{R}_+$, such that inequality

$$V(x) \geq \sigma_\gamma(x) + \inf_{r \in [-1,1]} \max_{u \in U} V(Ax + Br - Bu) \tag{14}$$

holds for all $x \in \mathbb{R}^m$ (see Theorem 1). We refer to inequality (14) as the Bellman inequality, and to a function V satisfying (14) as the value function.

B. Numerical Solutions to the Bellman Inequality

In this section, we outline our approach for numerical computation of the value function V and the control function K_r. We can simplify the problem of searching for a solution to inequality (14) by instead finding a solution $V \geq 0$ to the inequality

$$V(x) \geq \sigma_\gamma(x) + \min_{r \in \Gamma_r} \max_{u \in U} V(Ax + Br - Bu), \ \forall x \in \mathbb{R}^m \tag{15}$$

where Γ_r is a finite subset of $[-1,1]$. Since for every function $g : [-1,1] \mapsto \mathbb{R}$, we have

$$\inf_{r \in [-1,1]} g(r) \leq \min_{r \in \Gamma_r} g(r), \tag{16}$$

a solution V of (15) is also a solution of (14). In the remainder of this section we focus on finding a solution to (15).

A control invariant set of system (9), with respect to Γ_r, is formally defined as a subset $\mathcal{I} \subset \mathbb{R}^m$ such that:

$$\forall x \in \mathcal{I}, \ \exists r \in \Gamma_r : \ Ax + Br - Bu \in \mathcal{I}, \ \forall u \in U. \tag{17}$$

Furthermore, a strong invariant set of system (9), with respect to Γ_r, is defined as a subset $\mathcal{I} \subset \mathbb{R}^m$ such that:

$$\forall x \in \mathcal{I} : \ Ax + Br - Bu \in \mathcal{I}, , \ \forall r \in \Gamma_r, \ \forall u \in U. \tag{18}$$

Ideally we would like to have a bounded invariant set, so that the search for V satisfying the Bellman inequality is restricted to a bounded region of the state space. If $\max|\mathrm{eig}(A)| < 1$, then a bounded set $\mathcal{I}$ satisfying (18) is guaranteed to exist. However, if $\max|\mathrm{eig}(A)| = 1$, then there does not exist a bounded set $\mathcal{I}$ satisfying (17), due to the assumption that the smallest element in the set U is less than -1 and the largest element is greater than 1. The case when $\max|\mathrm{eig}(A)| = 1$ presents difficulties, since we cannot search for a numerical solution to (15) over an unbounded state space. However, for the case that there is only one pole on the unit circle, we will establish in Theorem 2 that the value function is zero for all x outside a certain bounded region. Hence, the numerical search for V satisfying (15) needs to be carried out only over a bounded subset of the state space. Next, uniform grids are created for the state space. In this paper, these are uniformly-spaced, discrete subsets of the Euclidean space, and are defined precisely as follows. The set

$$\mathbb{G} = \{i\Delta \mid i \in \mathbb{Z}\} \tag{19}$$

is a grid on $\mathbb{R}$, where $D = 1/\Delta$ is a positive integer. The corresponding grid on $\mathcal{I}$ is

$$\Gamma = \mathbb{G}^m \cap \mathcal{I}. \tag{20}$$

Furthermore, we define $\Gamma_r = \{r_1,\ r_2, \cdots, r_L\}$ as

$$\Gamma_r = \mathbb{G} \cap [-1, 1].$$

The next step is to create a finite-dimensional parameterization of V. In this paper, the search is performed over the class of *piecewise constant* (PWC) functions assuming a constant value over a *tile*. A *tile* in $\mathbb{G}^n$, $n \in \mathbb{N}$ is defined as the smallest hypercube formed by 2^n points on the grid, and thus, has $2n$ faces (the faces are hypercubes of dimension $n-1$). By convention, we assume that the n faces that contain the lexicographically smallest vertex are closed, and the rest are open. The union of all such tiles covers $\mathbb{R}^n$ and their intersection is empty. Let T_i denote the i^{th} tile over the grid $\mathbb{G}^m$, and $\mathcal{T}$ the set of all tiles that lie within $\mathcal{I}$, and N_T the number of all such tiles:

$$\mathcal{T} = \{T_i \mid i \in \{1, 2, \cdots, N_T\}\}.$$

The PWC parameterization of V is as follows

$$V(x) = V_i,\ \forall x \in T_i,\ i \in \{1, 2, \cdots, N_T\} \tag{21}$$

where $V_i \in \mathbb{R}_+$. We then search for a solution $V : \mathcal{I} \mapsto \mathbb{R}_+$ of (15) for all $x \in \mathcal{I}$ within the class of PWC functions defined in (21). The corresponding PWC control function $K_r : \mathcal{I} \mapsto \Gamma_r$ is given by

$$K_r(x) = \arg\min_{r \in \Gamma_r} \max_{u \in U, \bar{x} \in T(x)} V(A\bar{x} + Br - Bu),\ \forall x \in \mathcal{I}. \tag{22}$$

where $T(x) = T_i$ for $x \in T_i$. In the next subsection we show how to search and certify functions V and K_r satisfying (15) and (22).

C. Searching for Numerical Solutions

The Bellman inequality (15) is solved via value iteration. The algorithm is initialized at $\Lambda_0(x) = 0$, for all $x \in \mathcal{T}$, and at stage $k+1$ it computes a PWC function $\Lambda_{k+1} : \mathcal{T} \mapsto \mathbb{R}_+$ satisfying

$$\Lambda_{k+1}(x) = \max\left\{0,\ \sigma_\gamma(x) + \min_{r \in \Gamma_r} \max_{u \in U, \bar{x} \in T(x)} \Lambda_k(A\bar{x} + Br - Bu)\right\}. \tag{23}$$

At each stage of the iteration, Λ_{k+1} is computed and certified to satisfy (23) for all $x \in \mathcal{T}$ as follows:

1) For every $i \in \{1, 2, \cdots, N_T\}$ and $j \in \{1, 2, \cdots, L\}$, define

$$\sigma_i = \sup_{x \in T_i} \sigma_\gamma(x),$$

$$Y_{ij} = \{Ax + Br_j - Bu \mid x \in T_i,\ r_j \in \Gamma_r,\ u \in U\},$$

and find all the tiles that intersect with Y_{ij}

$$\Theta_{ij} = \{p \mid T_p \cap Y_{ij} \neq \{\emptyset\},\ p \in \{1, 2, \cdots, N_T\}\}.$$

2) Let

$$v_s = \Lambda_k(x),\ x \in T_s,\ s \in \{1, 2, \cdots, N_T\}.$$

Compute

$$v_{ij} = \max_{s \in \Theta_{ij}} v_s.$$

3) For every tile $x \in T_i$ compute PWC functions:

$$\Lambda_{k+1}(x) = \max\left\{0,\ \sigma_i + \min_j v_{ij}\right\}.$$

When the iteration converges, it converges pointwise to a limit $\Lambda : \mathcal{T} \mapsto \mathbb{R}_+$, where the limit satisfies, for all $x \in \mathcal{T}$, the equality

$$\Lambda(x) = \max\left\{0,\ \sigma_\gamma(x) + \min_{r \in \Gamma_r} \max_{u \in U, \bar{x} \in T(x)} \Lambda(A\bar{x} + Br - Bu)\right\}. \tag{24}$$

The largest γ for which (23) converges is found through line search. We take $V(x) = \Lambda(x)$, for all $x \in \mathcal{T}$. The associated suboptimal control law is a PWC function defined over all tiles T_i in the control invariant set $\mathcal{I}$ that satisfies (22).

IV. Theoretical Statements

In this section, we show that under some technical assumptions, the value function in (14) is zero beyond a bounded region. However, we first present a theorem that establishes the link between the full information feedback control problem and the Bellman inequality (14). Note that in this section we use subscript notation for values of sequences at specific time instances instead of the bracket notion used elsewhere in the paper.

Theorem 1: Let X be a topological space, Ω be a compact metric space, U be a finite set, and $f : X \times \Omega \times U \mapsto X$ and $\sigma : X \mapsto \mathbb{R}$ be continuous functions. Then the following statements are equivalent:

(i)
$$V_\infty(\bar{x}) \overset{\text{def}}{=} \sup_{\tau \in \mathbb{Z}_+} V_\tau(\bar{x}) < \infty, \quad \forall \bar{x} \in X, \tag{25}$$

where $V_\tau : X \mapsto \mathbb{R}_+$ is defined by

$$V_\tau(\bar{x}) = \max_{\theta_0} \inf_{r_0} \max_{u_0, \theta_1} \cdots \inf_{r_{\tau-2}} \max_{u_{\tau-2}, \theta_{\tau-1}} \sum_{n=0}^{\tau-1} h_{n+1}\sigma(x_n), \tag{26}$$

with $r_n,\ u_n,\ \theta_n$ restricted by $r_n \in \Omega,\ u_n \in U,\ \theta_n \in \{0, 1\}$ and $x_n,\ h_n$ defined by

$$x_{n+1} = f(x_n, r_n, u_n), \quad x_0 = \bar{x}, \quad \forall n \in \mathbb{Z}_+ \tag{27}$$
$$h_{n+1} = \theta_n h_n, \quad h_0 = 1, \quad \forall n \in \mathbb{Z}_+. \tag{28}$$

(ii) The sequence of functions $\Lambda_k : X \mapsto \mathbb{R}_+$ defined by

$$\Lambda_0(x) \equiv 0$$

$$\Lambda_{k+1}(x) = \max\left\{0,\ \sigma(x) + \inf_{r \in \Omega} \max_{u \in U} \Lambda_k(f(x, r, u))\right\} \tag{29}$$

converges pointwise to a limit $\Lambda_\infty : X \mapsto \mathbb{R}_+$.

(iii) There exists a function $V : X \mapsto \mathbb{R}_+$ such that

$$V(x) = \max\left\{0,\ \sigma(x) + \inf_{r\in\Omega}\max_{u\in U} V(f(x,r,u))\right\} \tag{30}$$

for every $x \in X$.

(iv) There exists a function $V : X \mapsto \mathbb{R}_+$ such that

$$V(x) \geq \sigma(x) + \inf_{r\in\Omega}\max_{u\in U} V(f(x,r,u)), \quad \forall x \in X. \tag{31}$$

Moreover, if (i)$-$(iv) hold, then V_∞ is a solution of (30) and

$$V_\infty = \Lambda_\infty \geq V_k = \Lambda_k, \quad \forall k \in \mathbb{Z}_+ \tag{32}$$

$$V \geq V_\infty. \tag{33}$$

Furthermore, if the sequence of functions $R_n : X \mapsto \Omega$ are such that

$$\max_{u\in U} V_\infty(f(x, R_n(x), u)) \leq 2^{-n}\epsilon + \inf_{r\in\Omega}\max_{u\in U} V_\infty(f(x,r,u)) \tag{34}$$

then,

$$\sup_{\tau}\max_{\{u_k\}_{k=0}^{\tau-2}} \sum_{n=0}^{\tau-1}\sigma(x_n) \leq \epsilon + V_\infty(x_0) \tag{35}$$

subject to $x_{n+1} = f(x_n, R_n(x_n), u_n)$.

Proof: Omitted due to space constraints, please see the full paper on arXiv. ■

Definition 1: Let ν be a non-zero vector in $\mathbb{R}^m$. A *cylinder* of radius β and axis ν is defined as:

$$\mathcal{C}_\beta(\nu) = \left\{p \in \mathbb{R}^m : \inf_{t\in\mathbb{R}} |p - t\nu| \leq \beta\right\}. \tag{36}$$

The following theorem establishes that the value function is zero for all x outside a certain bounded region.

Theorem 2: Let $U \subset \mathbb{R}$ be a fixed finite set. Consider the system defined by equation (9), where $x \in \ell_+(\mathbb{R}^m)$, $r \in \ell_+([-1,1])$, $u \in \ell_+(U)$, and the pair (A, B) is controllable. Suppose that A has at most one eigenvalue on the unit circle. Let e_1 denote the eigenvector corresponding to the eigenvalue of A that is on the unit circle. Let V be defined by (25) and σ be BIBO. If the set

$$S_0 = \{x \in \mathcal{C}_\beta(e_1) : \sigma(x) > -1\}, \quad \forall\beta \in \mathbb{R}_+ \tag{37}$$

is bounded, then there exists $\tilde{\beta} \in \mathbb{R}_+$ such that the set

$$M = \{x \in \mathcal{C}_{\tilde{\beta}}(e_1) : V(x) \neq 0\} \tag{38}$$

is bounded.

Proof: Omitted due to space constraints, please see the full paper on arXiv. ■

V. NUMERICAL EXAMPLE

Consider the example in [19], were the dynamical system L_G (5)$-$(6) has transfer function

$$H(z) = \frac{z+1}{z(z-1)}.$$

Let $U = \{-1.5, 0, 1.5\}$, $\phi(x) = |Cx|$, and $x = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T$. From [19], the strong invariant set $\mathcal{I}$ is given by

$$\mathcal{I} = \{x \in \mathbb{R}^2 \ : \ |x_1 - x_2| \leq 2.5\}. \tag{39}$$

Due to the pole at $z = 1$, the strong invariant set $\mathcal{I}$ given by (39) is unbounded and defines an infinite strip in $\mathbb{R}^2$. However, according to Theorem 2 we need to search for $V(x)$ only inside a bounded region within this infinite strip, since $V(x) = 0$ for all x outside a certain bounded region. The bounded region is found via trial and error. We select a grid spacing of $\Delta = 1/64$. Following the procedures outlined in subsections III-B and III-C, the largest γ for which the iteration in (23) converges to the limit Λ in (24), is $\gamma = 0.925$, which is a certified lower bound on the performance of any arbitrary ADC with respect to the specific performance measure selected. Figures 5, 6, and 7 show the value function V, the cross section of V, and the zero level set of V, respectively. Figures 8 and 9 show the control function and its cross section, respectively. The certified upper bound for the performance of the ADC designed in [19] with respect to the same performance criteria is 1.1875.

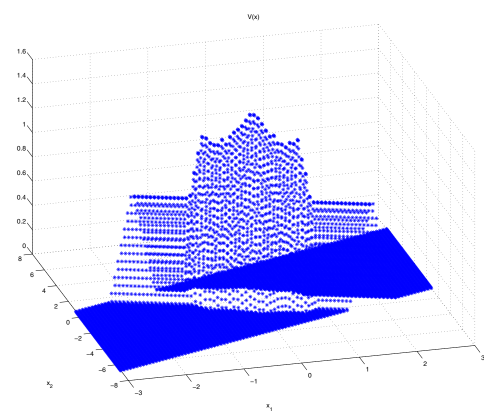

Fig. 5. Value Function $V(x)$ for Lower Bound

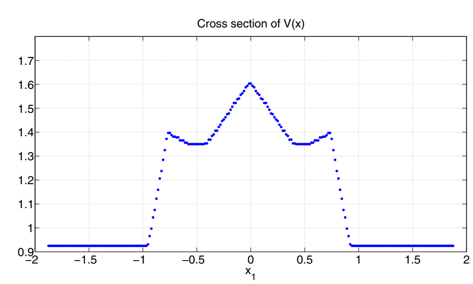

Fig. 6. Cross Section of $V(x)$ along $x_1 = -x_2$

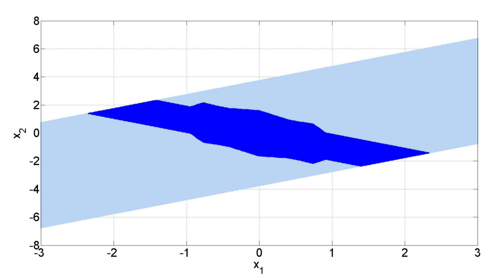

Fig. 7. Zero Level Set of $V(x)$

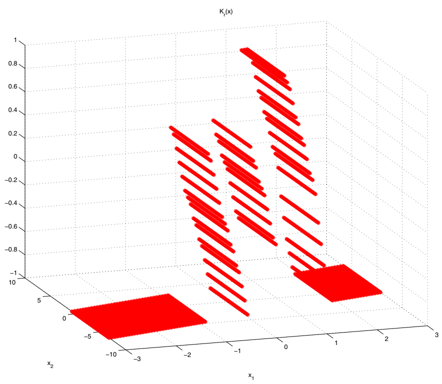

Fig. 8. Control $K_r(x)$ for Lower Bound

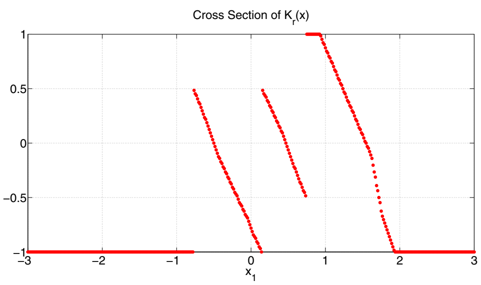

Fig. 9. Cross Section of Control $K_r(x)$ along $x_1 = x_2$

VI. CONCLUSION

In this paper, we studied performance limitations of Analog to Digital Converters (ADCs). The performance of an ADC was defined in terms of a measure that represents the worst case average intensity of the filtered input matching error. The passband of the shaping filter defines the frequency region in which the error is to be minimized. The problem of finding a lower bound for the performance of an ADC was associated with a full information feedback optimal control problem and formulated as a dynamic game in which the input of the ADC (control variable) played against the output of the ADC (quantized disturbance). Since the disturbances can exceed the control variable in magnitude, if the shaping filter has a pole on the unit circle, then there does not exist a bounded control invariant set, which presents a challenge for numerical computations. This challenge is overcome with theoretical results that show that the value function is zero beyond a bounded region, thus computations need to be done only over this bounded region. A numerical algorithm was presented that provided certified solutions to the underlying Bellman inequality in parallel with the control law; hence, certified lower bounds on the performance of arbitrary ADCs with respect to the adopted performance criteria.

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