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On quantum-classical equivalence for linear systems control problems and its application to quantum entanglement assignment

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***Abstract*— In this paper we consider general linear quantum systems with operator-valued variables. It is shown that a specific but fairly wide class of linear quantum control problems have exactly the same solutions to the corresponding classical one obtained simply by replacing the operator variables by the scalar variables. Based on this result, an application of the so-called covariance assignment control to the quantum system is given, which is a suitable approach for controlling quantum entanglement.**

I. INTRODUCTION

One of the most fundamental differences between quantum and classical systems is that any physical quantity is represented by an operator defined in an abstract Hilbert space in the quantum case. A typical example is that the pair of position and momentum operators $\hat{q}_t$ and $\hat{p}_t$ should satisfy the so-called *canonical commutation relation (CCR)* $\hat{q}_t\hat{p}_t - \hat{p}_t\hat{q}_t = \mathrm{i}\hbar$, which immediately implies that they are infinite-dimensional operators. Hence, it can be imagined that control theory for quantum systems with such operator-valued variables is drastically different from the classical one. Actually we are not allowed even just to simulate the dynamical control effect of $(\hat{q}_t, \hat{p}_t)$ in the phase space $\mathbb{R}^2$.

Now let us focus on a specific class of systems, linear quantum systems or equivalently Gaussian systems [1]. For this system a straightforward generalization of the above position and momentum operators are the system variables, and hence they satisfy the (generalized) CCR (see Eqs. (1) and (2) in Section II). The importance of this class of systems is maintained by the fact that it includes for instance linear optical networks [2], nano-mechanical oscillators [3], and trapped ions [4], all of which are vital test-beds with nontrivial quantum mechanical features appearing [5]. Remarkably, it was shown in [6], [7] that the LQG control problem for such a linear quantum system has exactly the same solution to the corresponding classical one obtained simply by replacing the operator variables by the scalar variables. This fact implies that, within the formalism of LQG control, the corresponding classical system serves as an equivalent representation for the quantum one. Clearly, such an equivalent classical system allows us to do various numerical simulations and obtain useful information in analyzing dynamical behavior of the system variables as well as synthesizing an efficient feedback controller.

The above-mentioned quantum-classical equivalence in linear case is actually observed in several situations [8], [9], [10], [11]. However, this does not mean that all linear control theory can directly be applied to quantum systems. The reason is as follows. As in the classical case, a Gaussian quantum state is fully characterized only by the first and second moments; thanks to this feature the CCR can equivalently be replaced by the so-called *uncertainty relation* $V + \hbar\mathrm{i}\Theta_N/2 \geq 0$, where V is the covariance matrix (See Section II for the detailed description). Clearly, in classical case the non-negativity of V is only required, thus in this sense the set of quantum Gaussian states is a special class of classical Gaussian states. Hence, we expect that a linear control theory with states satisfying the uncertainty relation only works for quantum systems. This observation further implies that, in performing a numerical simulation for the corresponding classical system, a fixed initial state cannot be specified, because this means $V = 0$ at $t = 0$. Consequently, our question is now posed; *do we really need this additional constraint on the state to apply a linear control theory to the quantum case?*

The first contribution of this paper is to show that the answer to the above question is "No" in special but enough general cases. More specifically, the (optimal) controllers for quantum and the corresponding classical systems are exactly the same for linear control problems with control performance evaluated only at the steady state if the linear system to be controlled is Hurwitz stable. Furthermore, in this case, any numerical simulation for that classical system, even trajectories in the phase space with a fixed initial state, makes sense for the quantum system.

The second contribution is to show that the *covariance assignment control theory* [12], [13], [14] can properly be applied to the quantum case. The purpose of this control is to (i) stabilize the system via feedback and then (ii) drive the system to a steady state with a given desirable covariance matrix. Hence, the above quantum-classical equivalence result holds for this control problem. That is, a covariance assigning controller for a classical system indeed achieves the same control goal for the corresponding quantum one. The significance of this application clearly appears for the problems of controlling *entanglement*, which is one of the most crucial properties in order to accomplish various quantum information technologies [15]. Indeed, in Gaussian case, it is known that the covariance matrix fully characterizes the entangling structure of a state as well as its quantity [16], [17], [18].

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This paper is organized as follows: In Section II a general description of quantum Gaussian states and linear quantum systems is given. The quantum-classical equivalence will be discussed in Section III. Section IV is devoted to study the application of covariance assignment control to the entanglement control.

We will use the following notations; All (infinite-dimensional) linear operators are denoted with hat, e.g., $\hat{x}$. We denote $\hat{1}$ by the identity operator, but it is omitted when trivial. $\top$, $*$, and $\dagger$ represent the matrix transpose, the element-wise conjugate operation, and the complex conjugation, respectively; i.e., $(\hat{M}_{k,l})^\dagger = (\hat{M}^*_{l,k})$. I_N and O_N are the N-dimensional identity and zero matrices, respectively. $\otimes$ denotes the tensor or the Kronecker product.

II. Preliminary

In this section, we introduce basic notions of a quantum Gaussian state and a linear quantum system.

A. Uncertainty relation

The *canonical conjugate pairs* $\hat{q}$ and $\hat{p}$ are unbounded linear operators that satisfy the following canonical commutation relation (CCR):

$$\hat{q}\hat{p} - \hat{p}\hat{q} = \mathrm{i}\hbar\hat{1},$$

where $\hat{1}$ represents the identity operator. We will deal with the generalized canonical conjugate pairs

$$\hat{\boldsymbol{x}} := (\hat{q}_1, \hat{p}_1, \cdots, \hat{q}_N, \hat{p}_N)^\top. \tag{1}$$

Note that $(\hat{q}_k, \hat{p}_k)$ and $(\hat{q}_j, \hat{p}_j)$ live in different Hilbert spaces for $j \neq k$. That is, $\hat{q}_k$ is a shorthand for

$$\hat{q}_k := \underbrace{\hat{1} \otimes \cdots \otimes \hat{1} \otimes \overbrace{\hat{q}}^{k\text{th}} \otimes \hat{1} \otimes \cdots \otimes \hat{1}}_{N\text{-fold tensor products}},$$

and the same rule is applied for $\hat{p}_k$. Hence the CCR is generalized to $\hat{q}_k\hat{p}_l - \hat{p}_l\hat{q}_k = \mathrm{i}\hbar\delta_{k,l}, k, l = 1, \cdots, N$, where $\delta_{k,l}$ is the Kronecker's delta. This is summarized in a single equation as

$$\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^\top - (\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^\top)^\top = \mathrm{i}\hbar\Theta_N, \tag{2}$$
$$\Theta_N := I_N \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

We here give a direct consequence of the CCR, which will become an equivalent representation of the above difference in Gaussian case. In quantum theory, a physical quantity is described by a suitable self-adjoint operator called an *observable*. The measurement results of an observable distribute probabilistically [19]. In particular, the expectation of the measurement results of an observable $\hat{X}$ is given by $\mathbb{P}_{\hat{\rho}}(\hat{X}) := \mathrm{Tr}(\hat{\rho}\hat{X})$, where $\hat{\rho}$ is a positive semidefinite operator satisfying $\mathrm{Tr}(\hat{\rho}) = 1$. This is called a *density operator* or simply called a *state*, because a density operator contains all the information determining the statistics of the measurement results. Now the covariance matrix of Eq. (1) given a suitable state $\mathbb{P}_{\hat{\rho}}$ is calculated as

$$\begin{aligned}\mathbb{P}_{\hat{\rho}}(\Delta\hat{\boldsymbol{x}}\Delta\hat{\boldsymbol{x}}^\top) =& \frac{1}{2}\mathbb{P}_{\hat{\rho}}\left(\Delta\hat{\boldsymbol{x}}\Delta\hat{\boldsymbol{x}}^\top + (\Delta\hat{\boldsymbol{x}}\Delta\hat{\boldsymbol{x}}^\top)^\top\right) \\ &+ \frac{1}{2}\mathbb{P}_{\hat{\rho}}\left(\Delta\hat{\boldsymbol{x}}\Delta\hat{\boldsymbol{x}}^\top - (\Delta\hat{\boldsymbol{x}}\Delta\hat{\boldsymbol{x}}^\top)^\top\right) \\ =& V(\hat{\boldsymbol{x}}) + \mathrm{i}\frac{\hbar}{2}\Theta_N \geq 0,\end{aligned} \tag{3}$$

where $\Delta\hat{\boldsymbol{x}} := \hat{\boldsymbol{x}} - \mathbb{P}_{\hat{\rho}}(\hat{\boldsymbol{x}})$. This is the so-called uncertainty relation. In particular, when $N = 1$ Eq. (3) reduces to $\mathbb{P}_{\hat{\rho}}(\Delta\hat{q}^2)\mathbb{P}_{\hat{\rho}}(\Delta\hat{p}^2) \geq \hbar^2/4$, implying that the canonical conjugate pairs cannot be specified simultaneously. In other words, statistical uncertainty must have a strict lower bound in quantum case. We call $V(\hat{\boldsymbol{x}})$ the symmetrized covariance matrix of $\hat{\boldsymbol{x}}$. Also a real symmetric positive matrix X is called quantum if it satisfies $X + \hbar\mathrm{i}\Theta_N/2 \geq 0$.

B. Gaussian state

A Gaussian state is a state that is fully characterized by only the first and second moments, similar to the classical case (see for instance [1]). To define it let us introduce the *Wigner function*:

$$f_{\hat{\rho}}(\boldsymbol{x}) := \frac{1}{(2\pi)^{2N}}\int_{\mathbb{R}^{2N}} \mathrm{Tr}\left[\hat{\rho}e^{\mathrm{i}\boldsymbol{k}^\top\Theta_N\hat{\boldsymbol{x}}}\right]e^{\mathrm{i}\boldsymbol{k}^\top\Theta_N\boldsymbol{x}}\mathrm{d}\boldsymbol{k},$$

where $\hat{\rho}$ is a given state. $f_{\hat{\rho}}(\boldsymbol{x})$ is in one-to-one correspondence to $\hat{\rho}$ [21]. A state $\hat{\rho}$ is called Gaussian if the corresponding Wigner function is Gaussian:

$$f_{\hat{\rho}}(\boldsymbol{x}) = \frac{\hbar^N}{\pi^N}\frac{1}{\sqrt{\det(V)}}\exp\left(-\frac{(\boldsymbol{x}-\boldsymbol{\mu})^\top V^{-1}(\boldsymbol{x}-\boldsymbol{\mu})}{2}\right),$$

with $\boldsymbol{\mu}$ a real $2N$-dimensional vector and $V \in \mathbb{R}^{2N\times 2N}$ a quantum matrix. In particular in this case we have

$$\mathbb{P}_{\hat{\rho}}(\hat{\boldsymbol{x}}) = \int_{\mathbb{R}^{2N}} \boldsymbol{x}f_{\hat{\rho}}(\boldsymbol{x})\mathrm{d}\boldsymbol{x} = \boldsymbol{\mu},$$
$$V(\hat{\boldsymbol{x}}) = \int_{\mathbb{R}^{2N}} (\boldsymbol{x}-\boldsymbol{\mu})(\boldsymbol{x}-\boldsymbol{\mu})^\top f_{\hat{\rho}}(\boldsymbol{x})\mathrm{d}\boldsymbol{x} = V.$$

That is, $\boldsymbol{\mu}$ and V are exactly the mean and the covariance matrix for the Gaussian state $\hat{\rho}$, respectively.

The condition of V to be a quantum matrix is necessary for $\boldsymbol{x}$ to satisfy the CCR. Conversely, given a Gaussian state we always construct Eq. (3), implying the CCR. That is, in Gaussian case the uncertainty relation $V + \hbar\mathrm{i}\Theta_N/2 \geq 0$ holds for a quantum state while $V \geq 0$ for a classical one, and this is only a difference between quantum and classical states.

C. Linear quantum system

A linear quantum system with linear measurements is represented by the following set of equations (see Fig. 1):

$$\mathrm{d}\hat{\boldsymbol{x}}_t = \hbar A\hat{\boldsymbol{x}}_t\mathrm{d}t + \hbar B_u u_t\mathrm{d}t + \hbar\Theta_N B(\boldsymbol{\alpha}\mathrm{d}t + \mathrm{d}\hat{\boldsymbol{W}}_t), \tag{4}$$
$$\mathrm{d}\hat{\boldsymbol{Y}}_t = 2\Theta_M B^\top\hat{\boldsymbol{x}}_t\mathrm{d}t + \boldsymbol{\alpha}\mathrm{d}t + \mathrm{d}\hat{\boldsymbol{W}}_t, \tag{5}$$
$$\begin{aligned}\mathrm{d}\boldsymbol{y}_t =& \sqrt{H}DS\mathrm{d}\hat{\boldsymbol{Y}}_t + \sqrt{H}\mathrm{d}\boldsymbol{w}_t \\ =& C\hat{\boldsymbol{x}}_t\mathrm{d}t + \sqrt{H}DS(\boldsymbol{\alpha}\mathrm{d}t + \mathrm{d}\hat{\boldsymbol{W}}_t) + \sqrt{H}\mathrm{d}\boldsymbol{w}_t,\end{aligned} \tag{6}$$

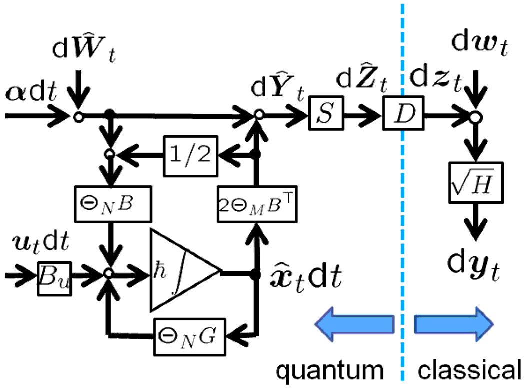

Fig. 1. Block diagram of the linear quantum system with linear measurements.

where $\hbar$ is the Dirac constant. Note that $\hat{\boldsymbol{x}}_0$ satisfies (2). All parameters in the above equations are defined as follows:

$$
\begin{aligned}
&A := \Theta_N(G + B\Theta_M B^\top),\ C := 2\sqrt{H}DS\Theta_M B^\top,\\
&G = G^\top \in \mathbb{R}^{2N\times 2N},\quad B = \mathbb{R}^{2N\times 2M},\quad B_u \in \mathbb{R}^{2N\times m},\\
&D \in \{D \in \mathbb{R}^{p\times 2M} \mid DD^\top = I_p,\ D\Theta_M D^\top = O_p\},\\
&S \in \{S \in \mathbb{R}^{2M\times 2M} \mid S\Theta_M S^\top = \Theta_M\},\\
&H = (DSV_{\rm qn}S^\top D^\top + V_{\rm m})^{-1},\\
&V_{\rm m} \geq 0,\quad V_{\rm qn} \in \{V \in \mathbb{R}^{2M\times 2M} \mid V + {\rm i}\Theta_M \geq 0\}.
\end{aligned}
$$

N, M, and m are positive integers and p is a positive integer satisfying $p \leq M$. Now we describe the variables: The system variable $\hat{\boldsymbol{x}}_t$ is the generalized canonical conjugate pairs (1). $\hat{\boldsymbol{W}}_t$ is the $2M$-dimensional vector with canonical conjugate pairs as well, though this represents the noise; that is, $\hat{\boldsymbol{W}}_t$ is the *quantum Wiener noise* with 0 means and symmetric covariance matrix $V_{\rm qn}t$ defined in the following *quantum Ito rule* (see [8], [20] for instance):

$$
{\rm d}\hat{\boldsymbol{W}}_t{\rm d}\hat{\boldsymbol{W}}_t^\top = (V_{\rm qn} + {\rm i}\Theta_M){\rm d}t,\quad {\rm d}\hat{\boldsymbol{W}}_t{\rm d}t = 0. \tag{7}
$$

$\boldsymbol{\alpha} \in \mathbb{R}^{2M}$ is the external quantum input, e.g., amplitude of an input laser, while $u_t \in \mathbb{R}^m$ is the classical control input. $\hat{\boldsymbol{Y}}_t$ and $\hat{\boldsymbol{Z}}_t$ are the system output and the transmitted output through some physical device, respectively. z_t is the classical signal transfered through the detector D. $\boldsymbol{w}_t$ is a classical measurement Wiener noise with 0 mean and covariance matrix $V_{\rm m}t$. We can only observe the classical output signal $z_t + \boldsymbol{w}_t$ and $\boldsymbol{y}_t$ denotes the normalized one; i.e., ${\rm d}\boldsymbol{y}_t({\rm d}\boldsymbol{y}_t)^\top = I_p{\rm d}t$.

Eq. (4) preserves the CCR, i.e., we have $\hat{\boldsymbol{x}}_t\hat{\boldsymbol{x}}_t^\top - (\hat{\boldsymbol{x}}_t\hat{\boldsymbol{x}}_t^\top)^\top = {\rm i}\hbar\Theta_N,\ \forall t$. Clearly, in classical case the corresponding algebraic relation must be $\hat{\boldsymbol{x}}_t\hat{\boldsymbol{x}}_t^\top - (\hat{\boldsymbol{x}}_t\hat{\boldsymbol{x}}_t^\top)^\top = 0$. Hence in this sense the quantum dynamics (4) differs from a classical dynamics.

III. Equivalent classical system

In this section, we provide a classical system that is equivalent to the quantum system (4), (5), and (6) in the sense that both dynamics satisfy the uncertainty relation (3), at least at steady state.

As mentioned in the previous section, Eq. (4) preserves the CCR: $\hat{\boldsymbol{x}}_t\hat{\boldsymbol{x}}_t^\top - (\hat{\boldsymbol{x}}_t\hat{\boldsymbol{x}}_t^\top)^\top = {\rm i}\hbar\Theta_N,\ \forall t$. This directly means that the uncertainty relation is always satisfied, i.e., $V_t + \hbar{\rm i}\Theta_N/2 \geq 0,\ \forall t$. In particular, if V_∞ exists, or equivalently if A is Hurwitz, we have $V_\infty + \hbar{\rm i}\Theta_N/2 \geq 0$. Now, as in the classical case, it is straightforward to see that the time evolution of $V_t = V(\hat{\boldsymbol{x}}_t)$ is given by the following Lyapunov differential equation:

$$
\frac{\rm d}{{\rm d}t}V_t = \hbar AV_t + \hbar V_t A^\top + \hbar^2\Theta_N BV_{\rm qn}B^\top\Theta_N^\top. \tag{8}
$$

When A is Hurwitz, this equation has a unique steady solution V_∞, which does not depend on V_0. Therefore, combining the above facts we conclude that, when A is Hurwitz, V_∞ is a quantum matrix even if V_0 is not. That is, the "quantumness" in the sense of uncertainty relation comes from the specific structure of the system matrices, or equivalently the block diagram structure depicted in Fig. 1.

Based on the above observation we here consider the following classical stochastic system:

$$
\begin{aligned}
{\rm d}\boldsymbol{x}_t &= \hbar A\boldsymbol{x}_t{\rm d}t + \hbar B_u u_t{\rm d}t + \hbar\Theta_N B(\boldsymbol{\alpha}{\rm d}t + {\rm d}\boldsymbol{v}_t), && (9)\\
{\rm d}\boldsymbol{Y}_t &= 2\Theta_M B^\top\boldsymbol{x}_t{\rm d}t + \boldsymbol{\alpha}{\rm d}t + {\rm d}\boldsymbol{v}_t, && (10)\\
{\rm d}\boldsymbol{y}_t &= \sqrt{H}DS{\rm d}\boldsymbol{Y}_t + \sqrt{H}{\rm d}\boldsymbol{w}_t. && (11)
\end{aligned}
$$

This is exactly the classical system obtained just by replacing the canonical conjugate variables in Eqs. (4), (5), and (6) by the corresponding scalar variables. That is, $\boldsymbol{x}_t \in \mathbb{R}^{2N}$, $\boldsymbol{x}_0 \in \mathbb{R}^{2N}$, $\boldsymbol{Y}_t \in \mathbb{R}^{2M}$, $\boldsymbol{v}_t \in \mathbb{R}^{2M}$, and $\boldsymbol{w}_t \in \mathbb{R}^p$ are all classical Gaussian random variables with the same means and covariance matrices, respectively. The point is that, the covariance matrix for this system, V_t^c, obeys exactly the same Lyapunov equation, i.e.,

$$
\frac{\rm d}{{\rm d}t}V_t^c = \hbar AV_t^c + \hbar V_t^c A^\top + \hbar^2\Theta_N BV_{\rm qn}B^\top\Theta_N^\top, \tag{12}
$$

thus consequently V_t^c has the same property as V_t. This means that, when V_0^c is a quantum matrix, V_t^c is also quantum for all $t \geq 0$ despite the lack of CCR in this case. Moreover, as in the quantum case, if A is Hurwitz, V_∞^c exists and is quantum even when V_0^c is not. That is, for the classical system (9), (10), and (11) with A Hurwitz, the "quantumness" is automatically satisfied at least at steady state. Hence, our conclusion is that this is an equivalent classical system to the linear quantum system, in the sense that both systems satisfy the quantum statistics at steady state without respect to the initial condition. This implies that the solution to a certain control problem for the classical system (9), (10), and (11) is at the same time the solution to the corresponding quantum control problem. In particular, independence on the initial condition is significant, because a controller synthesis is usually carried out without taking the initial condition into account. Moreover, because in the long time limit the uncertainty relation is automatically recovered, we can perform a numerical simulation of the dynamical behavior of $\boldsymbol{x}_t$ with a fixed initial state $\boldsymbol{x}_0$, which means the violation of the uncertainty relation at $t = 0$.

Remark 1: If A is not Hurwitz, we need to set V_0^c to be quantum. For example, when $B = 0$ we have $\det(V_t^c) = \det(V_0^c)$, implying that the uncertainty relation is violated for all time if it is at $t = 0$.

Remark 2: From the *quantum filtering theory* [22], we can construct the optimal estimate of $\hat{\boldsymbol{x}}_t$ in the sense of least mean squared error. This is given by the *quantum conditional expectation* $\hat{\pi}_t(\hat{\boldsymbol{x}}) = \mathbb{P}(\hat{\boldsymbol{x}}_t|\mathcal{Y}_t)$, where $\mathcal{Y}_t = \sigma\{y_s \mid s \leq t\}$ is the σ-algebra representing the measurement data up to time t. The optimal estimate can be computed recursively using the following *quantum Kalman filter*:

$$\begin{aligned} \mathrm{d}\hat{\pi}_t(\hat{\boldsymbol{x}}) =& \hbar A\hat{\pi}_t(\hat{\boldsymbol{x}})\mathrm{d}t + \hbar(B_u u_t + \Theta_N B\boldsymbol{\alpha})\mathrm{d}t \\ &+ (V_t C^\top + F)(\mathrm{d}\hat{\boldsymbol{y}}_t - C\hat{\pi}_t(\hat{\boldsymbol{x}})\mathrm{d}t - \sqrt{H}DS\boldsymbol{\alpha}\mathrm{d}t), \quad (13) \\ \frac{\mathrm{d}}{\mathrm{d}t}V_t^{\mathrm{R}} =& \hbar A V_t^{\mathrm{R}} + \hbar V_t^{\mathrm{R}} A^\top + \hbar^2 \Theta_N B V_{\mathrm{qn}} B^\top \Theta_N^\top \\ &- (V_t^{\mathrm{R}} C^\top + F)(V_t^{\mathrm{R}} C^\top + F)^\top, \quad (14) \end{aligned}$$

where $F := \hbar\Theta_N B V_{\mathrm{qn}} S^\top D^\top \sqrt{H}$ and V_t^{R} is the estimation error covariance matrix. V_t^{R} is a quantum matrix when V_0^{R} is quantum. The vital point is that, for the classical system (9), (10), and (11), the optimal estimate of $\boldsymbol{x}_t$ follows the same Kalman filter as above. Therefore, we can again stress the classical-quantum equivalence discussed before in the sense that the output statistics are also compatible. However, it should be noted that only for the smoothing problem there is no such a correspondence; see [23], [24] for more detailed description.

Remark 3: In classical case, a state-space representation has a freedom with respect to the similarity transformation $\boldsymbol{x}'_t = T\boldsymbol{x}_t$ with T any invertible matrix. However, for Eqs. (9), (10), and (11) to be an equivalent classical system to the quantum system, T must be *symplectic*, i.e., it has to satisfy $T\Theta_N T^\top = \Theta_N$, because by the similarity transformation the uncertainty relation changes to $V_t^c + \mathrm{i}\frac{\hbar}{2}T\Theta_N T^\top \geq 0$. However note this is not a constraint for controller synthesis.

IV. Application : Covariance Control and Entanglement Generation

Now we apply a classical control method to the linear quantum systems. Consider the problem of entanglement generation for two linear quantum systems. It is one of the main issues of quantum information technologies [15], and it is known that entanglement can be characterized by a covariance matrix for quantum Gaussian state [17], [25]. Hence, a problem of entangled state generation can be reduced to a problem to achieve specific covariance matrices, which is well known as the covariance assignment control problem in control theory [12], [13], [14]. We here apply this controller synthesis for the linear quantum systems.

A. Problem setting

Let $V := V_{\hat{\rho}}(\hat{\boldsymbol{x}}) \in \mathbb{R}^{4\times 4}$ be a covariance matrix under the state $\hat{\rho}$, and we divide V into 4 block matrices,

$$V = \left[\begin{array}{c|c} V_{11} & V_{12} \\ \hline V_{12}^\top & V_{22} \end{array}\right], \quad V_{11}, V_{12}, V_{22}, \in \mathbb{R}^{2\times 2}.$$

A given Gaussian state $\hat{\rho}$ is called entangled, if $\tilde{\nu}_-(V) < \frac{\hbar}{2}$ holds, where

$$\tilde{\nu}_-(V) := \frac{1}{\sqrt{2}}\sqrt{\tilde{\Delta}(V) - \sqrt{\tilde{\Delta}(V)^2 - 4\det(V)}}$$

and $\tilde{\Delta}(V) := \det(V_{11}) + \det(V_{22}) - 2\det(V_{12})$. One of the measures of Gaussian entanglement is so called *logarithmic negativity* [18] defined by

$$\mathrm{LN}_\hbar(V) := \max\{-\ln(\tilde{\nu}_-(V)/(\hbar/2)),\ 0\}.$$

The logarithmic negativity is widely used in physics literatures [25], [26]. The goal of this section is to generate a V satisfying $\mathrm{LN}_\hbar(V) > 0$ by classical feedback control.

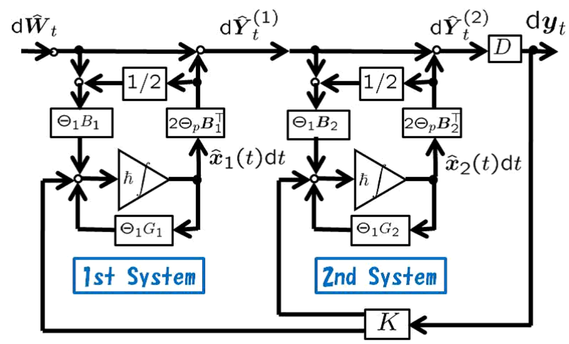

Fig. 2. Feedback controlled linear quantum systems connected by cascade.

Consider a linear quantum systems in Fig. 2 with $\hbar = 1$. The dynamics of the system is represented by

$$\begin{cases} \mathrm{d}\hat{\boldsymbol{x}}_t = A\hat{\boldsymbol{x}}_t\mathrm{d}t + K\mathrm{d}\boldsymbol{y}_t + \Theta_2 B\mathrm{d}\hat{\boldsymbol{W}}_t \\ \mathrm{d}y_t = C\hat{\boldsymbol{x}}_t\mathrm{d}t + D\mathrm{d}\hat{\boldsymbol{W}}_t \end{cases}$$

with

$$\begin{aligned} A &= \Theta_2 \begin{bmatrix} G_1 + B_1\Theta_p B_1^\top & O_2 \\ 2B_2\Theta_p B_1^\top & G_2 + B_2\Theta_p B_2^\top \end{bmatrix}, \\ C &= 2D\Theta_p B^\top,\ B = \begin{bmatrix} B_1^\top & B_2^\top \end{bmatrix}^\top, \end{aligned}$$

where $\hat{\boldsymbol{x}}_t = (\hat{\boldsymbol{x}}_1(t)^\top,\ \hat{\boldsymbol{x}}_2(t)^\top)^\top$, $G_1 = G_1^\top$, $G_2 = G_2^\top \in \mathbb{R}^{2\times 2}$ and $B_1, B_2 \in \mathbb{R}^{2\times 2p}$, $D \in \mathbb{R}^{p\times 2p}$ and $K \in \mathbb{R}^{4\times p}$. The time evolution of the state covariance matrix $V_t := V(\hat{\boldsymbol{x}}_t)$ is expressed as

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}V_t =& (A + KC)V_t + V_t(A + KC)^\top \\ &+ (KD + \Theta_2 B)(KD + \Theta_2 B)^\top. \quad (15) \end{aligned}$$

Note that the control input u_t depends on the information $\mathcal{Y}_{t+\mathrm{d}t}$ in this setting. We should emphasize that V_t always satisfies the uncertainty relation with direct feedback control input if V_0 is a quantum matrix. This is ensured by physical realization theorem [8], and the following relation holds:

$$\begin{aligned} &(A + KC)\Theta_2 + \Theta_2(A + KC)^\top \\ &+ 2(\Theta_2 B + KD)\Theta_p(\Theta_2 B + KD)^\top = O_4. \end{aligned}$$

Also note that there exists an unique Gaussian state $\hat{\rho}$ for any vector $\mu \in \mathbb{R}^{2N}$ and any quantum matrix $V \in \mathbb{R}^{2N\times 2N}$.

B. Entanglement generation by covariance assignment controller

Consider a following problem; for a given positive definite matrix V, to find a necessary and sufficient condition for the existence of a feedback gain K which assigns the V as a steady solution of Eq. (15). To this end, letting $\frac{\mathrm{d}}{\mathrm{d}t}V_t = O_4$ in (15), we have

$$\begin{aligned}&-(K+VC^\top+F)(K+VC^\top+F)^\top\\ =&AV+VA^\top+\Theta_2BB^\top\Theta_2^\top-(VC^\top+F)(VC^\top+F)^\top\end{aligned} \tag{16}$$

where $F := \Theta_2 BD^\top \in \mathbb{R}^{4\times p}$. It is seen from [12], [14] that the necessary and sufficient condition of the existence of the gain K satisfying (16) is the existence of $L \in \mathbb{R}^{4\times p}$ which satisfies

$$\begin{aligned}O_4 =&AV+VA^\top+\Theta_2BB^\top\Theta_2^\top\\ &-(VC^\top+F)(VC^\top+F)^\top+LL^\top\end{aligned} \tag{17}$$

If the above condition holds, then the gain K is set as

$$K = LU - VC^\top - F,$$

where $U \in \mathbb{R}^{p\times p}$ is any orthogonal matrix. If it is possible to choose an assignable positive definite matrix V with $\mathrm{LN}_1(V) > 0$, the static controller K given above equation generates an entangled state.

Note that it is impossible to assign any positive definite matrix as a steady state covariance matrix. A class of the assignable positive definite matrix is given by solutions of Eq. (17) with respect to V for each $L \in \mathbb{R}^{4\times p}$ and, at least, a class of assignable matrices is included in a class of quantum matrices. It is an open problem that the relationship between the assignable class of Gaussian states and the class of entangled Gaussian states. Furthermore, it is hard to find parameter L for the given desirable positive matrix, since Eq. (17) is an equality condition. We will show a simple method to find a set of assignable and entangled matrices in the next subsection.

Mancini and Wiseman [27] proposed a method to a generate quantum Gaussian entanglement by optimal feedback control in the special case of linear quantum systems. Their method is a generation of the most nearest covariance matrix to a covariance matrix of maximal entangled Gaussian state. Although their method may be attractive, the nearest covariance matrix does not imply entanglement because the notion of neighborhood of entangled Gaussian states is not clear yet. In contrast with this, the covariance assignment controller derived by our method assigns a desirable covariance matrix if there exists L satisfying (17).

Remark 4: We can check numerically whether an assignable entangled state exists or not. Let $V(L)$ be a solution of Eq. (17) for a given $L \in \mathbb{R}^{4\times p}$. Then if $\min_L \tilde{\nu}_-(V(L))$ is strictly less than $1/2$, then there is at least an assignable entangled state by feedback control. Though this is a nonlinear optimization problem, the problem becomes tractable when $p \geq 4$ and $LL^\top > O_4$. The positive definiteness of $LL^\top$ implies the following matrix inequality:

$$\begin{aligned}O_4 >&P(A-FC)+(A-FC)^\top P\\ &+P(\Theta_2BB^\top\Theta_2^\top-FF^\top)P-C^\top C,\end{aligned} \tag{18}$$

where $P := V^{-1}$. Applying the Schur complement (e.g., see [14]) to (18), we can formulate the following numerical optimization problem:

$$\begin{aligned}&\min_P \tilde{\nu}_-(P^{-1})\\ &\text{s.t. } P = P^\top > O_4,\\ &\begin{bmatrix}P(A-FC)+(A-FC)^\top P-CC^\top & P\sqrt{R}\\ \sqrt{R}P & -I\end{bmatrix} < O_8,\end{aligned}$$

where $R := \Theta_2BB^\top\Theta_2^\top - FF^\top$ and $R \geq O_4$. Although this is also a nonlinear optimization problem, its constraints are LMIs.

C. Numerical examples

As we mentioned above, it is difficult to find assignable positive definite matrices in general. Here, we show that there exists a system which has assignable entangled states.

Consider two controlled linear quantum systems which connected cascade as in Fig. 2 with following parameters:

$$G_1 = G_2 = I_2,\ B_1 = \Theta_1,\ B_2 = \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix},\ D = \begin{pmatrix}1 & 0\end{pmatrix}.$$

Since (A, B) is controllable and (A, C) is observable, there exists a unique positive definite real symmetric solution of Eq. (17) with $L = 0$. We can see from the comparison theorem (see, e.g., [28]) that there exists a unique positive definite real symmetric matrix solution of Eq. (17) for any fixed $L \in \mathbb{R}^{4\times 1}$. Therefore, the assignable covariance matrices are characterized by the free parameter L and we can check numerically whether the intersection between the sets of assignable matrices and entangled states is nonempty or not.

Fortunately, we can see the existence of the intersection. Set the free parameter L as

$$\begin{aligned}L =&a\begin{pmatrix}0 & \cos(\theta) & \sin(\theta) & 0\end{pmatrix}^\top,\\ a =&0.1, 1, 2, \quad \theta = 5, 10, \cdots, 180 \text{ [deg]},\end{aligned}$$

and calculate the logarithmic negativity of resultant assignable covariance V by solving (17) for each L.

The result is illustrated in Fig. 3. Fig. 3 is a plot of logarithmic negativities of the feedback controlled Gaussian states as a function of the parameter θ. The solid red line, the chained green line and the dashed blue line in Fig. 3 describe $a = 0.1$, $a = 1$ and $a = 2$, respectively. We can see that the logarithmic negativity decreases as $|L|$ increase. This fact can be interpreted by one of another criteria of the quantum entanglement [17] : a quantum matrix $V \in \mathbb{R}^{4\times 4}$ is entangled if and only if it satisfies

$$\begin{aligned}&\frac{\hbar^2}{4}(\det(V_{11})+\det(V_{22})+2\det(V_{12}))\\ \leq&\frac{\hbar^4}{16}+\det(V) < \frac{\hbar^2}{4}\tilde{\Delta}(V).\end{aligned} \tag{19}$$

Roughly speaking, V increases as $|L|$ increases. A trivial facts $\det(bV) = b^4 \det(V)$ and $\tilde{\Delta}(bV) = b^2 \Delta(\tilde{V})$ for $b \in \mathbb{R}$ imply that large $|L|$ easily breaks the above entanglement condition. We can also see that a decrease of the logarithmic negativity around $\theta = \pi/4$ is less than others. $\theta = \pi/4$ means that information of the output is equivalently fed into each system. It is intuitive to keep the correlation between two systems.

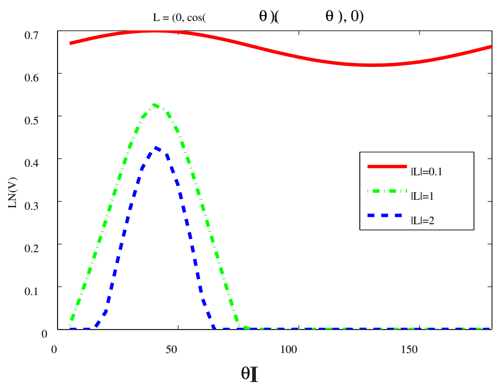

Fig. 3. Plot of logarithmic negativities of the feedback controlled Gaussian states as a function of the parameter θ. The solid red line, the chained green line and the dashed blue line describe $a = 0.1$, $a = 1$ and $a = 2$, respectively.

Though it seems that $|L| = 0$ is better than $|L| > 0$ for entanglement generation because the control gain with $|L| = 0$ must be generate the largest entanglement, we should note that a large entanglement does not imply an useful entanglement as mentioned in [26]. To explain this fact, consider atoms with two level energy states in cavities. These atoms represent qubits, which are fundamental for quantum information technologies. It is important to generate a quantum entanglement of qubits for quantum information processing. The atoms interact with cavity modes, which is represented as a linear quantum system. If the cavity modes have a quantum entanglement, then it transfers to a quantum entanglement of qubits via interaction. McHugh et.al. [26] showed that a large quantum entanglement of cavity modes does not imply a large resultant quantum entanglement of qubits. Since it is not clear which covariance matrix is better to generate the large quantum entanglement of qubits, the free parameter L with $|L| > 0$ give a degree of freedom to assign useful covariance matrices.

V. Conclusion

In this paper, we have provided a classical system that is equivalent to a linear quantum system in the sense that both dynamics satisfy the uncertainty relation at least at steady state. This result allows us to apply a solution for a specific linear classical control problem to the corresponding quantum case, directly. Also for such an equivalent classical system we can carry out various numerical simulations, which should be useful in analyzing dynamical behavior of the system variables as well as synthesizing an efficient feedback controller. Note that the initial condition need not satisfy the uncertainty relation in these simulations; this is practically a convenient fact because we can set a certain fixed initial condition in the simulation. Finally, based on this quantum-classical equivalence result, an application of the covariance assignment control to the quantum case was shown. We believe this will be a powerful approach to the problems of controlling quantum entanglement.

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