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On the Swing-Up of the Pendubot Using Virtual Holonomic Constrains

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***Abstract*— We investigate the problem of stabilizing energy level sets for Euler-Lagrange systems subject to virtual holonomic constraints. We present an energy level set stabilization technique with a guaranteed domain of attraction which preserves the invariance of the constraint manifold. As an illustration of the theory, we present a controller which swings up the Pendubot system while guaranteeing that the unactuated link does not fall over during transient.**

I. Introduction

Recent work by Jessy Grizzle and collaborators on biped locomotion (see, e.g., [1], [2], [3], [4]) has transformed the classical view of the motion control problem, in which one first solves a motion planning problem to generate reference signals, and subsequently one designs a controller to asymptotically track these reference signals. Grizzle showed that the correct way to enforce a desired gait in a walking robot is to enforce by feedback desired relations between the joint angles of the robot. Such relations are called *virtual holonomic constraints (VHCs)* because they depend on the generalized coordinates of the robot, and not on its generalized velocities. Grizzle's work triggered interest of various researchers on virtual holonomic constraints as a paradigm for motion control. In particular, we refer the reader to the work of Shiriaev and collaborators in [5], [6], [7], [8], [9]. Inspired by Grizzle's work, in [10] we initiated a systematic investigation of virtual holonomic constraints for underactuated Euler-Lagrange systems. Our work in [10] provides answers to four questions:

1) When is a virtual holonomic constraint (VHC) feasible?
2) How to enforce a VHC via feedback?
3) How to systematically select VHCs that are feasible?
4) When are the constrained dynamics Euler-Lagrange?

In [11] we developed a technique to address a fifth problem:

5) How to stabilize a desired level set of the energy of the constrained system while simultaneously enforcing a VHC?

Then, we applied our theory to the pendubot system, depicted in Figure 1, which is a double-pendulum with actuator on the shoulder [12]. For this system, we enforced a VHC specifying what should be the angle of the second link as a function of the angle of the first link. Simultaneously, we stabilized the energy level set of the constrained motion containing the unstable high-high equilibrium. The result was intriguing: our feedback not only swings up the pendubot from the high-low to the high-high equilibrium, but it does so while guaranteeing that during transient the unactuated link does not fall over. The analysis in [11] guarantees local asymptotic stability of the target energy level set of the constrained system. In this paper, we present an alternative technique solving problem (5) above which has the advantage of yielding exponential stability of a target energy level set of the constrained motion with a guaranteed domain of attraction, or even global asymptotic stability. The technique we present is currently only applicable to systems with two degrees-of-freedom.

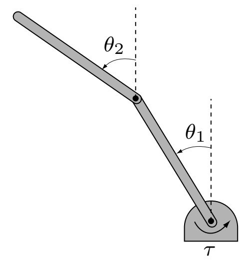

Fig. 1. The pendubot.

This paper is organized as follows. The virtual holonomic constraint theory of [10] is reviewed in Section II. In Section III we present a novel approach to stabilize a level set of the energy for the reduced system describing the motion on the virtual constrained manifold. In Section IV this approach is applied to the pendubot system.

II. Virtual holonomic constraints

In this section we review the theory of [10].

Consider an Euler-Lagrange system

$$D(q)\ddot{q} + C(q,\dot{q})\dot{q} + \nabla P(q) = B\tau, \tag{1}$$

with n degrees-of-freedom and $n-1$ controls. The matrix B is assumed to have full rank $n-1$. Consider a **virtual holonomic constraint (VHC)** of the form

$$\mathrm{col}(q_1,\dots,q_{n-1}) = \mathrm{col}(\phi_1(q_n),\dots,\phi_{n-1}(q_n)) = \phi(q_n),$$

where $q_n \in S^1$ is an angular configuration variable parametrizing the constraint. More generally, one could define a VHC to be an implicit relation $h(q) = 0$ but the explicit description above is sufficient and convenient for our purposes. Throughout this paper, we let $\hat{\phi}(q_n) = \mathrm{col}(\phi(q_n), q_n)$, so that we can conveniently express the constraint as $q = \hat{\phi}(q_n)$.

Definition 2.1: A VHC $q = \hat{\phi}(q_n)$ is **feasible** if the set

$$\Gamma = \{(q,\dot{q}) : \mathrm{col}(q_1,\dots,q_{n-1}) = \phi(q_n),\\ \mathrm{col}(\dot{q}_1,\dots,\dot{q}_{n-1}) = \phi'(q_n)\dot{q}_n\},$$

is controlled invariant, i.e., if it can be made invariant by a suitable feedback $\tau(q,\dot{q})$. We call the set Γ the **constraint manifold**.

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Γ is a two-dimensional embedded submanifold of the state space, and being parametrized by $(q_n, \dot{q}_n)$, it is diffeomorphic to the cylinder $S^1 \times \mathbb{R}$. The controlled invariance of the constraint manifold expresses the fact that whenever the configuration variable $q(0)$ is initialized on the constraint, and its initial velocity $\dot{q}(0)$ is tangent to the constraint, a suitable feedback makes the resulting solution $q(t)$ satisfy the constraint for all t.

Proposition 2.2 ([10]): A VHC $q = \hat{\phi}(q_n)$ is feasible if

$$(\forall q_n \in S^1)\ \operatorname{Im}(D(\hat{\phi}(q_n))\hat{\phi}'(q_n)) \cap \operatorname{Im}(B) = \{0\}$$

or, equivalently, if

$$B^{\perp} D(\hat{\phi}(q_n))\hat{\phi}'(q_n) \neq 0, \tag{2}$$

where $B^{\perp}$ is a nonzero row vector such that $B^{\perp} B = 0$. Moreover, the output function $e = \operatorname{col}(q_1, \ldots, q_{n-1}) - \phi(q_n)$ yields a vector relative degree $\{2, \cdots, 2\}$ on Γ, and therefore the constraint manifold Γ is locally exponentially stabilizable.

A feedback which exponentially stabilizes Γ is

$$\tau(q,\dot{q}) = \left\{[I_{n-1}\ \ -\phi'(q_n)]D^{-1}(q)B(q)\right\}^{-1} \Big[-k_1 e - k_2 \dot{e} + \phi''(q_n)\dot{q}_n^2 + [I_{n-1}\ \ -\phi'(q_n)]D^{-1}(q)(C(q,\dot{q})\dot{q} + \nabla P(q))\Big],$$

where $k_1, k_2 > 0$ are design parameters and $e = \operatorname{col}(q_1, \ldots, q_{n-1}) - \phi(q_n)$, $\dot{e} = \operatorname{col}(\dot{q}_1, \ldots, \dot{q}_{n-1}) - \phi'(q_n)\dot{q}_n$. A VHC $q = \hat{\phi}(q_n)$ satisfying (2) will be called **regular**. Hence, regular VHC's are feasible. The mechanical interpretation of the regularity property is this. The generalized momentum of a solution $(q(t), \dot{q}(t))$ satisfying the virtual constraint at all time is $D(\hat{\phi}(q_n))\hat{\phi}'(q_n)\dot{q}_n$. The regularity condition implies that, on Γ, it is always possible to choose τ such that the generalized momentum is compatible with motion on Γ.

There is a systematic way to *generate* regular holonomic constraints as solutions of a scalar ordinary differential equation. The idea is to select $n-2$ of the $n-1$ required functions in $\phi(q_n)$, for instance $\phi_2(q_n), \ldots, \phi_{n-1}(q_n)$, and find a function $\phi_1(q_n)$ satisfying the equation $B^{\perp} D(\hat{\phi}(q_n))\hat{\phi}'(q_n) = \delta(q_n)$, where $\delta(q_n)$ is a nonzero function $S^1 \to \mathbb{R}\backslash\{0\}$ to be assigned. The latter equation can be rewritten as

$$f_1(\phi_1, q_n)\frac{d\phi_1}{dq_n} + f_2(\phi_1, q_n) = \delta(q_n). \tag{3}$$

The above is a T-periodic ordinary differential equation for ϕ_1, where T is the period of the angular variable q_n. If, for a given $\delta(q_n) : S^1 \to \mathbb{R}\backslash\{0\}$, (3) has a T-periodic solution $\phi_1(q_n)$, then this function together with the functions $\phi_2(q_n), \ldots, \phi_{n-1}(q_n)$ forms a regular holonomic constraint. For this reason, we call (3) a **virtual constraint generator (VCG)**. The issue then becomes whether, for a given initial condition, it is possible to choose $\delta \neq 0$ such that the solution of (3) is T-periodic. The answer to this question for the case when the ODE (3) has no singularities is contained in the next

Proposition 2.3 ([10]): Consider equation (3) and suppose that $f_1 \neq 0$. Fix an initial condition $\phi_1(q_{n0}) = \phi_0$. There exists a C^1 function $\delta(q_n) : S^1 \to \mathbb{R}\backslash\{0\}$ such that the solution $\phi_1(q_n)$ is T-periodic if and only if the solution when $\delta = 0$ is not T-periodic, and in this case $\delta(q_n)$ can be chosen as follows. Choose a C^1 function $\mu(q_n) : S^1 \to \mathbb{R}\backslash\{0\}$ and let $\delta(q_n) = \epsilon\mu(q_n)$. Then, there exists a unique $\epsilon \neq 0$ such that the solution of (3) is T-periodic.

The proof of sufficiency is found in Lemma 3.1 of [10], while that of necessity is obvious and is omitted.

Once a regular VHC has been found, the motion on the virtual constraint manifold is found by left-multiplying both sides of (1) by $B^{\perp}$, letting $q = \hat{\phi}(q_n)$, $\dot{q} = \hat{\phi}'(q_n)\dot{q}_n$, $\ddot{q} = \hat{\phi}'(q_n)\ddot{q}_n + \hat{\phi}''(q_n)\dot{q}_n^2$, and using the fact that $B^{\perp} D(\hat{\phi}(q_n))\hat{\phi}'(q_n) = \delta(q_n) \neq 0$. Doing so, one obtains

$$\ddot{q}_n = -\frac{B^{\perp}(\hat{\phi}(q_n))}{\delta(q_n)}\Big[D\,\hat{\phi}''(q_n)\dot{q}_n^2 + C\,\dot{q} + \nabla P\Big]_{\substack{q = \hat{\phi}(q_n),\\ \dot{q} = \hat{\phi}'(q_n)\dot{q}_n}}.$$

Using the structure of the matrix C, the above can be put in the form

$$\ddot{q}_n = \Psi_1(q_n) + \Psi_2(q_n)\dot{q}_n^2. \tag{4}$$

As pointed out earlier, Γ is parametrized by $(q_n, \dot{q}_n)$, and so the system above describes the dynamics of the system on the virtual constraint manifold. Note that system (4) is unforced. This is because the original system (1) has degree of underactuation one, and all control directions are used to make Γ invariant. As shown in [13], the constrained dynamics (4) are not, in general, Euler-Lagrange or Hamiltonian. However, under the following conditions they are in fact Euler-Lagrange:

- C1 $D(q)$, $P(q)$, and $B(q)$ in the original system (1) are even functions.
- C2 $\phi_2(q_n), \ldots, \phi_{n-1}(q_n)$ are chosen to be odd functions.
- C3 In Proposition 2.3, the initial condition is chosen to be $\phi_1(0) = 0$, and $\mu(q_n)$ to be an even function.

Throughout the next section we will assume that conditions C1-C3 above hold. Under these conditions, the Lagrangian function is $\mathcal{L}(q_n, \dot{q}_n) = \frac{1}{2}M(q_n)\dot{q}_n^2 - V(q_n)$, where

$$\begin{aligned} M(q_n) &= \exp\left\{-2\int_0^{q_n} \Psi_2(\tau)d\tau\right\} \\ V(q_n) &= -\int_0^{q_n} \Psi_1(\mu)M(\mu)d\mu. \end{aligned} \tag{5}$$

The total energy of the system evolving on the constraint manifold is

$$E(q_n, \dot{q}_n) = \frac{1}{2}M(q_n)\dot{q}_n^2 + V(q_n). \tag{6}$$

III. Energy level stabilization on constraint manifold

Suppose we have found a regular VHC, $q = \hat{\phi}(q_n)$, chosen according to the procedure reviewed in the previous section, and such that the motion on the constraint manifold in equation (4) is Euler-Lagrange with energy $E(q_n, \dot{q}_n) = (1/2)M(q_n)\dot{q}_n^2 + V(q_n)$. The objective now is, for a given constant $E_0 \in \mathbb{R}$, to stabilize the set $\Lambda \subset \Gamma$ given by $\Lambda = \{(q, \dot{q}) \in \Gamma : E(q_n, \dot{q}_n) = E_0\}$. Λ is the union of a finite number of phase curves of (4). Let $\underline{V} = \min_{q_n \in S^1} V(q_n)$ and $\bar{V} = \max_{q_n \in S^1} V(q_n)$. Then, for all $E_0 > \bar{V}$, Λ is the union of two closed curves parametrized by q_n with opposite orientations: $\dot{q}_n = \pm\sqrt{(2/M)(E_0 - V)}$. Such motions correspond to complete revolutions of the angular variable q_n, and therefore we call them **rotations.** For all $E_0 \in [\underline{V}, \bar{V}]$, if $V'(q_n) \neq 0$ for all $q_n \in V^{-1}(E_0)$, then Λ is the union of a finite number of closed phase curves homeomorphic to the circle $q_n^2 + \dot{q}_n^2 = 1$. These solutions correspond to motions where q_n oscillates without performing complete revolutions, and therefore we call them **oscillations.** Finally, if $V'(q_n) = 0$ for some $q_n \in V^{-1}(E_0)$, then Λ is the union of a finite number of closed phase curves, some of which contain equilibria.

Henceforth, we focus on the stabilization of a connected component of Λ and we replace Λ by the connected component of interest. Since the reduced dynamics in (4) are unforced, it is impossible to stabilize Λ while, at the same time, preserving the invariance of Γ. In [11], we presented an approach to stabilize Λ which preserves the invariance of the constraint manifold, but dynamically changes its geometry in order to introduce in equation (4) a new control parameter. The stability analysis in [11] was local. In this paper, we present a different method which has the advantage of affording a simple stability analysis, and it allows one to draw conclusions about the domain of attraction of Λ. Throughout this section we assume that system (1) has two degrees-of-freedom, i.e., $n = 2$.

Suppose that through the ideas summarized in Section II we have found a regular VHC, $q = \hat{\phi}(q_n)$, which is odd so that the reduced dynamics on the constraint manifold are Euler-Lagrange. Select C^1 functions $f_a(q_n), f_b(q_n)$ defined on S^1 that are odd (i.e., $f_a(-q_n) = -f_a(q_n)$ and $f_b(-q_n) = -f_b(q_n)$), and modify the VHC as follows

$$q = \hat{\phi}(q_n) + a\hat{f}_a(q_n) + b\hat{f}_b(q_n) \ , \tag{7}$$

where a, b are scalars to be determined later and $\hat{f}_a(q_n) = \mathrm{col}(f_a(q_n), 0)$, $\hat{f}_b(q_n) = \mathrm{col}(f_b(q_n), 0)$. This constraint is obviously odd for all a and b. Recall that $n = 2$ and consider the output function $e = q_1 - [\phi(q_n) + af_a(q_n) + bf_b(q_n)]$. We have

$$\dot{e} = \dot{q}_1 - [\phi'(q_n) + af'_a(q_n) + bf'_b(q_n)]\dot{q}_n - \dot{a}f_a(q_n) + \dot{b}f_b(q_n).$$

By setting

$$\dot{a} = f_b(q_n)v, \ \dot{b} = -f_a(q_n)v \ , \tag{8}$$

where v is a new control input, we obtain that $\dot{e}$ does not depend on v,

$$\dot{e} = \dot{q}_1 - [\hat{\phi}'(q_n) + af'_a(q_n) + bf'_b(q_n)]\dot{q}_n.$$

Next, on the set $\{e = \dot{e} = 0\}$ we have

$$\ddot{e} = f(q_n, \dot{q}_n, a, b, v) + \{[1 \ \ -(\phi' + af'_a + bf'_b)]D^{-1}B\}^{-1}\tau,$$

where f is a smooth function. Since, by construction, $\phi(q_n)$ was chosen so that $\{[1 - \phi']D^{-1}B\} \neq 0$, it follows that there exist $\bar{a}, \bar{b} > 0$ such that $\{[1 \ -(\phi' + af'_a + bf'_b)]D^{-1}B\} \neq 0$ for all $|a| < \bar{a}$, $|b| < \bar{b}$. Thus, for small enough a, b and for any v, system (1) with input τ the output e has relative degree 2. Define the virtual constraint manifold for system (1) augmented with the compensator (8) as follows

$$\begin{aligned}\tilde{\Gamma} = &\{(q, \dot{q}, a, b) : q_1 = \phi(q_n) + af_a(q_n) + bf_b(q_n),\\ &\dot{q}_1 = [\phi'(q_n) + af'_a(q_n) + bf'_b(q_n)]\dot{q}_n, (a, b) \in W\},\end{aligned}$$

where W is a neighborhood of $(a, b) = (0, 0)$ contained in $\{(a, b) : |a| < \bar{a}, |b| < \bar{b}\}$. This manifold is controlled invariant because we have shown that, on it, the system with input τ and output e has relative degree 2, and the feedback v can be designed so that the set W is positively invariant.

The dynamics on $\tilde{\Gamma}$ are found, once again, by left-multiplying (1) by $B^{\perp}$ and by letting

$$\begin{aligned}q_1 &= \phi(q_n) + af_a(q_n) + bf_b(q_n),\\ \dot{q}_1 &= [\phi'(q_n) + af'_a(q_n) + bf'_b(q_n)]\dot{q}_n,\\ \ddot{q}_1 &= [\phi'(q_n) + af'_a(q_n) + bf'_b(q_n)]\ddot{q}_n + [\phi''(q_n) + af''_a(q_n)\\ &\quad + bf''_b(q_n)]\dot{q}_n^2 + [f_b(q_n)f'_a(q_n) - f_a(q_n)f'_b(q_n)]\dot{q}_n v.\end{aligned} \tag{9}$$

From (8) and (9) we obtain the dynamics on the virtual constraint manifold $\tilde{\Gamma}$

$$\begin{aligned}\dot{a} &= f_b(q_n)v\\ \dot{b} &= -f_a(q_n)v\\ \ddot{q}_n &= \Psi_1(q_n, a, b) + \Psi_2(q_n, a, b)\dot{q}_n^2 + \Psi_3(q_n, a, b)\dot{q}_n v,\end{aligned} \tag{10}$$

where Ψ_i, $i = 1, 2, 3$ ar suitable C^1 functions and Ψ_1, Ψ_2 are odd with respect to their first argument. We denote by x the state of system (10), i.e. $x = (q_n, \dot{q}_n, a, b)$. Similarly to before, set

$$E(q_n, \dot{q}_n, a, b) = \frac{1}{2}M(q_n, a, b)\dot{q}_n^2 + V(q_n, a, b),$$

where now

$$\begin{aligned}M(q_n, a, b) &= \exp\left\{-2\int_0^{q_n} \Psi_2(\tau, a, b)d\tau\right\}\\ V(q_n, a, b) &= -\int_0^{q_n} \Psi_1(\mu, a, b)M(\mu, a, b)d\mu.\end{aligned}$$

The fact that Ψ_1 and Ψ_2 are odd with respect to their first argument implies that the M and V are well-defined functions on $S^1 \times \mathbb{R} \times \mathbb{R}$ in that they are periodic with respect to their first argument q_n.

One can readily verify that

$$\dot{E}(x) = vh(x) \ , \tag{11}$$

where $h(x) = M\Psi_3\dot{q}_n^2 + \partial_a E(x)f_b(q_n) \ - \partial_b E(x)f_a(q_n)$. Note that system (4) is obtained from (10) by setting $a, b = v = 0$. The control objective for system (10) is to stabilize the set $\tilde{\Lambda} \subset \tilde{\Gamma}$ given by

$$\tilde{\Lambda} = \{(q, \dot{q}, a, b) \in \tilde{\Gamma} : E(q_n, \dot{q}_n, a, b) = E_0, \ a = 0, \ b = 0\}.$$

We also require the stabilizer to render $\tilde{\Gamma}$ invariant for the closed-loop system. In light of Siebert-Florio's reduction principle for asymptotic stability of compact sets (see [14], [15]), the two conditions below are necessary and sufficient to stabilize the set $\tilde{\Lambda}$ while guaranteeing closed-loop invariance of $\tilde{\Gamma}$:

(i) $\tilde{\Gamma}$ is asymptotically stable for the closed-loop system,
(ii) $\tilde{\Lambda}$ is asymptotically stable relative to $\tilde{\Gamma}$, i.e., the set

$$\gamma = \{(q_n, \dot{q}_n, a, b) : E(q_n, \dot{q}_n, a, b) = E_0, a = b = 0\}$$

is asymptotically stable for (10).

Condition (i) is already met by the input output linearizing feedback $\tau(q, \dot{q}, a, b)$. We only have to design $v(q_n, \dot{q}_n, a, b)$ enforcing (ii).

Let E_0 be a desired value for energy E and define the following Lyapunov function (L is a positive real constant)

$$V(q_n, \dot{q}_n, a, b) = \frac{1}{2}\big((E(x) - E_0)^2 + L^{-1}(a^2 + b^2)\big) \ . \quad (12)$$

Remark that, when $v = 0$, the derivative of V with respect to (10) is zero, since a, b and E are constant.

Taking (11) into account, we obtain

$$\begin{aligned} \dot{V} = v((E - E_0)h(x) + L^{-1}af_b(q_n) \\ -L^{-1}bf_a(q_n)) = vg(x) \ , \end{aligned}$$

where g is given by the following scalar product

$$g(x) = \langle \begin{pmatrix} E - E_0 \\ a \\ b \end{pmatrix} , \begin{pmatrix} h(x) \\ L^{-1}f_b(q_n) \\ -L^{-1}f_a(q_n) \end{pmatrix} \rangle \ . \quad (13)$$

Setting the control law for v

$$v(x) = -\lambda g(x) \ , \quad (14)$$

where $\lambda > 0$ is a gain constant, it follows that

$$\dot{V}(x) = -\lambda g^2(x) \ . \quad (15)$$

Condition ii) above is satisfied if

$$\lim_{t \to +\infty} V(x(t)) = 0 \ , \quad (16)$$

since this implies that E converges to E_0 and a, b converge to 0.

The following proposition presents a condition that guarantees that there exists parameters L (in (12)) and λ in (14) for which a required energy level E_0 is asymptotically stabilized. We need the following notation. Function $w_E(t) = (q_{n,E}(t), \dot{q}_{n,E}(t), 0, 0)$ is the solution of (10) with $v = 0$, initial state $x_E(0) = (0, \dot{q}_n(0), 0, 0)$, where $\dot{q}_n(0)$ is such that $E(x_E(t)) = E$, $\forall t \geq 0$. In other words, $w_E(t)$ is a periodic trajectory of constant energy E.

Proposition 3.1: Let $x_0 = (q_{n,0}, \dot{q}_{n,0}, 0, 0)$ be an assigned initial conditions for system (10) and let E_0 be a desired value for total energy.

If for each value of $E \in \{E \in \mathbb{R} \big| |E - E_0| \leq |E(x_0) - E_0|\}$ functions $f_a(q_{n,e}(t))$, $f_b(q_{n,e}(t))$ and $h(w_E(t))$ are periodic and linearly independent then, for every initial state of (10), there exist sufficiently small values K, L such that system (10) with control law (14) satisfies

$$\lim_{t \to +\infty} V(x(t)) = 0 \ .$$

Sketch of the proof: The proof is based on the application of lemma 3.2. Consider the potential function V defined in (12). Let L be sufficienty low such that $L^{-1}(\bar{a}^2 + \bar{b}^2) > V(x_0)$, this guarantees that $|a(t)| < \bar{a}$, $|b(t)| < \bar{b}$, $\forall t \geq 0$ and that the set $\mathcal{D} = \{x | V(x) \leq V(x_0)\}$ is positively invariant for any choice of $\lambda > 0$.

Equation (15) is in the form (19) with $G(x) = g(x)^2$, where $g(x)$ is given in (13). Using the notations of lemma 3.2, for any $z \in \mathcal{D}$, set $x_z(t) = (q_n(t), \dot{q}_n(t), a, b)$. Remark that the last two components (a,b) of x_z are constants and that $E(q_n(t), \dot{q}_n(t))$ is constant. Hence

$$\begin{aligned} &\int_0^{T_z} G(x_z)dt \\ &= \begin{pmatrix} E - E_0 \\ la \\ lb \end{pmatrix}^T M(z) \begin{pmatrix} E - E_0 \\ la \\ lb \end{pmatrix} , \end{aligned} \quad (17)$$

where $l = \sqrt{L^{-1}}$ and M is the Gramian given by

$$M(z) = \int_0^{T_z} \begin{pmatrix} h(x_z(t)) \\ lf_b(q_n(t)) \\ -lf_a(q_n(t)) \end{pmatrix} \begin{pmatrix} h(x_z(t)) \\ lf_b(q_n(t)) \\ -lf_a(q_n(t)) \end{pmatrix}^T dt \ .$$

Form (17) is positive definite if $M(z)$ is nonsingular, or, equivalently, if functions $h(x_z(t))$, $f_b(q_n(t))$, $f_a(q_n(t))$ are linearly independent. Since $\mathcal{D}$ is compact, if these three functions are linearly independent for every $z \in \mathcal{D}$ one can satisfy hypothesis (20) setting $\chi = \min_{z \in \mathcal{D}}\{\mu(M(z))\}$, (where $\mu(M(z))$ denotes the minimum eigenvalue of $M(z)$).

It remains to show that $M(z)$ is nonsingular for every $z \in \mathcal{D}$. By hypothesis, these functions are independent for $a = b = 0$. By continuity, there exist $\hat{a}$, $\hat{b}$ such that they are still independent for any $a \in [-\hat{a}, \hat{a}]$, $b \in [-\hat{b}, \hat{b}]$. If L is chosen again sufficiently small such that $L^{-1}(\hat{a}^2 + \hat{b}^2) > V(x_0)$, then $|a| < \hat{a}$, $|b| < \hat{b}$, $\forall (q_n, \dot{q}_n, a, b) \in \mathcal{D}$. This implies that functions $h(x_z(t))$, $f_b(q_n(t))$, $f_a(q_n(t))$ are independent for every value that E, a, b assume in set $\mathcal{D}$. ■

A. Lemma used in proposition 3.1

Lemma 3.2: Let $\mathcal{D} \in \mathbb{R}^n$ be compact and positively invariant for the family of dynamic systems dependent on $\lambda \in \mathbb{R}$

$$\dot{x} = f_\lambda(x) \ , \quad (18)$$

with $f : \mathcal{D} \to \mathbb{R}^n$.

For any $z \in \mathbb{R}^n$ let x_z be the solution of

$$\begin{cases} \dot{x}_z = f_0(x_z) \\ x_z(0) = z \ , \end{cases}$$

and assume that x_z is periodic of period $T_z > 0$. Let $V, G : \mathcal{D} \to \mathbb{R}$ be differentiable functions such that $\forall x \in \mathcal{D}$

$$\dot{V}(x) = -\lambda G(x) \ . \quad (19)$$

Assume finally that there exists $\chi > 0$ such that $\forall z \in \mathcal{D}$

$$\int_0^{T_z} G(x_z(t))dt > \chi V(z) \ . \quad (20)$$

Then, there exists λ, sufficiently small, such that $\forall z \in \mathcal{D}$, the solution of

$$\begin{cases} \dot{x} = f_\lambda(x_z) \\ x(0) = z \ , \end{cases}$$

satisfies

$$\lim_{t \to \infty} V(x(t)) = 0 \ .$$

Proof: Let $z \in \mathcal{D}$ and let $x_{\lambda,z}$ be the solution of (18) with initial condition z. By (19) it follows that

$$V(x_{\lambda,z}(T_z)) - V(x_{\lambda,z}(0)) = -\lambda D_\lambda(z) \ ,$$

where

$$D_\lambda(z) = -\int_0^{T_z} G(x_{\lambda,z}(t))dt \ .$$

Function D is continuous on λ, uniformly with respect to $z \in \mathcal{D}$ (since $\mathcal{D}$ is compact), moreover, by (20), $\forall z \in \mathcal{D}$, $D_0(z) > \chi V(z)$. Therefore there exist $\bar{\lambda}$, sufficiently small such that

$$D_{\bar{\lambda}}(z) > \frac{\chi}{2} V(z), \ \forall z \in \mathcal{D} \ .$$

This implies that, $\forall z \in \mathcal{D}$

$$V(x_{\bar{\lambda},z}(T_z)) - V(x_{\bar{\lambda},z}(0)) \leq -\bar{\lambda}\frac{\chi}{2} V(z) \ .$$

A consequence of this is that $\forall z \in \mathcal{D}$, $\lim_{t\to\infty} V(x_{\bar{\lambda},z}(t)) - V(x_{\bar{\lambda},z}(0)) = 0$. ■

IV. Application to the Pendubot swing-up problem

Consider the Pendubot in Figure 1. The configuration variables are $q = (\theta_1, \theta_2)$. Assuming, for simplicity of exposition, that the masses and lengths of the two links are equal and unitary, and neglecting friction, the pendubot model reads as $D(q)\ddot{q} + C(q,\dot{q})\dot{q} + \nabla P(q) = B\tau$, where

$$D(q) = \begin{bmatrix} 2 & \cos(\theta_1 - \theta_2) \\ \cos(\theta_1 - \theta_2) & 1 \end{bmatrix}$$

$$C(q,\dot{q}) = \begin{bmatrix} 0 & \sin(\theta_1 - \theta_2)\dot{\theta}_2 \\ -\sin(\theta_1 - \theta_2)\dot{\theta}_1 & 0 \end{bmatrix}$$

$$P(q) = 2g\cos\theta_1 + g\cos\theta_2, B = \begin{bmatrix} 1 \\ 0 \end{bmatrix} .$$

The energy of the pendubot is $\mathcal{H}(q,\dot{q}) = \frac{1}{2}\dot{q}^\top D(q)\dot{q} + P(q)$. When $\tau = 0$, the pendubot has four equilibria, depicted in Figure 2. The equilibria lie on different level sets of the energy. For the solution to this problem, we point out, in particular, the work in [16], where an energy-based controller is introduced with a complete stability analysis. Recently, [7] used virtual holonomic constraints to determine periodic orbits of the pendubot in which the elbow oscillates without performing complete revolutions. Then, using a technique of transverse linearization, they stabilized the periodic orbits in question.

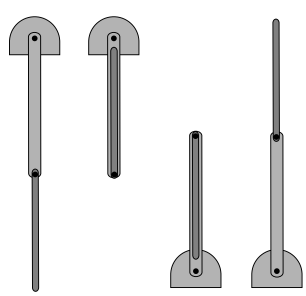

Fig. 2. Equilibrium configurations.

Here, we focus on the following control problem:

Low-high to high-high swing-up problem. Design a feedback law yielding the following two properties:

1) **Swing-up:** For any neighborhood U of the high-high equilibrium, there exists a punctured neighborhood V of the low-high equilibrium such that for each initial condition in V, the solution enters U in finite time.
2) **Boundedness:** For any initial condition in V, the solution has the property that $\theta_2(t) \in (-\pi, \pi)$ for all $t \geq 0$. In other words, the unactuated link does not fall over.

The pendubot system satisfies condition C1 in Section II. We look for a constraint $\theta_2 = \phi(\theta_1)$ with the following properties:

- $\phi(0) = \phi(\pi) = 0$, so that the second link is high when the first link is either low or high.
- The image $\phi(S^1) \subset (-\pi, \pi)$, so that the second link doesn't fall over as the first link revolves.

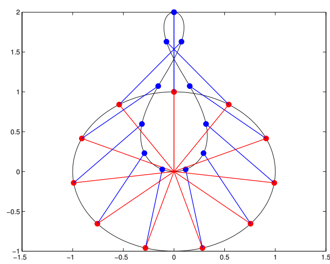

Fig. 3. Configurations of the double pendulum with the virtual constraint obtained by setting $\mu(\theta_1) = 1$.

For the pendubot we have $B^\perp = [0 \ 1]$ and the VCG is given by $\frac{d\phi}{d\theta_1} = -\cos(\theta_1 - \phi(\theta_1)) + \delta(\theta_1)$. The solution with zero initial condition and with $\delta = 0$ is not 2π-periodic, so we can apply Proposition 2.3. We should select a 2π-periodic function $\mu(\theta_1) \neq 0$, set $\delta(\theta_1) = \epsilon\mu(\theta_1)$, and find the unique value of ϵ guaranteeing that the solution with zero initial condition is 2π-periodic. In order to meet condition C3, we must select $\mu(\theta_1)$ to be even. If we set $\mu = 1$, then we find $\epsilon = 1 - \sqrt{2}$ and the virtual constraint

$$\theta_2 = \phi(\theta_1) = \theta_1 + 2\arctan[\tan(-\theta_1/2)(1+\sqrt{2})] \quad (21)$$

depicted on Figure 3. This constraint has the required properties. As predicted by the theory of [10], the motion of the pendubot on the constraint manifold is Euler-Lagrange. The phase portrait of the dynamics on the constraint manifold is depicted in Figure 4. The level sets of E inside the shaded

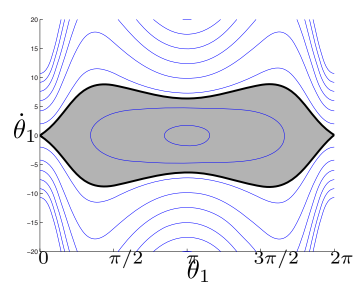

Fig. 4. Energy level sets for double pendulum on the VHC $\theta_2 = \theta_1 + 2\arctan[\tan(-\theta_1/2)(1+\sqrt{2})]$.

region of Figure 4 are oscillations, while the ones outside the shaded region correspond to rotations.

Swinging up the pendulum to the high-high equilibrium corresponds to stabilizing the level set of the energy bounding the shaded region, for which $E_0 = 0$. This level set is neither an oscillation nor a rotation and the corresponding trajectory is not a periodic orbit. For this reason we set E_0 slightly smaller than zero.

We return to the VCG with $\mu(\theta_1) = 1$, $\frac{d\phi}{d\theta_1} = -\cos(\theta_1 - \phi(\theta_1)) + \epsilon$. We apply the method presented in section III. We choose the functions appearing in (7) as $f_a(\theta_1) = \sin(5\theta_1)$, $f_b(\theta_1) = \sin(7\theta_1)$. In this case one can set $\bar{a} = 0.01$, $\bar{b} = 0.01$. To stabilize the VHC $\theta_2 = \phi(\theta) + af_a(\theta_1) + bf_b(\theta_1)$, the physical input τ of the pendubot is designed to input-output linearize the system with output $e = \theta_2 - \phi(\theta_1) - af_a(\theta_1) - bf_b(\theta_1)$ (which has relative degree 2). The parameters a, b are varied with input v accordingly to (8).

The input v of the dynamic compensator affects the shape of the virtual constraint manifold. We use the feedback v defined in (14) to stabilize the energy $E_0 = -0.1$, with $\lambda = 10^{-5}$. In potential function (12) we set $L = 1/60000$. Numerically, we checked that the hypotheses of Proposition 3.1 are satisfied for every periodic orbit with energy $E < 0$, with the only exception of the one that corresponds to the low-high equilibrium of the pendubot.

The swing-up controller switches to a linear stabilizing controller when $\|(\theta_1, \dot{\theta}_1, \theta_2, \dot{\theta}_2)\| < 0.3$.

Figure 5 shows the value of total energy $E(t)$ during the swing-up phase and figure 7 shows the angle $\theta_2(t)$ during swing-up (dashed line) and equilibrium stabilization (dotted line). Figure 6 shows the value of Lyapunov function V, and figure 8 shows the corresponding phase portait in plane $(\theta_1, \dot{\theta}_1)$.

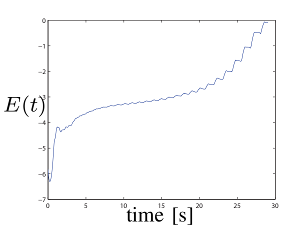

Fig. 5. Total energy $E(t)$ for the pendubot example.

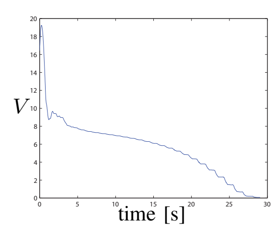

Fig. 6. Value of potential function $V(x(t))$for the pendubot example.

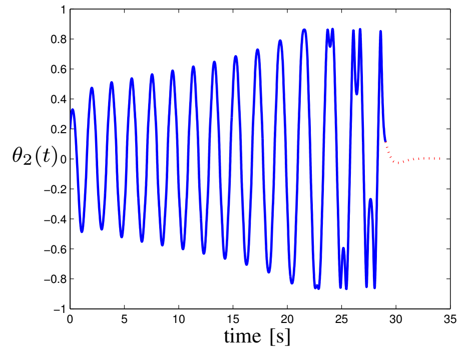

Fig. 7. Angle $\theta_2(t)$ for the pendubot example.

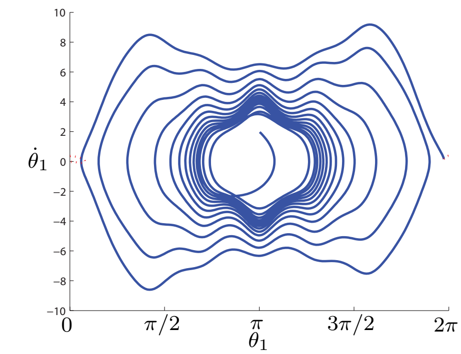

Fig. 8. Phase portrait of $(\theta_1, \dot{\theta}_1)$ for the pendubot example.

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