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Design of distributed esitimators over arbitrary causal networks

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Abstract— **We consider the problem of designing distributed estimators that are realizable over a given causal communication network. We consider the following two cases - 1) Given an interconnected plant over a given network, we design a distributed estimator with sub-units interacting over the same network such that each sub-unit estimates the states of the corresponding sub-system of the plant; 2) Given a general plant, we design a distributed estimator on a given network such that each sub-unit estimates the whole state vector of the plant by exchanging information with other sub-units. In the first case, we model the problem as a special case of distributed controller design problem discussed in our previous work. In the second case, we use the structure of the distributed estimator to decompose the problem into n sub-problems which can be solved separately, given that the plant satisfies certain detectability assumptions. The solutions of these sub-problems are finally combined together to form a distributed estimator that is realizable over the given network.**

Index Terms— Distributed estimation, Interconnected systems, Network realizability, H_2 filtering

I. Introduction

The problem of distributed estimation has been studied for networked systems and sensor networks using various distributed algorithms like local Kalman filtering (LKF), distributed Kalman filtering (DKF) [1], [2], gossip interactive Kalman filtering (GIKF) [3] to estimate the states of a plant at each node of a sensor network by inter-sensor communication of local observations. In the consensus based approaches like DKF [1], [2], [4], [5], the estimation algorithms are designed to perform distributed fusion of sensor measurements and covariance data which is then provided to micro-Kalman filters that provide estimates at each node.

In this paper, we approach the distributed estimation problem by posing it as a distributed control problem. Similar approaches were used when the underlying information pattern is partially nested [6]. In the case when the nodes communicate over a causal network interconnection, we provided the theory to design optimal distributed controller that can be implemented over the given causal network [7]. Posing the distributed estimation problem as a distributed control problem and using the results from [7], we solve two different distributed estimation problems.

This paper is presented in the following format. Section II reviews the results given in [7], [8] regarding the state-space and input-output representations for interconnected systems and the notion of network realizability over causal networks. Section III poses two distributed estimation problems: 1) to design distributed estimators for interconnected systems such that each sub-unit of the estimator estimates the states of a corresponding sub-system of the given plant, 2) to design distributed estimators for any general plant such that each sub-unit of the estimator estimates the state vector of the given plant. These two problems are analyzed and solved in Section IV and Section V. The main contribution of this paper is that, given a causal network interconnection we provide the best linear distributed estimator (for both the above mentioned problems) that can be realized over the given causal network. The proposed methodologies work for any arbitrary network configuration as long as the detectability conditions are satisfied. With the framework provided in this paper, it is also easy to extend the results to distributed H_∞ filtering. A thorough comparison with other distributed estimation algorithms will be provided in a future work.

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II. Interconnected systems

In this paper, we follow the notation used in [7] to describe the various matrices and the interconnected systems. In [7], [8], we provided a framework to describe the properties of interconnected systems and introduced the notion of realizability over causal communication networks. In order to describe the design of network realizable distributed estimators, we provide a brief summary of the results on realizability of systems over causal networks, in this section. The structural properties of state-space and input-output representations of interconnected systems over causal networks and the results corresponding to the notion of network realizability are vital to the problem formulation and solutions provided in the later part of this paper.

A group of plants or sub-systems interacting over a communication network is termed as an *Interconnected system*. Fig. 1 depicts an interconnected system where $\{P_i\}_{i=1,2,3}$ are sub-systems interacting over a communication network that can be described by a pseudograph $\mathscr{G} = (\mathscr{V}, \mathscr{A})$ where $\mathscr{V} = \{1,2,3\}$ is the vertex-set and $\mathscr{A} = \{(1,1),(1,2),(2,1),(2,2),(2,3),(3,3)\}$ is the arc-set.

A. State-space and Input-output descriptions of interconnected systems

Following the discussion in [7], we note that any discrete-time, causal, finite-dimensional linear time-invariant (FDLTI) interconnected system built on a causal network interconnection with an underlying graph $\mathscr{G}$ has a state-space realization (A,B,C,D) where the state-space matrices are *structured according to* $\mathscr{G}$. The set of such state-space realizations

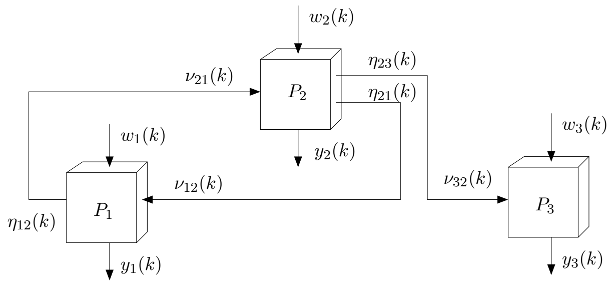

Fig. 1. A simple example of an interconnected system made of 3 different sub-systems interacting over a communication network

is denoted by $\mathfrak{S}(\mathscr{G},\mathscr{P}_x,\mathscr{P}_u,\mathscr{P}_y)$, where $\mathscr{P}_x$, $\mathscr{P}_u$ and $\mathscr{P}_y$ correspond to the state, input and output partitions. Note that a block-matrix $A = [A_{ij}]_{i,j}$ is structured according to $\mathscr{G} = (\mathscr{V},\mathscr{A})$ if A_{ij} is a zero matrix when ever $(j,i) \notin \mathscr{A}$.

Similarly, we note that the transfer function matrix of any interconnected system over a given causal network interconnection $\mathscr{G}$ is of the form $P(z) = [P_{ij}(z)]_{i,j}$ where

$$P_{ij}(z) = \begin{cases} 0 & \text{if there exists no path from} \\ & \quad j \text{ to } i \text{ on the graph } \mathscr{G}, \\ z^{-l(j,i)+1}H_{ij}(z) & \text{if the path length } l(j,i) \geq 1, \\ H_{ij}(z) & \text{if } i = j \end{cases} \tag{1}$$

where $H_{ij}(z)$ is a real-rational proper transfer function matrix. The set of all such transfer function matrices $[P_{ij}(z)]_{i,j}$ is denoted by $\mathfrak{T}(\mathscr{G},\mathscr{P}_u,\mathscr{P}_y)$.

B. Realizing interconnected systems over the given network

A system is said to be *realizable over a causal network* if it can be implemented as n individual sub-systems (with their local states, inputs and outputs corresponding to each node of the network) that pass messages to each other along the directed links while respecting the causal network interconnection and maintaining internal stability.

In this section, the network realizability property of the elements of $\mathfrak{S}(\mathscr{G},\mathscr{P}_x,\mathscr{P}_u,\mathscr{P}_y)$ and $\mathfrak{T}(\mathscr{G},\mathscr{P}_u,\mathscr{P}_y)$ is reviewed through Lemma 1 and Theorem 2, respectively.

Lemma 1: Given a causal network interconnection $\mathscr{G}$ and the partitions $\mathscr{P}_x$, $\mathscr{P}_u$ and $\mathscr{P}_y$, any system $Q \in \mathfrak{S}(\mathscr{G},\mathscr{P}_x,\mathscr{P}_u,\mathscr{P}_y)$ is realizable over the network $\mathscr{G}$.

We define $\mathfrak{S}(\mathscr{G},\mathscr{P}_u,\mathscr{P}_y) := \bigcup_{\mathscr{P}_x \in \mathbb{N}^n} \mathfrak{S}(\mathscr{G},\mathscr{P}_x,\mathscr{P}_u,\mathscr{P}_y)$. Since any realizable system on the network $\mathscr{G}$, with input and output partitions as $\mathscr{P}_u$ and $\mathscr{P}_y$, is an element of $\mathfrak{S}(\mathscr{G},\mathscr{P}_x,\mathscr{P}_u,\mathscr{P}_y)$ for some state partition $\mathscr{P}_x$ and any element of $\mathfrak{S}(\mathscr{G},\mathscr{P}_x,\mathscr{P}_u,\mathscr{P}_y)$ is realizable over $\mathscr{G}$ for any $\mathscr{P}_x$, we can say that the set $\mathfrak{S}(\mathscr{G},\mathscr{P}_u,\mathscr{P}_y)$ represents the set of all discrete-time, causal FDLTI systems that are realizable over a causal network interconnection $\mathscr{G}$. We denote the set of all stable systems in $\mathfrak{S}(\mathscr{G},\mathscr{P}_u,\mathscr{P}_y)$ by $\mathfrak{S}^s(\mathscr{G},\mathscr{P}_u,\mathscr{P}_y)$.

Theorem 2: Given a network represented by a directed pseudograph $\mathscr{G}$ and the input and output partitions, $\mathscr{P}_u$ and $\mathscr{P}_y$, any bounded-input bounded-output (BIBO) stable system $Q(z) \in \mathfrak{T}(\mathscr{G},\mathscr{P}_u,\mathscr{P}_y)$ is realizable over the network $\mathscr{G}$ for some state partition $\mathscr{P}_x$.

We denote the set of all stable real-rational proper transfer function matrices in $\mathfrak{T}(\mathscr{G},\mathscr{P}_u,\mathscr{P}_y)$ by $\mathfrak{T}^s(\mathscr{G},\mathscr{P}_u,\mathscr{P}_y)$. One can show that $\mathfrak{S}^s(\mathscr{G},\mathscr{P}_u,\mathscr{P}_y)$ and $\mathfrak{T}^s(\mathscr{G},\mathscr{P}_u,\mathscr{P}_y)$ are equivalent and represent the same set of all stable interconnected systems realizable over $\mathscr{G}$ given the input and output partitions $\mathscr{P}_u$ and $\mathscr{P}_y$.

III. Distributed estimation problems

In this paper, we consider the problem of designing linear, distributed estimators that can be realized over a given causal communication network. By a distributed estimator, we mean a stable network realizable filter that can estimate the states of a given plant using the measurements, such that the estimates stay "close" to the states of the plant even in the presence of process and measurement noise. Following are the two distributed estimation problems that we analyze in this paper.

1) We consider an interconnected plant over a causal communication network and design a distributed estimator that can be realized over the same network. The objective of this problem is to make each sub-unit of the distributed estimator estimate the states of the corresponding sub-system of the interconnected plant by exchanging information with other estimator sub-units.
2) We consider a general plant and design a distributed estimator that can be realized over a given causal communication network. The objective of this problem is to make each sub-unit of the distributed estimator estimate the complete state vector of the plant.

In the following sections, the above mentioned distributed estimation problems are formulated and analyzed to estimate the states of a given plant by minimizing the effect of external disturbances and measurement noise. We shall make some assumptions about detectability of the plant dynamics to assure the existence of a distributed estimator.

IV. Distributed H_2 filtering for interconnected systems

In this section, the first problem posed in Section III will be addressed. Let the interconnected system P be made of sub-systems $\{P_i\}_i$ interacting over a causal communication network $\mathscr{G}$. Let $x_i(k)$ be the state vector and $w_i(k)$ denote the process and measurement noise vector corresponding to P_i at time instant k. Due to the interactions over the network $\mathscr{G}$, the dynamics of sub-system P_i can be given by the following state-space equations

$$\begin{aligned} x_i(k+1) &= A_i x_i(k) + B_i w_i(k) + B_i^\nu \nu_i(k), \\ y_i(k) &= C_i x_i(k) + D_i w_i(k), \\ \eta_i(k) &= C_i^\eta x_i(k) + D_i^\eta w_i(k) \qquad \forall\, i \end{aligned} \tag{2}$$

where $\eta_i(k) = [\eta_{ij}(k)]_{j \in \mathscr{N}_i^+ \setminus \{i\}}$ and $\nu_i(k) = [\nu_{ij}(k)]_{j \in \mathscr{N}_i^- \setminus \{i\}}$ denote the messages passed onto, and received from, the network $\mathscr{G}$ by sub-system P_i at time instant k. In this paper, we assume that the communication links are delay-free and noiseless. So, $\eta_{ij}(k) = \nu_{ji}(k)$ for all $(i,j) \in \mathscr{A}$, which corresponds to the message passed from P_i to P_j at time instant k.

Combining the above equations corresponding to the dynamics and the messages exchanged by the sub-systems $\{P_i\}_i$, we get the state equations corresponding to the interconnected system P as follows

$$\begin{aligned} x_i(k+1) &= A_i x_i(k) + B_i w_i(k) \\ &\qquad + B_i^{\nu}[C_{ji}^{\eta} x_j(k) + D_{ji}^{\eta} w_j(k)]_{j \in \mathscr{N}_i^- \setminus \{i\}} \\ &= \sum_{j \in \mathscr{N}_i^-} A_{ij} x_j(k) + \sum_{j \in \mathscr{N}_i^-} B_{ij} w_j(k) \qquad \forall\, i \end{aligned} \tag{3}$$

where A_{ij} and B_{ij} are appropriately defined for all i and j. Note that A_{ij} and B_{ij} are zero matrices when $(j,i) \notin \mathscr{A}$. The equations in (2) and (3) can be written in a simpler form as

$$\begin{aligned} x(k+1) &= Ax(k) + Bw(k), \\ y(k) &= Cx(k) + Dw(k) \end{aligned} \tag{4}$$

where $x(k) = [x_i(k)]_i$, $y(k) = [y_i(k)]_i$ and $w(k) = [w_i(k)]_i$ are the state, measurement and disturbance vectors corresponding to the interconnected plant P. Note that A and B matrices are structured according to $\mathscr{G}$ while C and D matrices are block-diagonal.

Corresponding to this interconnected system, we design a distributed estimator (as shown in Fig. 2) such that each sub-unit E_i estimates the states of the sub-system P_i by exchanging messages over the same causal network $\mathscr{G}$.

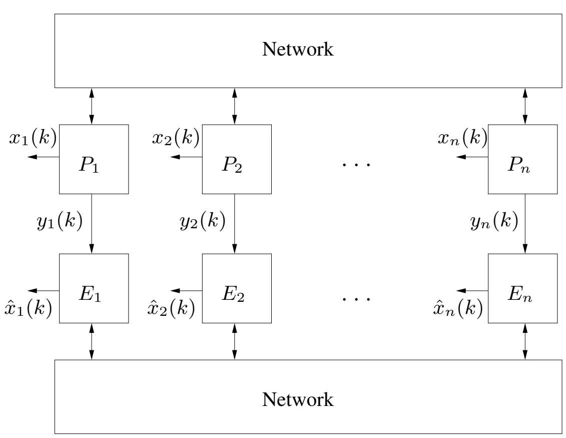

Fig. 2. Interconnected plant P and a distributed estimator E in terms of their sub-units $\{P_i\}_i$ and $\{E_i\}_i$.

This problem can easily be converted into a distributed control problem discussed in [7] by treating estimates as control inputs and writing an equivalent generalized plant's sub-systems G_i as follows

$$\begin{aligned} x_i(k+1) &= \sum_{j \in \mathscr{N}_i^-} A_{ij} x_j(k) + \sum_{j \in \mathscr{N}_i^-} B_{ij} w_j(k), \\ z_i(k) &= x_i(k) - u_i(k), \\ y_i(k) &= C_i x_i(k) + D_i w_i(k) \qquad \forall\, i \end{aligned} \tag{5}$$

where $u_i(k) = \hat{x}_i(k)$ is the state estimate and $z_i(k)$ represents the estimation error corresponding to P_i at time instant k, for all i. Pictorially, we can view the problem as Fig. 3 where G is the generalized plant, corresponding to the interconnected system P, with a state-space representation given by

$$\begin{bmatrix} x(k+1) \\ z(k) \\ y(k) \end{bmatrix} = \begin{bmatrix} A & B & 0 \\ I & 0 & -I \\ C & D & 0 \end{bmatrix} \begin{bmatrix} x(k) \\ w(k) \\ u(k) \end{bmatrix} \tag{6}$$

where $A = [A_{ij}]_{i,j}$, $B = [B_{ij}]_{i,j}$ are structured according to the network $\mathscr{G}$; and $C = \textbf{diag}[C_i]_i$, $D = \textbf{diag}[D_i]_i$ are block diagonal matrices; and E is stable and network realizable over the given causal network $\mathscr{G}$.

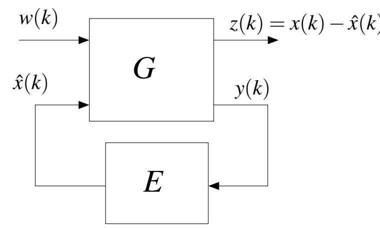

Fig. 3. An equivalent model using a generalized plant G in a feedback interconnection with the distributed estimator E.

Our objective in this paper is to design a distributed H_2 estimator for an interconnected plant P which can be interpreted as design of a stable and network realizable controller E for the generalized plant G that minimizes the closed-loop system norm $\|T_{zw}\|_2 = \|\textbf{lft}(G,E)\|_2$. Thus the distributed H_2 filtering problem can be written as

$$\begin{aligned} \min \quad & \|T_{zw}\|_2 \\ \text{subject to} \quad & E \in \mathfrak{S}^s(\mathscr{G}, \mathscr{P}_y, \mathscr{P}_u), \\ & T_{zw} \text{ is stable.} \end{aligned} \tag{7}$$

Note that the estimation problem does not require stabilization of the given plant and thus the problem in (7) is much simpler than the corresponding distributed controller design problem discussed in [7].

A. Parametrization of all network realizable estimators

Using the methodology given in [7], we parametrize the set of all possible stable estimators that are realizable over the network $\mathscr{G}$ for a given interconnected plant P using the following theorem.

Theorem 3: Given an interconnected system P over a causal network $\mathscr{G}$ with a state-space representation given by (4). Given P is detectable and given a matrix L structured according to $\mathscr{G}$, and partitioned accordingly, such that $A+LC$ is stable. Then the set of all stable and network realizable estimators that keeps the estimation error $(x(k) - \hat{x}(k))$ bounded is given by

$$E = \textbf{lft}(J,Q)$$

where Q is FDLTI, causal, stable and realizable over the given network $\mathscr{G}$ and

$$J = \left[\begin{array}{c|cc} A+LC & -L & 0 \\ \hline I & 0 & -I \\ -C & I & 0 \end{array}\right]. \tag{8}$$

B. Stability conditions for constructing L

In Theorem 3, we assumed the existence of a matrix L that is structured according to $\mathscr{G}$ such that $A+LC$ is Schur-stable. One can treat this as a *network detectability* condition.

Lemma 4: Given matrices A and C that are partitioned according to $(\mathscr{P}_x, \mathscr{P}_x)$ and $(\mathscr{P}_y, \mathscr{P}_x)$, respectively, there exists a matrix L that is structured according to $\mathscr{G}$ and partitioned according to $(\mathscr{P}_x, \mathscr{P}_y)$ such that $A+LC$ is Schur-stable if the following feasibility problem has a solution

$$\begin{aligned} &\min && 1 \\ &\text{subject to} && \begin{bmatrix} M & A'G+C'R \\ (A'G+C'R)' & G+G'-M \end{bmatrix} > 0, \\ &&& G \text{ is block-diagonal and partitioned} \\ &&& \quad \text{according to } (\mathscr{P}_x, \mathscr{P}_x), \\ &&& R' \text{ is structured according to } \mathscr{G} \text{ and} \\ &&& \quad \text{partitioned according to } (\mathscr{P}_y, \mathscr{P}_x). \end{aligned} \tag{9}$$

Proof: The proof of this lemma follows the stability test for discrete-time systems, as given in [9]. ∎

C. Optimal distributed H_2 filter

From the previous sections, we know that the set of FDLTI, causal and stable systems that are realizable over a causal communication network $\mathscr{G}$ are given by $\mathfrak{S}^s(\mathscr{G}, \mathscr{P}_y, \mathscr{P}_u)$ or $\mathfrak{T}^s(\mathscr{G}, \mathscr{P}_y, \mathscr{P}_u)$. We consider $Q(z) \in \mathfrak{T}^s(\mathscr{G}, \mathscr{P}_y, \mathscr{P}_u)$ to parametrize all stabilizing controllers realizable over $\mathscr{G}$. If there exists matrix L with the properties described in the hypothesis of Theorem 3, the set of all closed-loop transfer matrices from $w(k)$ to $z(k)$ can be obtained using Theorem 3 and the results from [10] as

$$T_{zw} = \mathbf{lft}(T,Q) = \{T_{11} + T_{12}QT_{21} : \; Q \in \mathfrak{T}^s(\mathscr{G}, \mathscr{P}_y, \mathscr{P}_u)\} \tag{10}$$

where T is given by

$$T = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} = \left[\begin{array}{cc|cc} A & 0 & B & 0 \\ 0 & A+LC & -B-LD & 0 \\ \hline 0 & -I & 0 & I \\ 0 & -C & D & 0 \end{array}\right] = \left[\begin{array}{c|cc} A+LC & -B-LD & 0 \\ \hline -I & 0 & I \\ -C & D & 0 \end{array}\right]. \tag{11}$$

Note that (11) corresponds to

$$T_{11} = \left[\begin{array}{c|c} A+LC & -B-LD \\ \hline -I & 0 \end{array}\right], \; T_{12} = I,$$

$$T_{21} = \left[\begin{array}{c|c} A+LC & -B-LD \\ \hline -C & D \end{array}\right], \; T_{22} = 0.$$

Since the closed-loop transfer matrix is simply an affine function of the controller parameter matrix Q while the delay and sparsity constraints on the transfer function of Q are linear, we can rewrite the distributed H_2 problem in (7) as a convex optimization problem

$$\begin{aligned} &\min && \|T_{11} + QT_{21}\|_2 \\ &\text{subject to} && Q \in \mathfrak{T}^s(\mathscr{G}, \mathscr{P}_y, \mathscr{P}_u). \end{aligned} \tag{12}$$

Vectorization techniques can be applied to write the optimization problem in (12) as an equivalent unconstrained problem. To represent the vectorization of a transfer function matrix, we make a slight change of notation for representing the matrices. Instead of treating Q_{ij} as a sub-matrix of Q, we consider Q_{ij} to be the element of the matrix Q in the i^{th} row and j^{th} column.

Let $\mathbf{vec}(\mathfrak{T}^s(\mathscr{G}, \mathscr{P}_y, \mathscr{P}_u)) = \{\mathbf{vec}(Q) | Q \in \mathfrak{T}^s(\mathscr{G}, \mathscr{P}_y, \mathscr{P}_u)\}$ denote the set of vectorized elements of $\mathfrak{T}^s(\mathscr{G}, \mathscr{P}_y, \mathscr{P}_u)$. If $\mathscr{P}_u = \{\mathscr{P}_u^1, \ldots, \mathscr{P}_u^n\}$ denotes the output partition, then denote $n_u = \sum_i \mathscr{P}_u^i$ to represent the total number of outputs. Similarly, denote n_y to represent the total number of inputs. It can be seen that $\mathbf{vec}(\mathfrak{T}^s(\mathscr{G}, \mathscr{P}_y, \mathscr{P}_u)) \in \mathscr{RH}_\infty^{n_u n_y \times 1}$ is a sub-space due to the delay and sparsity constraints imposed by the network $\mathscr{G}$. Let a denote the total number of elements of $Q \in \mathfrak{T}^s(\mathscr{G}, \mathscr{P}_y, \mathscr{P}_u)$ that are not constrained to be zero. It can be shown that there exists a matrix $H \in \mathscr{R}_p^{n_u n_y \times a}$ whose columns form an orthonormal basis for $\mathbf{vec}(\mathfrak{T}^s(\mathscr{G}, \mathscr{P}_y, \mathscr{P}_u))$. Thus, we know that

$$Q \in \mathfrak{T}^s(\mathscr{G}, \mathscr{P}_y, \mathscr{P}_u) \Longleftrightarrow \mathbf{vec}(Q) = Hx \quad \text{for some } x \in \mathscr{RH}_\infty^{a\times 1}.$$

Note that H contains the delay and sparsity constraints imposed by the causal network interconnection $\mathscr{G}$. Using the results of vectorization, we get that

$$\begin{aligned} \|T_{11} + QT_{21}\|_2 &= \|\mathbf{vec}(T_{11} + QT_{21})\|_2 \\ &= \left\|\mathbf{vec}(T_{11}) + (T_{21}^t \otimes I)\,\mathbf{vec}(Q)\right\|_2 \\ &= \left\|\mathbf{vec}(T_{11}) + (T_{21}^t \otimes I)Hx\right\|_2 \end{aligned}$$

Thus, we can pose the problem (12) as an unconstrained H_2 problem

$$\begin{aligned} &\min && \left\|\mathbf{vec}(T_{11}) + (T_{21}^t \otimes I)Hx\right\|_2 \\ &\text{subject to} && x \in \mathscr{RH}_\infty^{a\times 1} \end{aligned} \tag{13}$$

which can be solved using standard techniques. Let $x^\star$ denote the solution of the optimization problem (13). Then the corresponding optimal $Q^\star$ is given by $Q^\star = \mathbf{vec}^{-1}(Hx^\star)$. Since $Q^\star \in \mathfrak{T}^s(\mathscr{G}, \mathscr{P}_y, \mathscr{P}_u)$, we can obtain a realization $(A_Q^\star, B_Q^\star, C_Q^\star, D_Q^\star)$ (using Theorem 2) that satisfies the constraints imposed by the causal network interconnection $\mathscr{G}$ and the corresponding controller is given by $E^\star = \mathbf{lft}(J, Q^\star)$, where J is given by (8). From Theorem 3, we can see that $E^\star$ thus designed is the optimal estimator that is realizable over the given causal network $\mathscr{G}$.

V. Distributed filtering for general plants

In this section, the second problem posed in Section III will be addressed. Consider a general plant P with the following dynamics

$$\begin{aligned} x(k+1) &= Ax(k) + Bw(k), \\ y_i(k) &= C_i x(k) + D_i w(k) \qquad \forall\, i \in \mathscr{V} \end{aligned} \tag{14}$$

where $\{y_i(k)\}_i$ are n measurement vectors corresponding to n sensor units. In the case (shown in Fig. 4) when each sub-unit E_i has to estimate the full state vector $x(k)$ of the plant, we define estimation error vectors $z_i(k) = x(k) - \hat{x}_i(k)$ for all i, where $\hat{x}_i(k)$ is the estimate provided by E_i. The objective is to keep the estimation error vector "close" to zero even in the presence of external disturbance.

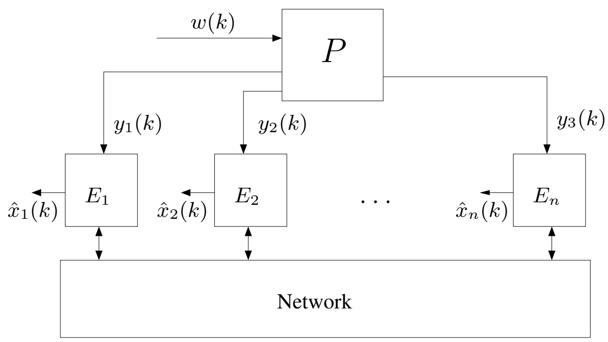

Fig. 4. A distributed estimation problem with a general plant and a distributed estimator.

The distributed H_2 filtering problem in this case can be formulated as a distributed control problem by treating the estimates as control inputs to a generalized plant G with the following dynamics

$$\begin{bmatrix} x(k+1) \\ z_1(k) \\ \vdots \\ z_n(k) \\ y(k) \end{bmatrix} = \begin{bmatrix} A & B & 0 & \dots & 0 \\ I & 0 & -I & & \\ \vdots & \vdots & & \ddots & \\ I & 0 & & & -I \\ C & D & 0 & \dots & 0 \end{bmatrix} \begin{bmatrix} x(k) \\ w(k) \\ u_1(k) \\ \vdots \\ u_n(k) \end{bmatrix} \tag{15}$$

where $u_i(k) = \hat{x}_i(k)\ \forall\, i$, $y(k) = [y_i(k)]_i$, $C = \mathbf{vert}[C_i]_i$ and $D = \mathbf{vert}[D_i]_i$. Thus the distributed H_2 filtering problem can be written as

$$\begin{aligned} &\min && \|T_{zw}\|_2 \\ &\text{subject to} && E(z) \in \mathfrak{T}^s(\mathscr{G}, \mathscr{P}_y, \mathscr{P}_u),\ T_{zw} \text{ is stable} \end{aligned} \tag{16}$$

where $T_{zw} = \mathbf{lft}(G, E)$ is the closed-loop map from the disturbance $w(k)$ to the overall estimation error vector $z(k) = [z_i(k)]_i$. Using the structure of G and $E(z)$, the optimization problem in (16) can equivalently be expressed as n independent problems given by

$$\begin{aligned} &\min && \|T_{z_i w}\|_2 \\ &\text{subject to} && E_i(z) = \mathbf{hor}[E_{ij}(z)]_{j \in \mathscr{V}}, \\ & && E_{ij}(z) \text{ satisfying (1)},\ T_{z_i w} \text{ is stable} \end{aligned} \tag{17}$$

where $T_{z_i w}$ is the closed-loop map from $w(k)$ to $z_i(k)$, for all i. Based on the delay and sparsity constraints given in (1), we define transfer function matrices (of size $(\mathscr{P}_y^j, \mathscr{P}_y^j)$)

$$M_{ij}(z) = \begin{cases} 0 & \text{if there exists no path from} \\ & \quad j \text{ to } i \text{ on the graph } \mathscr{G}, \\ z^{-l(j,i)+1} I & \text{if the path length } l(j,i) \geq 1, \\ I & \text{if } i = j \end{cases} \tag{18}$$

Since $E(z) = [E_i(z)]_i \in \mathfrak{T}^s(\mathscr{G}, \mathscr{P}_y, \mathscr{P}_u)$, the equations (1) and (18) can be used to write

$$\begin{aligned} E_i(z) &= \mathbf{hor}[E_{ij}(z)]_j = \mathbf{hor}[H_{ij}(z) M_{ij}(z)]_j \\ &= \mathbf{hor}[H_{ij}(z)]_j\, \mathbf{diag}[M_{ij}(z)]_j := H_i(z) M_i(z) \end{aligned} \tag{19}$$

where $H_{ij}(z)$ is a stable real-rational proper transfer function matrix. Note that $M_i(z)$ is dependent only on the network $\mathscr{G}$ and is always diagonal and stable because it has only delay terms of the form of z^{-r} (for some $r \geq 0$) or 0 on the diagonal. Using this separation, we can treat $\tilde{P}_i := M_i P$ as the new plant and define a corresponding generalized plant $\tilde{G}_i$ given by

$$\begin{bmatrix} x(k+1) \\ x_M^i(k+1) \\ z_i(k) \\ \tilde{y}_i(k) \end{bmatrix} = \begin{bmatrix} A & 0 & B & 0 \\ B_M^i C & A_M^i & B_M^i D & 0 \\ I & 0 & 0 & -I \\ D_M^i C & C_M^i & D_M^i D & 0 \end{bmatrix} \begin{bmatrix} x(k) \\ x_M^i(k) \\ w(k) \\ u_i(k) \end{bmatrix} \tag{20}$$

where $u_i(k) = \hat{x}_i(k)$ and $M_i = (A_M^i, B_M^i, C_M^i, D_M^i)$. In the following theorem, we show that each of the n sub-problems in (17) can be converted into an unconstrained optimization problem and can be solved using standard techniques.

Theorem 5: Given the plant P and a network interconnection $\mathscr{G}$, the optimization problem given by (17) is equivalent to

$$\begin{aligned} &\min && \|\tilde{T}_{z_i w}\|_2 \\ &\text{subject to} && H_i(z) \in \mathscr{R}_p \cap \mathscr{RH}_\infty,\ \tilde{T}_{z_i w} \text{ is stable} \end{aligned} \tag{21}$$

where $\tilde{T}_{z_i w} = \mathbf{lft}(\tilde{G}_i, H_i)$.

Let $H_i^\star(z)$ be the solution of the optimization problem (21). Then we can construct $E_i^\star(z) = H_i^\star(z) M_i(z)$ using (19). After computing $E_i^\star(z)$ for all i, we obtain the estimator $E^\star(z) = [E_i^\star(z)]_i$ which, by construction, belongs to the set $\mathfrak{T}^s(\mathscr{G}, \mathscr{P}_y, \mathscr{P}_u)$ and is realizable over $\mathscr{G}$ by Theorem 2.

A. Example

In this section, we present a simple example to describe the first problem posed in Section III. We consider an interconnected system made of 3 sub-systems interacting over a communication network represented by a directed psuedograph $\mathscr{G}$ as shown in Fig. 1. The dynamics of the three sub-systems and their interaction over the network is given by

$$\begin{bmatrix} x_1(k+1) \\ y_1(k) \\ \eta_{12}(k) \end{bmatrix} = \left[\begin{array}{cc|ccc|c} 0.1 & 0.2 & 0.5 & 0.4 & 0 & 0 \\ 0.7 & -0.5 & 0.1 & 0.2 & 0 & 1 \\ \hline 0.4 & 0 & 0 & 0 & 1 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 & 0 & 0 \end{array}\right] \begin{bmatrix} x_1(k) \\ w_1(k) \\ v_{12}(k) \end{bmatrix},$$

$$\begin{bmatrix} x_2(k+1) \\ y_2(k) \\ \eta_{21}(k) \\ \eta_{23}(k) \end{bmatrix} = \left[\begin{array}{c|cc|cc} -0.6 & 0.3 & 0 & 1 & -1.4 \\ \hline 0.5 & 0 & 1 & 0 & 0 \\ \hline 0.8 & 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 \end{array}\right] \begin{bmatrix} x_2(k) \\ w_2(k) \\ v_{21}(k) \end{bmatrix},$$

$$\begin{bmatrix} x_3(k+1) \\ y_3(k) \end{bmatrix} = \left[\begin{array}{c|cc|c} 0.4 & 0.6 & 0 & 0.3 \\ \hline 0.6 & 0 & 1 & 0 \end{array}\right] \begin{bmatrix} x_3(k) \\ w_3(k) \\ v_{32}(k) \end{bmatrix},$$

$$v_{12}(k) = \eta_{21}(k),\ v_{21}(k) = \eta_{12}(k),\ v_{32}(k) = \eta_{23}(k).$$

In this particular example, one can see that the individual sub-systems $\{P_i\}_i$ are stable while the interconnected system P is unstable. We design a network realizable distributed

$$
A_E = \left[\begin{array}{cccc|cccc|cccc}
0.013 & 0.2 & 0 & 0 & 0 & 0 & 0 & -0.044 & 0 & 0 & 0 & 0 \\
-0.031 & -0.5 & 0 & 0 & 0 & 0 & 0 & 0.11 & 0 & 0 & 0 & 0 \\
-0.375 & 0 & -0.382 & -0.273 & 0.379 & -0.723 & 0.747 & 0.046 & 0 & 0 & 0 & 0 \\
-0.058 & 0 & -0.19 & -0.11 & -0.132 & -0.434 & 0.075 & -0.191 & 0 & 0 & 0 & 0 \\
\hline
0.026 & 0 & -0.855 & 1.236 & -0.739 & -0.231 & -0.234 & 0.383 & 0 & 0 & 0 & 0 \\
-0.473 & 0 & -0.297 & 0.772 & 0.308 & -0.342 & 1.126 & 0.065 & 0 & 0 & 0 & 0 \\
0.287 & 0 & 0.013 & -0.034 & -0.024 & -0.045 & 0.308 & 0.008 & 0 & 0 & 0 & 0 \\
-0.044 & -0.7 & 0 & 0 & 0 & 0 & 0 & 0.154 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.092 & 0.493 & -0.01 & -0.126 & -0.637 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.141 & 0.542 & -0.367 & -0.278 & -1.356 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.064 & -0.385 & -0.506 & 0.365 & 0.201
\end{array}\right]
$$

$$
B_E = \left[\begin{array}{c|c|c}
0.219 & 0.088 & 0 \\
1.829 & 1.38 & 0 \\
0.937 & -0.091 & 0 \\
0.146 & 0.383 & 0 \\
\hline
-0.066 & -0.766 & 0 \\
1.82 & -0.129 & 0 \\
-0.718 & -0.016 & 0 \\
2.61 & -1.508 & 0 \\
\hline
0 & 0.6 & 0.667 \\
0 & -0.184 & -0.822 \\
0 & -0.283 & -0.904 \\
0 & 0.128 & 0.641
\end{array}\right],\;
C_E = \left[\begin{array}{cc|c|c}
0.932 & -0.009 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0.186 & 0.141 & -0.151 & 0 \\
-0.271 & -0.436 & 0.18 & 0 \\
\hline
-0.098 & 1.084 & 0.019 & 0 \\
0.057 & -0.293 & 0.092 & 0 \\
0.602 & 1.899 & -0.010 & 0 \\
0.017 & -0.011 & 0.268 & 0.048 \\
\hline
0 & 0 & 0 & 0.822 \\
0 & 0 & 0 & 0.393 \\
0 & 0 & 0 & 0.503 \\
0 & 0 & 0 & 0.455
\end{array}\right]',\;
D_E = \left[\begin{array}{c|c|c}
0.170 & -0.033 & 0 \\
0.024 & 0.022 & 0 \\
\hline
0 & 1.464 & 0 \\
\hline
0 & -0.097 & 0.297
\end{array}\right]
$$

estimator $E = (A_E, B_E, C_E, D_E)$ using the methodology given in Section IV. Note that the coefficients of A_E, B_E, C_E and D_E are truncated to 3 decimals due to lack of space. The optimal performance cost is given by $\|T_{zw}\|_2 = 2.2126$.

To check the performance of this estimator, we consider a process and measurement noise $w(k) \sim \mathcal{N}(0, I)$. It is known that the H_2 norm of a transfer function is equal to the average power of the output signal when the input is a zero-mean Gaussian vector with covariance as I. The average power of the estimation error signal $z(k)$ (as shown in Fig. 5) was found to be 2.2123 when simulated for 10000 time steps.

In this example, we show that a network realizable distributed estimator can be designed to even estimate the states of an unstable interconnected system using the approach discussed in this paper. A thorough comparison with other distributed estimation techniques will be considered in future.

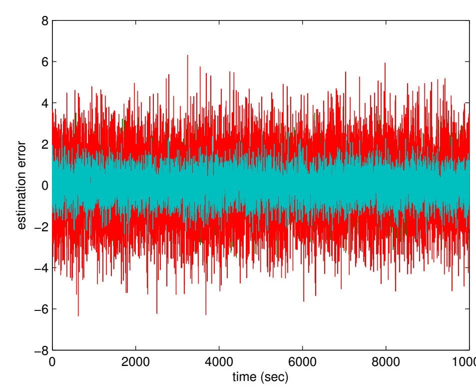

Fig. 5. Estimation error vector $z(k)$.

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