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Constrained interpolation-based control for polytopic uncertain systems

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Abstract— **In this paper, we consider the regulation problem for uncertain and time-varying linear discrete-time systems with bounded input and bounded state.**

By using an interpolation technique it is shown that, the convex hull of a set of invariant ellipsoids is also invariant. Feasibility and robustly and asymptotically stable closed-loop behavior are assured by minimizing an appropriate objective function. Moreover we show that the control value can be computed by solving nonlinear equations.

I. Introduction

In this paper, we consider the regulaton problem for discrete-time linear systems with state and input constraints, subject to parametric uncertainty. Control problems for such systems have attracted tremendous attention in recent years because of their practical significance and theoretical challenges, see [1], [2], [3] and the references therein.

For estimating the domain of attraction, two typical types of invariant sets are the invariant ellipsoids [4] and invariant polytopes [5], [6], corresponding to quadratic Lyapunov functions and polyhedral Lyapunov functions, respectively.

The analysis methods resulting from quadratic functions are widely used due to computational efficiency via linear matrix inequalities (LMI) and the complexity is fixed. However, quadratic functions have a rather restricted shape, which may be very conservative in typical problems. The methods based on invariant polytopes may yield non-conservative results, if the number of vertices or half-planes is allowed to be arbitrarily large. However, for high dimensional systems, the number of vertices or half-planes may increase without bound.

In the control community, the problem of estimating the domain of attraction of linear systems under saturated state feedback have been addressed by many researches by means of the Lyapunov theory and the LMI framework. For example, the circle and Popov criteria was used in [7]. In [8] polyhedral Lyapunov functions were considered and a piecewise quadratic Lyapunov function technique was proposed in [9]. In [10] a novel polytopic model of the saturation nonlinearity was employed. Based on this development several interesting results have been reported for estimating the region of attraction [11], [12], [13].

In this paper, the convex hull of a family of quadratic functions is used for estimating the stability domain for a constrained control system. This is motivated by problems arising from the estimation of the domain of attraction of stable dynamics and the control design which aims to enlarge such a domain of attraction. In order to briefly describe the class of problems suppose that we have a set of invariant ellipsoids and an associated set of feedback control laws. The question whether the convex hull of this set of ellipsoids is invariant and how to construct a control law for this region is one of our objectives.

This problem was already introduced in [14]. For the continuous linear time invariant systems, it was shown that a convex hull of invariant ellipsoids is also invariant. A continuous feedback law was constructed based on the gradient of the function or on a given set of linear feedback laws.

Here we extend the above result to discrete-time linear time-varying systems. We show how a control value can be computed by using an interpolation based technique, thereby making invariant the *convex hull* of invariant ellipsoids.. At each time instant, an LMI problem is solved. In the limiting case of two feedback gains, the use of an LMI solver can be avoided by solving a nonlinear equation.

Notation: Throughout the paper, the superscript T stands for matrix transposition. A positive definite (negative definite) square matrix A is denoted by $A \succ 0$ ($A \prec 0$). The non-degenerate $E(P)$ ellipsoid in $\mathbb{R}^{n_x}$ with the center at the origin is defined as follows:

$$E(P) = \{x \in \mathbb{R}^{n_x} : x^T P^{-1} x \leq 1\},\ P \succ 0$$

For a matrix $F \in \mathbb{R}^{n \times n_x}$, denote the ith row of F as f_i and define the symmetric polyhedral set L as follows:

$$L(F) = \{x \in \mathbb{R}^{n_x} : |f_i x| \leq 1\ \ \forall i = 1, \dots, n\}$$

A function $sat(u)$ represents the actuator saturation defined as:

$$sat(u) = \begin{cases} -u_{max}, & \text{if } u \leq -u_{max} \\ u, & \text{if } -u_{max} \leq u \leq u_{max} \\ u_{max} & \text{if } u \geq u_{max} \end{cases}$$

with u_{max} being the saturation level.

The paper is organized as follows. Section 2 is concerned with the problem formulation. Section 3 is dedicated to an invariant set construction. Section 4 is devoted to an interpolation technique while simulation results are evaluated in Section 5 before drawing the conclusions.

II. Problem formulation

Consider the problem of regulating to the origin the following discrete-time linear time-varying system:

$$x(t+1) = A(t)x(t) + B(t)sat(u(t)) \tag{1}$$

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where $x(t) \in \mathbb{R}^{n_x}$ and $u(t) \in \mathbb{R}^{m_u}$ are respectively the measurable state and the input, and with given matrices A_i and B_i, the matrices $A(t) \in \mathbb{R}^{n_x \times n_x}$ and $B(t) \in \mathbb{R}^{n_x \times m_u}$ satisfy:

$$\begin{cases} A(t) = \sum_{i=1}^{s} \alpha_i(t) A_i, B(t) = \sum_{i=1}^{s} \alpha_i(t) B_i, \\ \alpha_i(t) \geq 0, \forall i = 1, \dots, s, \\ \sum_{i=1}^{s} \alpha_i(t) = 1. \end{cases} \tag{2}$$

Both the state vector $x(t)$ and the control vector $u(t)$ are subject to constraints:

$$\begin{cases} x(t) \in X = L(F) \\ u(t) \in U, U = \{u : |u| \leq u_{max}\} \end{cases} \forall t \geq 0 \tag{3}$$

where the matrix F and the vector u_{max} are assumed to be constant with $u_{max} \geq 0$ such that the origin is contained in the interior of X and U. Here the inequalities are elementwise.

In this paper, we assume that the states of the system are measurable.

III. Invariant sets

The aim of this section is twofold. Firstly we give a method for estimating the domain of attraction under the given feedback gain $u = sat(Kx)$ and then an approach for computing the feedback gain K is proposed.

Recall the following definitions:

Definition 1: An ellipsoid $E(P)$ is robustly positively invariant (RPI) [15], [5] with respect to system (1) if and only if for all $x(t) \in E(P)$, one has:

$$x^T(t+1) P^{-1} x(t+1) - x(t)^T P^{-1} x(t) \leq 0 \tag{4}$$

where $x(t+1) = A(t)x(t) + B(t)sat(Kx(t))$.

Clearly, the ellipsoid $E(P)$ may not be contractive, in the sense that the system trajectories may not be converge to the origin. In order to ensure that $x(t) \to 0$, we require that the increment of the Lyapunov function is strictly negative, namely:

$$x^T(t+1) P^{-1} x(t+1) - x(t)^T P^{-1} x(t) < 0$$

In this case $E(P)$ is called a contractively positively invariant ellipsoid. It is obvious that $E(P)$ is inside the domain of attraction of the origin with respect to the closed loop dynamics.

Definition 2: An invariant ellipsoid $E(P)$ is feasible with respect to constraints (3) if and only if $E(P) \subset X$.

Note that the feasibility of the ellipsoid $E(P)$ is easily checked by solving an LMI problem. It is well known [4], [3] that for a given row vector $f_0 \in R^{1 \times n_x}$ the ellipsoid $E(P)$ is a subset of $L(f_0)$ if and only if $f_0 P f_0^T \leq 1$, or by using the Schur complements, this condition can be rewritten as:

$$\begin{pmatrix} 1 & f_0 P \\ P f_0^T & P \end{pmatrix} \succeq 0$$

Denote Ω as the set of $m_u \times m_u$ diagonal matrices whose diagonal elements are either 1 or 0. For example, if $m_u = 2$, then,

$$\Omega = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

There are 2^{m_u} elements in Ω. Denote each element of Ω as E_i, $i = 1, 2, \dots, 2^{m_u}$ and define $E_i^- = I - E_i$ where I is the identity matrix. Clearly if $E_i \in \Omega$ then $E_i^- \in \Omega$. Given two matrices $K, H \in \mathbb{R}^{m_u \times n_x}$

$$E_i K + E_i^- H, \forall i = 1, 2, \dots, 2^{m_u}$$

is the set of matrices formed by choosing some rows from K and the rest from H.

Theorem 1: [10] Given an ellipsoid $E(P)$ and a feedback gain K, if there exists a matrix $H \in \mathbb{R}^{m_u \times n_x}$ such that for all $i = 1, \dots, s$ and all $j = 1, \dots, 2^{m_u}$

$$(A_i + B_i(E_j K + E_j^- H))^T P^{-1} (A_i + B_i(E_j K + E_j^- H)) - P^{-1} \prec 0 \tag{5}$$

and $\forall x \in E(P) : |Hx| \leq u_{max}$ and $E(P) \subset X$ then $E(P)$ is a contractive feasible invariant set with an associated feedback gain K.

Using the Schur complements, condition (5) can be rewritten as:

$$\begin{pmatrix} P^{-1} & (A_i + B_i(E_j K + E_j^- H))^T \\ (A_i + B_i(E_j K + E_j^- H)) & P \end{pmatrix} \succ 0$$

for all $i = 1, \dots, s$ and all $j = 1, \dots, 2^{m_u}$ or by using the Schur complements again, one obtains:

$$\begin{pmatrix} P & (A_i + B_i(E_j K + E_j^- H))P \\ P(A_i + B_i(E_j K + E_j^- H))^T & P \end{pmatrix} \succ 0 \tag{6}$$

for all $i = 1, \dots, s$ and all $j = 1, \dots, 2^{m_u}$.

By denoting $G = HP$, it is obvious that the above condition is an LMI problem, for which nowadays, there exist several effective solvers, see for example [16], [17].

For one particular case, when $E_j = 0$ and $E_j^- = I$, one has:

$$\begin{pmatrix} P & (A_i + B_i H)P \\ P(A_i + B_i H)^T & P \end{pmatrix} \succ 0 \tag{7}$$

for all $i = 1, \dots, s$. It is clear that system (1) is asymptotically stable under the feedback gain $u = Hx$ and the ellipsoid $E(P)$, resulting from problem (7) is contractively feasibly invariant, meaning that for any $x(t) \in E(P)$, one has $x(t+1) \in E(P)$ and $|Hx(t)| \leq u_{max}$.

Unlike procedures described in [10], here we propose another scheme for computing a feedback gain K and an invariant ellipsoid $E(P)$. In a first stage, a feedback gain H together with an invariant ellipsoid $E(P)$ are computed which aims to maximize some convex objective function $J(P)$, for example $traceP$. This can be done by using the following LMI optimization problem:

$$J = \max_{P,G} trace(P)$$
$$\text{s.t.} \begin{cases} \begin{pmatrix} P & A_iP + B_iG \\ PA_i^T + G^TB_i^T & P \end{pmatrix} \succ 0, \forall i = 1, \ldots, s \\ \begin{pmatrix} u_{max}^2 & G \\ G' & P \end{pmatrix} \succeq 0, \\ \begin{pmatrix} 1 & f_iP \\ Pf_i^T & P \end{pmatrix} \succeq 0, \forall i = 1, \ldots, n \end{cases} \tag{8}$$

Remark 1: We can develop LMI conditions for choosing the largest invariant ellipsoid $E(P)$ with respect to some reference point x_0, that means the set $E(P)$ is the one that includes θx_0, where θ is a scaling factor. In fact $\theta x_0 \in E(P)$ implies $\theta^2 x_0^T P^{-1} x_0 \leq 1$ or by using the Schur complements:

$$\begin{pmatrix} \frac{1}{\theta^2} & x_0^T \\ x_0 & P \end{pmatrix} \succeq 0$$

By choosing different x_0, say x_0^i, for $i = 1, 2, \ldots, q$ one can obtain q optimized invariant ellipsoids.

By solving the optimization problem (8) one gets the gain $H = GP^{-1}$ and the invariant ellipsoid $E(P)$. In the second stage, based on the gain H and the ellipsoid $E(P)$, a feedback gain K which aims to maximize some contraction factor $1 - g$, is computed. Following the proof of Theorem 1 which can be found in [10], it is clear that the ellipsoid $E(P)$ for the following system:

$$x(t+1) = A(t)x(t) + B(t)sat(Kx(t))$$

is contractive invariant with the contraction factor $1 - g$ if

$$(A_i + B_i(E_jK + E_j^-H))^T P^{-1}(A_i + B_i(E_jK + E_j^-H)) - P^{-1} \prec -gP^{-1} \tag{9}$$

for all $i = 1, \ldots, s$ and all $E_j \in \Omega$ such that $E_j \neq 0$.

This problem can be converted into an LMI condition as:

$$J = \max_{g,K} g$$
$$\text{s.t.} \begin{pmatrix} (1-g)P^{-1} & (A_i + B_i(E_jK + E_j^-H))^T \\ (A_i + B_i(E_jK + E_j^-H)) & P \end{pmatrix} \succ 0 \tag{10}$$

for all $i = 1, \ldots, s$ and all $E_j \in \Omega$ such that $E_j \neq 0$. Recall here the only unknown parameters are K and g; the matrices P and H being given.

Remark 2: The proposed two-stage control design presented here benefits of global uniqueness properties of the solution. This is due to the one-way dependence of the two (prioritized) objectives: the *trace* maximization precedes the associated contraction factor.

From this point on, it is assumed that, using the results in this section, one obtains q ellipsoids $E(P_i), \forall i = 1, \ldots, q$ with q feedback gains $K_i, \forall i = 1, \ldots, p$. Denote Ξ as the convex hull of ellipsoids $E(P_i), \forall i = 1, 2, \ldots, q$. It is clear that $\Xi \subseteq X$ as a consequence of the fact that $E(P_i) \subset X$. In the remaining of the paper we will be interested in the selection of the control action such that the set Ξ to be controlled invariant.

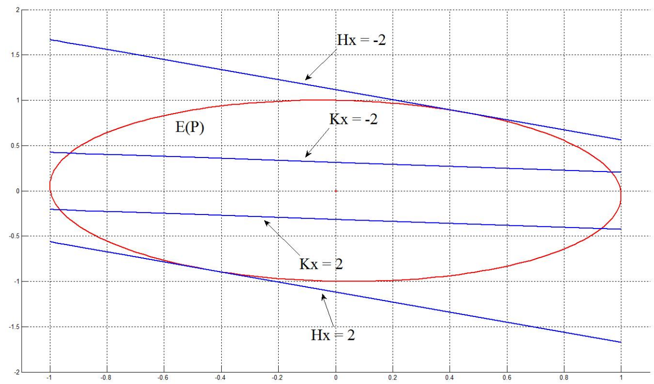

Fig. 1. Contractive invariant set and different feedback gains for example 1

IV. Interpolation based control for polytopic uncertain systems

A. Interpolation based control - Algorithm 1

Any state $x(t)$ in Ξ can be decomposed as follows:

$$x(t) = \sum_{i=1}^{q} \lambda_i x_i \tag{11}$$

where $x_i \in E(P_i), \forall i = 1, 2, \ldots, q$.

λ_i are interpolating coefficients, that satisfy:

$$\begin{cases} 0 \leq \lambda_i \leq 1, \forall i = 1, \ldots, q \\ \sum_{i=1}^{q} \lambda_i = 1 \end{cases}$$

Consider the following control law:

$$u(t) = \sum_{i=1}^{q} \lambda_i u_i \tag{12}$$

where u_i are the feasible control laws $u_i = sat(K_i x_i)$ corresponding to the ellipsoidal region $E(P_i)$.

Theorem 2: The above linear control (12) is feasible for all $x \in \Xi$.

Proof: Starting with the decomposition (11), the control law obtained by the corresponding convex combination of the control actions is leading to the expression in (12).

One has to prove that $|u(t)| \leq u_{max}$ and $x(t+1) = A(t)x(t) + B(t)u(t) \in \Xi$ for all $x \in \Xi$. It follows:

$$\begin{aligned} |u(t)| &= |\textstyle\sum_{i=1}^{q} \lambda_i u_i| \leq \sum_{i=1}^{q} |\lambda_i u_i| \\ &\leq \textstyle\sum_{i=1}^{q} \lambda_i |sat(K_i x_i)| \leq u_{max} \sum_{i=1}^{q} \lambda_i \leq u_{max} \end{aligned}$$

and

$$\begin{aligned} x(t+1) &= A(t)x(t) + B(t)u(t) \\ &= A(t) \textstyle\sum_{i=1}^{q} \lambda_i x_i + B(t) \sum_{i=1}^{q} \lambda_i u_i \\ &= \textstyle\sum_{i=1}^{q} \lambda_i (A(t)x_i + B(t)u_i) \\ &= \textstyle\sum_{i=1}^{q} \lambda_i (A(t)x_i + B(t)sat(K_i x_i)) \end{aligned}$$

We have $A(t)x_i + B(t)sat(K_i x_i) \in E(P_i) \subseteq \Xi, \forall i = 1, 2, \ldots, q$ which ultimately assures that $x(t+1) \in \Xi$. $\square$

For a given state $x(t)$, consider the following objective function:

$$J = \min_{\lambda_i, x_i} \sum_{i=1}^{q-1} \lambda_i \text{ s.t. } \begin{cases} x_i^T P_i^{-1} x_i \leq 1, \forall i = 1, \dots, q \\ \sum_{i=1}^{q} \lambda_i x_i = x \\ 0 \leq \lambda_i \leq 1 \\ \sum_{i=1}^{q} \lambda_i = 1 \end{cases} \tag{13}$$

Theorem 3: The control law using interpolation based on the objective function (13) guarantees robustly asymptotic stability for all initial state $x(0) \in \Xi$.

Proof: Let λ_i^o be the solutions of the optimization problem (13) and consider a positive function

$$V(x) = \sum_{i=1}^{q-1} \lambda_i^o(t), \forall x \in \Xi \setminus E(P_q)$$

$V(x)$ is a Lyapunov function candidate.

For any $x(t) \in \Xi$, one has $x(t) = \sum_{i=1}^{q} \lambda_i^o(t) x_i^o(t)$ and $u(t) = \sum_{i=1}^{q} \lambda_i^o(t) u_i(t)$. It follows that:

$$\begin{array}{l} x(t+1) = A(t)x(t) + B(t)u(t) \\ = A(t) \sum_{i=1}^{q} \lambda_i^o(t) x_i^o(t) + B(t) \sum_{i=1}^{q} \lambda_i^o(t) u_i^o(t) \\ = \sum_{i=1}^{q} \lambda_i^o(t)(A(t)x_i^o(t) + B(t)u_i^o(t)) \\ = \sum_{i=1}^{q} \lambda_i^o(t) x_i(t+1) \end{array}$$

where $x_i(t+1) = A(t)x_i^o(t) + B(t)u_i^o(t) \in E(P_i)$, $\forall i = 1, 2, \dots, q$.

By using the interpolation based on the optimization problem (13):

$$x(t) = \sum_{i=1}^{q} \lambda_i^o(t+1) x_i^o(t+1)$$

where $x_i^o(t+1) \in E(P_i)$. It follows that

$$\sum_{i=1}^{q-1} \lambda_i^o(t+1) \leq \sum_{i=1}^{q-1} \lambda_i^o(t)$$

and $V(x)$ is a non-increasing function.

The contractive positively invariant property of the ellipsoids $E(P_i)$ assures that there is no initial condition $x(0) \in \Xi \setminus E(P_q)$ such that $\sum_{i=1}^{q-1} \lambda_i^o(t) = \sum_{i=1}^{q-1} \lambda_i^o(0), \forall t \geq 0$. It follows that $V(x) = \sum_{i=1}^{q-1} \lambda_i^o(t)$ is a Lyapunov function for $x \in \Xi \setminus E(P_q)$.

The proof is complete by noting that inside $E(P_q)$ the feasible stabilizing controller $u = sat(K_q x)$ is contractive and thus the interpolation based controller assures asymptotic stability for all $x \in P_N$. □

Denote $r_i = \lambda_i x_i$. It is clear that $r_i^T P_i^{-1} r_i \leq \lambda_i^2$. The non-convex optimization problem (13) can be rewritten as follows:

$$J = \min_{\lambda_i, r_i} \sum_{i=1}^{q-1} \lambda_i \text{ s.t. } \begin{cases} r_i^T P_i^{-1} r_i \leq \lambda_i^2, \forall i = 1, \dots, q \\ \sum_{i=1}^{q} r_i = x \\ 0 \leq \lambda_i \leq 1 \\ \sum_{i=1}^{q} \lambda_i = 1 \end{cases}$$

or by using the Schur complements:

$$J = \min_{\lambda_i, r_i} \sum_{i=1}^{q-1} \lambda_i \text{ s.t. } \begin{cases} \begin{pmatrix} \lambda_i & r_i^T \\ r_i & \lambda_i P_i \end{pmatrix} \succeq 0, \forall i = 1, \dots, q \\ \sum_{i=1}^{q} r_i = x \\ 0 \leq \lambda_i \leq 1 \\ \sum_{i=1}^{q} \lambda_i = 1, \end{cases} \tag{14}$$

This is an LMI optimization problem.

In summary, at each time instant the interpolation based controller involves the following steps:

Algorithm 1:

1) For any state $x(t) \in \Xi$, solve the LMI problem (14). In the result, one gets $x_i^o(t) \in E(P_i)$ and λ_i^o, $\forall i = 1, \dots, q$.
2) For $x_i^o(t) \in E(P_i)$, one associates the control value $u_i^o = sat(K_i x_i^o)$.
3) The control value $u(t)$ is found as a convex combination of $u_i^o, \forall i = 1, \dots, q$: $u(t) = \sum_{i=1}^{q} \lambda_i^o u_i^o$.

B. Interpolation based control - Algorithm 2

For the case, when $q = 2$, the following properties can be exploited at the construction stage:

1) For $x \in E(P_2)$ the result of the optimal interpolation problem has a trivial solution $x_2^o = x$ and thus $\lambda_1 = 0$ and $\lambda_2 = 1$ in (14).
2) Let $x \in \Xi \setminus E(P_2)$ with a particular convex combination $x = \lambda_1 x_1 + \lambda_2 x_2$, where $x_1 \in E(P_1)$ and $x_2 \in E(P_2)$. If x_2 is strictly inside $E(P_2)$, one can set $x_2^o = \text{Fr}(E(P_2)) \cap \overline{x, x_2}$ (the intersection between the frontier of $E(P_2)$ and the line connecting x and x_2). Using convexity arguments $x = \lambda_1^o x_1^o + \lambda_2^o x_2^o$ with $\lambda_2^o \geq \lambda_2$ or $\lambda_1^o \leq \lambda_1$. In general terms, the optimal interpolation process leads to a solution (x_1^o, x_2^o) with $x_2^o \in \text{Fr}(E(P_2))$.
3) On the other hand, if x_1 is strictly inside $E(P_1)$, then by setting $x_1^o = \text{Fr}(E(P_1)) \cap \overline{x, x_1}$(the intersection between the frontier of $E(P_1)$ and the line connecting x and x_1) one can obtain $x = \lambda_1^o x_1^o + \lambda_2^o x_2^o$ with $\lambda_2^o \geq \lambda_2$ and $\lambda_1^o \leq \lambda_1$ leading to the conclusion that for the optimal solution $(x_1, x_2)^o$ we have $x_1^o \in \text{Fr}(E(P_1))$.

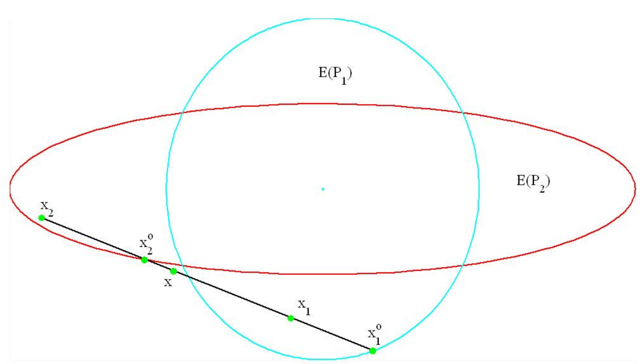

Fig. 2. Graphical illustration of the construction related to the discussion in point B1-B3

From the previous remark we conclude that, for all x such that $x \notin E(P_2)$, the interpolating coefficient λ_1 will reach a minimum in (14) if x is written as a convex combination of two points, one belonging to the frontier of $E(P_1)$ and the other on the frontier of $E(P_2)$. That means that the optimal solution satisfy $x_1^T P_1^{-1} x_1 = 1$ and $x_2^T P_2^{-1} x_2 = 1$. The problem (14) can be rewritten as:

$$J = \min_{\lambda_i, x_i} \lambda_1 \text{ s.t. } \begin{cases} x_1^T P_1^{-1} x_1 = 1,\, i = 1,2 \\ \lambda_1 x_1 + \lambda_2 x_2 = x \\ 0 \le \lambda_i \le 1,\, i = 1,2 \\ \lambda_1 + \lambda_2 = 1 \end{cases} \tag{15}$$

or by denoting $r_i = \lambda_i x_i$, $i = 1,2$, one has

$$J = \min_{\lambda_i, r_i} \lambda_1 \text{ s.t. } \begin{cases} \sqrt{r_i^T P_i^{-1} r_i} = \lambda_i,\, i = 1,2 \\ r_1 + r_2 = x \\ 0 \le \lambda_i \le 1,\, i = 1,2 \\ \lambda_1 + \lambda_2 = 1 \end{cases} \tag{16}$$

The problem (16) can be transformed into the following problem

$$J = \min \sqrt{r_1^T P_1^{-1} r_1} \text{ s.t. } \begin{cases} r_1 + r_2 = x \\ \sqrt{r_1^T P_1^{-1} r_1} + \sqrt{r_2^T P_2^{-1} r_2} = 1 \end{cases} \tag{17}$$

Define the Lagrangian as:

$$\begin{aligned} L(r_i, c_1, c_2) &= \sqrt{r_1^T P_1^{-1} r_1} + c_1^T (r_1 + r_2 - x) + \\ &+ c_2 \left(\sqrt{r_1^T P_1^{-1} r_1} + \sqrt{r_2^T P_2^{-1} r_2} - 1\right) \end{aligned}$$

The following is the set of conditions for a stationary point, which must be satisfied by an optimal solution r_i^o, c_1^o, c_2^o

$$\begin{aligned} (1 + c_2^o) \frac{P_1^{-1} r_1^o}{\sqrt{(r_1^o)^T P_1^{-1} r_1^o}} + c_1^o &= 0 \\ c_2^o \frac{P_2^{-1} r_2^o}{\sqrt{(r_2^o)^T P_2^{-1} r_2^o}} + c_1^o &= 0, \\ r_1^o + r_2^o &= x \\ \sqrt{(r_1^o)^T P_1^{-1} r_1^o} + \sqrt{(r_2^o)^T P_2^{-1} (r_2^o)} &= 1 \end{aligned}$$

or

$$\begin{gathered} (1 + c_2^o) \frac{P_1^{-1} r_1^o}{\sqrt{(r_1^o)^T P_1^{-1} r_1^o}} = c_2^o \frac{P_2^{-1} r_2^o}{\sqrt{(r_2^o)^T P_2^{-1} r_2^o}}, \\ \sqrt{(r_1^o)^T P_1^{-1} r_1^o} + \sqrt{(r_2^o)^T P_2^{-1} (r_2^o)} = 1 \\ r_1^o + r_2^o = x \end{gathered} \tag{18}$$

This is a system of nonlinear equations, for which there exist several numerical methods as for example Newton's method, fixed point iteration method and others [18].

In summary the interpolation based controller involves the following steps:

Algorithm 2:

- If $x \in E(P_2)$, set $u = sat(K_2 x)$.
- If $x \in \Xi \setminus E(P_2)$, then solve equation (18). The control value u is a convex combination of u_1^o, u_2^o, $u = \lambda_1^o u_1^o + \lambda_2^o u_2^o$, where $u_i^o = sat(K_i x_i^o), i = 1,2$.

V. Example

To show the effectiveness of the proposed approach, one example will be presented in this section. For this example, to find feedback gains we used `CVX`, a package for specifying and solving convex programs, [17].

A. Example 1

This example is taken from [19]. Consider the following uncertain discrete time system:

$$x(t+1) = A(t)x(t) + Bu(t) \tag{19}$$

where

$$A(t) = \alpha(t) A_1 + (1 - \alpha(t)) A_2$$

$$A_1 = \begin{pmatrix} 1 & 0.1 \\ 0 & 0.99 \end{pmatrix}, A_2 = \begin{pmatrix} 1 & 0.1 \\ 0 & 0 \end{pmatrix},$$

and $B(t) = (0 \;\; 0.0787)^T$.

At each sampling time $\alpha(t) \in [0,1]$ is an uniformly distributed pseudo-random number.

The constraints are $-1 \le x_1 \le 1$, $-1 \le x_2 \le 1$ and $-2 \le u \le 2$.

By using the results in Section 3, we have designed three feedback gains

$$\begin{aligned} K_1 &= (-0.6927 \;\; -6.3537), \\ K_2 &= (-5.1090 \;\; -6.8028), \\ K_3 &= (-9.2357 \;\; -7.2155) \end{aligned}$$

along with three ellipsoids

$$\begin{aligned} P_1 &= \begin{pmatrix} 1.0000 & -0.0545 \\ -0.0545 & 1.0000 \end{pmatrix} \\ P_2 &= \begin{pmatrix} 1.0000 & -0.4020 \\ -0.4020 & 1.0000 \end{pmatrix} \\ P_3 &= \begin{pmatrix} 0.0623 & -0.0453 \\ -0.0453 & 0.3409 \end{pmatrix} \end{aligned}$$

Figure 1 shows the invariant ellipsoid $E(P_1)$. This set is obtained by using (8) with the gain H_1. Then the matrices P_1 and H_1 are used in (10) to obtain the feedback gain K_1.

Figure 3 shows the convex hull of three ellipsoids Ξ and different trajectories of the closed loop system, depending on the realization of $\alpha(t)$ and the initial condition.

For the initial condition $x_0 = (-0.25 \;\; 1)^T$ Figure 4 shows the state trajectories, Figure 5 shows the input trajectory, Figure 6 shows the interpolating coefficient $\lambda_1 + \lambda_2$ as a positive non-increasing function of t and the realization of $\alpha(t)$.

VI. Conclusion

In this paper a novel interpolation scheme is introduced for time-varying and uncertain linear discrete-time plants with polyhedral state and control constraints. The interpolation is done between several local unconstrained robust optimal controls and described in two approaches. For the first approach, at each time instant a linear matrix inequality optimization problem is solved while the second approach is based on the resolution of a system of nonlinear equations in

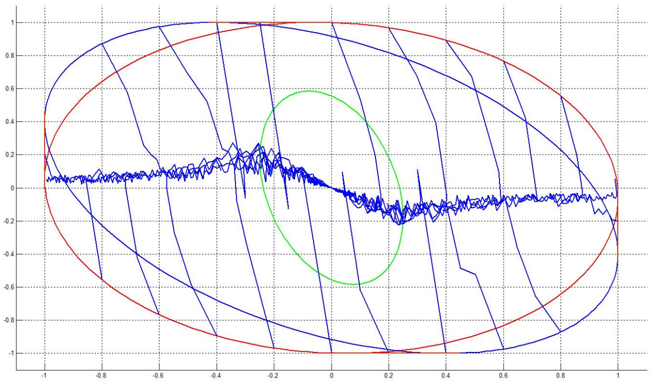

Fig. 3. Convex hull of ellipsoids and trajectories of the closed loop system for example 1

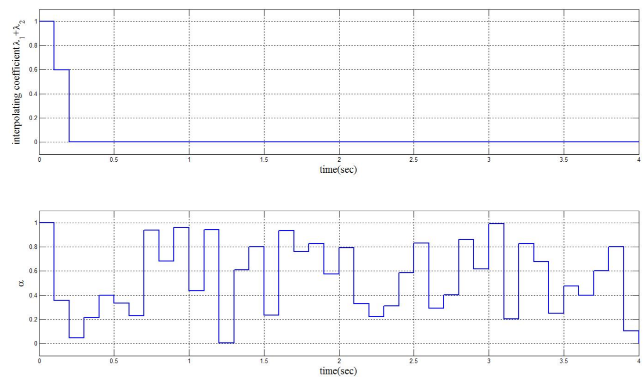

Fig. 6. The interpolating coefficient $\lambda_1 + \lambda_2$ and the realization of α for example 1

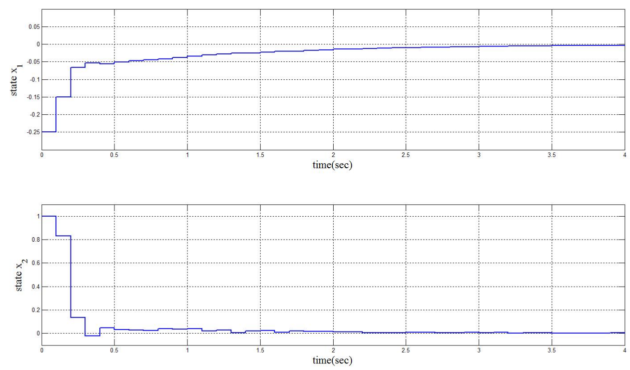

Fig. 4. State trajectory for example 1

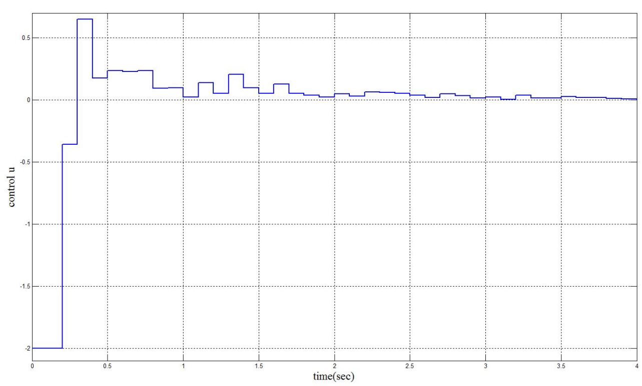

Fig. 5. Input trajectory for example 1

the case when the interpolation is done between two control laws.

The resulting interpolation based control assures the asymptotic stability in presence of constraints. Numerical example is presented to support the algorithms with illustrative simulations.

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