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A State Space Approach to the Parameterization of Output Regulating Controllers

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Abstract—**The output regulation problem is a classic problem which needs updated machinery to be effectively applied in a switched or blended controller setting. For stabilizing controllers approaches with Youla parameterization has been crucial to the solution of many complex controller switching problems. This paper seeks similar benefits by developing a state-space based parameterization of output regulating controllers. A critical step in this approach involves augmenting the plant with the unstabilizable exogenous input. Lastly, a controller switching example for an active suspension quarter car model is considered where a switched controller based on the developed parameterization framework is shown to significantly minimize switching transients in comparison to one with Youla parameterization.**

I. Introduction

THE output regulation problem, also known as the servomechanism problem, is a classic and important controls problem. It concerns developing controllers that not only internally stabilize the plant but also reject disturbances and track signals of which both are generated by an exogenous system. Applications of output regulation apply to various areas, such as performance regulation in the controller switching of hypersonic vehicles [14], power regulation in controller switching of wind turbines [8], and track-following servo systems of disk drives [1].

The study of output regulation began with the authors in [3] who laid out fundamental concepts such as the internal model principle. More recently, Saberi et al. [11] have written a comprehensive book on linear output regulation. Their techniques, based on transforming the controller synthesis problem with regulation constraints into a synthesis problem without constraints, are capable of solving complex controller synthesis problems in state-space involving multiple performance metrics and saturation constraints. However, not much detail is given towards controller parameterization and dealing with switched or blended control systems [10], [16], [4] which are important for accommodating changing objectives and operating conditions.

In terms of controller parameterization, Youla parameterization [15] has been very popular because it classifies of all stabilizing controllers in terms of a convenient stable controller parameters which have applications in terms of controller interpolation and optimization. Youla parameterization was initially studied in the transfer function context but has been carried over into the state space context to solve complex time-varying problems and to utilize powerful linear matrix inequality (LMI) techniques such as in Scherer's work [13] to design multiobjective controllers. Further examples, include how Hespana and Morse [5] leverage Youla parameterization to show that there always exists a switched controller that stabilizes a linear time-invariant plant for any arbitrarily fast switching signal [5]. The Youla parameterization framework has also contributed to developing H_2 and H_∞ synthesis [17] as well as L_1 robust control [12], distributed control [9], and robust controller switching [4].

Literature on parameterization of output regulating controllers [6], [7] is limited. The authors in [7] approach parameterization problem by classifying the set of Youla parameters which satisfy output regulation. The authors in [6] approach the parameterization problem through constructing an augmented plant embedded with a regulator and parameterizing the set of controllers which stabilize the augmented plant. These papers are limited in scope to the transfer function context which is inadequate in dealing with switched control systems and cannot utilize LMI optimization techniques.

There currently appears to be a gap in the literature for defining a convenient parameterization of output regulating controllers in the state space context. This paper seeks to fill the gap and provide new insight to this problem. The obvious approach is to incorporate the exogenous system into the P and parameterized the set of stabilizing controllers. However, the problem is that the exogenous system is defined as unstabilizable and thus cannot find a stabilizing solution to the augmented system, let alone tackle the parameterization problem. In the subsequent sections, we clarify how to get around this difficulty and explain why it is not a problem.

This paper is organized in the following way. Section 2 defines a set of regulating controller parameterization criteria. Section 3 discusses relevant controller parameterization and output regulation background. Section 4 develops a framework for output regulating controller parameterization. Section 5 presents a vehicle dynamics example presenting a seamless transition among controllers due to an output regulation guarantee and highlighting the potential of an output regulating controller parameterization framework. Section 6 presents some concluding remarks and directions for future research.

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For nomenclature, the notation $K_1 \sim K_2$ denotes that systems K_1 and K_2 are input-output equivalent. The two systems K_1 and K_2 are input-output equivalent if $\|K_1 - K_2\|_\infty = 0$ where $\|\cdot\|_\infty$ denotes the induced L_2 norm.

II. Problem Definition

Consider the plant

$$P:\left\{\begin{bmatrix}\dot{x}\\ \hline e\\ y\end{bmatrix}=\left[\begin{array}{c|cc}A & B_w & B_u\\ \hline C_e & D_{ew} & D_{eu}\\ C_y & D_{yw} & D_{yu}\end{array}\right]\begin{bmatrix}x\\ \hline w\\ u\end{bmatrix}\right. \tag{1}$$

driven by the exogenous system

$$P_E : \dot{w} = Sw \tag{2}$$

and interconnected with the controller

$$K:\left\{\begin{bmatrix}\dot{v}\\ \hline u\end{bmatrix}=\left[\begin{array}{c|c}A_K & B_K\\ \hline C_K & D_K\end{array}\right]\begin{bmatrix}v\\ \hline y\end{bmatrix}\right. \tag{3}$$

where $x\in\mathbb{R}^n$, $w\in\mathbb{R}^{n_w}$, $u\in\mathbb{R}^{n_u}$, $e\in\mathbb{R}^{n_e}$, $y\in\mathbb{R}^{n_y}$ and $x_K\in\mathbb{R}^{n_K}$. This interconnection forms the closed loop system denoted as $T_{ew}(K)$ via the lower fractional transformation (LFT) as shown in Fig. 1.

Augmenting the dynamics of P with P_E creates the following unstabilizable augmented plant

$$\overline{P}:\left\{\begin{bmatrix}\dot{\overline{x}}\\ \hline \dot{e}\\ y\end{bmatrix}=\left[\begin{array}{c|c}\overline{A} & \overline{B}_u\\ \hline \overline{C}_e & \overline{D}_{eu}\\ \overline{C}_y & \overline{D}_{yu}\end{array}\right]\begin{bmatrix}\overline{x}\\ \hline u\end{bmatrix}\right. \tag{4}$$

where,

$$\overline{x}=\begin{bmatrix}x\\ w\end{bmatrix}\quad\text{and}\quad\left[\begin{array}{c|c}\overline{A} & \overline{B}_u\\ \hline \overline{C}_e & \overline{D}_{eu}\\ \overline{C}_y & \overline{D}_{yu}\end{array}\right]=\left[\begin{array}{cc|c}A & B_w & B_u\\ 0 & S & 0\\ \hline C_e & D_{ew} & D_{eu}\\ C_y & D_{yw} & D_{yu}\end{array}\right].$$

The interconnection of $\overline{P}$ and K forms the closed loop system denoted as $\overline{T}_{ew}(K)$ and shown in Fig. 1.

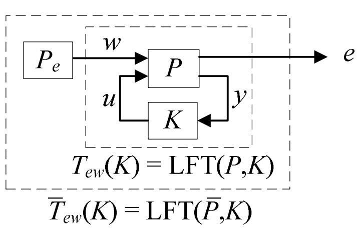


Fig 1. Closed loop system $T_{ew}(K) = LFT(P,K)$ and $\overline{T}_{ew} = LFT(\overline{P},K)$ formed via lower linear fractional transformation

In order to simplify the ensuing calculations, we make the following standard assumptions that D_{yu}=0 without loss of generality and,

A1) (A, B_u) is stabilizable;

A2) S is anti-Hurwitz stable;

A3) $(\overline{C}_y, \overline{A})$ is detectable;

A4) (Π,Γ) exists solving the regulator equation

$$\begin{cases}\Pi S = A\Pi + B_u\Gamma + B_w\\ 0 = C_e\Pi + D_{eu}\Gamma + D_{ew}\end{cases} \tag{5}$$

as done in [11]. Assumption (**A2**) states that the dynamics of P_E do not stabilize over time; otherwise, regulation is merely a consequence of internal stabilization. Assumption (**A1**) and (**A3**) assures that P can be internally stabilized. Adding (**A4**), there exists a regulating controller with a regulating strategy mapped from (Π,Γ).

An output regulating controller is defined as satisfying the following *output regulating controller criteria*:

B1) *Internal stability:* For $w(t) = 0$ the states of the closed loop system $T_{ew}(K)$ are exponentially stable for all $x(0)\in\mathbb{R}^n$.

B2) *Output Regulation:* For all $x(0)\in\mathbb{R}^n$ and $w(0)\in\mathbb{R}^{n_w}$, the output $e(t)$ of the closed loop system $T_{ew}(K)$ satisfies $e(t) \to 0$ as $t \to 0$.

Criterion (**B1**) requires that the closed loop system remain stable for all initial conditions when all external inputs are removed. Criterion (**B2**) ensures output regulation for any initial conditions.

The purpose of this paper is to find a convenient parameterization satisfying the following *output-regulating controller parameterization criteria* for all output regulating controllers with common (Π,Γ):

C1) For all output regulating controller K with common (Π,Γ), there exists a mapping to a stable control parameter Q, such that K and the parameterized controller $K(Q)$ are input-output equivalent.

C2) For all stable Q, $K(Q)$ satisfies the *output regulating controller criteria*.

Criterion (**C1**) ensures that parameterization does not affect the input-output traits of the original controller. Criterion (**C2**) asserts that any parameterized controller from this mapping is internally stabilizing and output regulating.

III. Background Information

A. LFT Parameterization Formulas

LFT parameterization provides a convenient way to characterize controllers in terms of control parameters. In the following formulas we explore how to parameterize a general controller and later apply these concepts to parameterizing an output regulating controller.

Consider a linear controller K parameterized by the controller parameter Q from Fig. 2a, where the central controller J and the controller parameter Q are defined as

$$J:\left\{\begin{bmatrix}\dot{\hat{x}}\\ \hline u\\ \tilde{y}\end{bmatrix}=\left[\begin{array}{c|cc}A_J & B_{J1} & B_{J2}\\ \hline C_{J1} & D_{J11} & D_{J12}\\ C_{J2} & D_{J21} & D_{J22}\end{array}\right]\begin{bmatrix}\hat{x}\\ \hline y\\ \tilde{u}\end{bmatrix}\right. \tag{6}$$

and

$$Q:\left\{\begin{bmatrix}\dot{q}\\ \hline \tilde{u}\end{bmatrix}=\left[\begin{array}{c|c}A_q & B_q\\ \hline C_q & D_q\end{array}\right]\begin{bmatrix}q\\ \hline \tilde{y}\end{bmatrix}\right.. \tag{7}$$

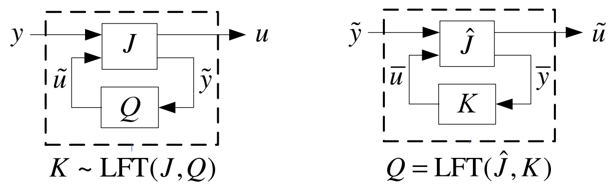
For any J and K of appropriate input-output dimensions, we would like to determine Q such that $K \sim LFT(J,Q)$. To find this, first assume D_{J12} and D_{J21} are square invertible and D_{J22}=0, then consider the system

$$\hat{J}:\left\{\begin{bmatrix}\dot{x}_{\hat{J}}\\ \hline \tilde{u}\\ \tilde{y}\end{bmatrix}=\left[\begin{array}{c|cc}\hat{A}_J & \hat{B}_{J1} & B_{J2}D_{J12}^{-1}\\ \hline \hat{C}_{J1} & \hat{D}_{J11} & D_{J12}^{-1}\\ -D_{J21}^{-1}C_{J2} & D_{J21}^{-1} & 0\end{array}\right]\begin{bmatrix}x_{\hat{J}}\\ \hline \tilde{y}\\ \breve{u}\end{bmatrix}\right. \tag{8}$$

where

$\hat{A}_J = A_J - \hat{B}_{J1}C_{J2} - B_{J2}D_{J12}^{-1}C_{J1}$, $\hat{B}_{J1} = B_{J1}D_{J21}^{-1} + B_{J2}\hat{D}_{J11}$,
$\hat{C}_{J1} = -(D_{J12}^{-1}C_{J1} + \hat{D}_{J11}C_{J2})$, and $\hat{D}_{J11} = -D_{J12}^{-1}D_{J11}D_{J21}^{-1}$.

Now consider the following lemma which claims that $Q = LFT(\hat{J}, K)$ satisfies $K \sim LFT(J,Q)$.



[a] [b]

Fig 2. [a] Input-output equivalent controller parameterization via LFT; [b] Controller parameter Q in terms of the original K

Lemma 1: Given an LTI system K and J with $D_{J22} = 0$ and square invertible D_{J12} and D_{J21}. Then $LFT(J, Q)$ is input-output equivalent to K for $Q = LFT(\hat{J}, K)$.

The proof, which can be found in Lemma 1 of [4], is associated with showing that this special construction of the systems of $\hat{J}$ and J allows the output of K at steady state to pass through both systems to the plant P.

B. Output Regulation Formulas

Output regulation is defined as driving the output $e(t)$ in (1) to zero given an anti-Hurwitiz stable input w(t). The following lemma presents the conditions for regulation a controller of form (3) must satisfy.

Lemma 2: Given K of the form in (3) that internally stabilizes P in (1) satisfying assumptions (**A1-A4**) and has a solution (Π,Θ) to the *extended regulator equation* in (9), then K is also output regulating for P.

$$\begin{bmatrix}\Pi\\ \Theta\end{bmatrix} S = \begin{bmatrix} A + B_u D_c C_y & B_u C_c \\ B_c C_y & A_c \end{bmatrix}\begin{bmatrix}\Pi\\ \Theta\end{bmatrix} + \begin{bmatrix} B_w + B_u D_c D_{yw} \\ B_c D_{yw}\end{bmatrix}$$
$$0 = \begin{bmatrix} C_e + D_{eu}D_c C_y & D_{eu}C_c \end{bmatrix}\begin{bmatrix}\Pi\\ \Theta\end{bmatrix} + \begin{bmatrix} D_{ew} + D_{eu}D_c D_{yw}\end{bmatrix} \tag{9}$$

Proof: Consider the closed loop system $\overline{T}_{ew}(K)$ and assume D_c = 0 without loss of generality. Then, apply a similarity transformation to this closed loop system such that the resulting alternate realization is

$$\begin{bmatrix} \dot{x} - \Pi\dot{w} \\ \dot{v} - \Theta\dot{w} \\ \dot{w} \\ \hline e \end{bmatrix} = \left[\begin{array}{ccc} A & B_u C_K & M_1 \\ B_K C_y & A_K & M_2 \\ 0 & 0 & S \\ \hline C_e & D_{eu}C_K & M_3 \end{array}\right]\begin{bmatrix} x - \Pi w \\ v - \Theta w \\ w \end{bmatrix}$$

where,

$$\begin{aligned} M_1 &:= A\Pi + B_u C_K \Theta + B_w - \Pi S = 0 \\ M_2 &:= B_K C_y \Pi + A_K \Theta + B_K D_{yw} - \Theta S = 0 \\ M_3 &:= C_e \Pi + D_{eu} C_K \Theta + D_{ew} = 0 \end{aligned} \tag{10}$$

The matrices M_1, M_2, and M_3 are exactly the *extended regulator equation* in (9) with D_c = 0. Thus these matrices are equal to zero and output regulation of $e(t)$ now relies on the internal stability of K which is given. □

Remark 1: The *extended regulator equation* in (9) reduces to the *regulator equation* in (5) with the following substitution:

$$\Gamma = D_C(C_y \Pi + D_{yw}) + C_C \Theta . \tag{11}$$

A controller's regulating strategy is said to be mapped from (Π,Γ) if it can be mapped from a (Π,Θ) satisfying (9).

IV. OUTPUT REGULATING CONTROLLER PARAMETERIZATION

In this section, a controller parameterization scheme for output regulating controllers is developed which satisfies criteria (**C1-C2**). First we prove the existence of a stable Q for every regulating controller K with common (Π,Γ), and then prove the existence of an input-output equivalent parameterized controller based on that Q. Lastly for completeness, we show for every stable Q there exists a unique regulating parameterized controller with regulating strategy (Π,Γ).

A. Input-Output Equivalence of $K(Q)$ and K

A Youla-like parameterization approach on the augmented plant $\overline{P}$ is taken. This approach differs from traditional Youla parameterization because we relax the criterion that $\overline{A} + \overline{B}\overline{F}$ is exponentially stable to merely A+BF being exponentially stable. The relaxation of this criterion causes $Q := LFT(\hat{J}, K)$ to be unstable; and thus, requires a reformulation of an admissible control parameter.

First, we construct an observer based J and $\hat{J}$ for $\overline{P}$ in the form of (6) and (8), respectively, where (Π,Θ_o) satisfies (9) for the observer-based J. The resulting systems are as follows:

$$J : \left\{ \begin{bmatrix} \dot{\hat{\overline{x}}} \\ \hline u \\ \tilde{y} \end{bmatrix} = \left[\begin{array}{c|cc} \overline{A}_o & \overline{B}_o & \overline{B}_u \\ \hline \overline{C}_o & \overline{D}_0 & I \\ -\overline{C} & I & 0 \end{array}\right]\begin{bmatrix} \hat{\overline{x}} \\ \hline y \\ \tilde{u} \end{bmatrix} \right. \tag{12}$$

$$\hat{J} : \left\{ \begin{bmatrix} \dot{x}_{\hat{J}} \\ \hline \tilde{u} \\ \breve{y} \end{bmatrix} = \left[\begin{array}{c|cc} \overline{A} & \overline{B}_o & \overline{B}_u \\ \hline -\overline{C}_o & -\overline{D}_o & I \\ \overline{C}_y & I & 0 \end{array}\right]\begin{bmatrix} x_{\hat{J}} \\ \hline \tilde{y} \\ \breve{u} \end{bmatrix} \right. \tag{13}$$

where,

$$\left[\begin{array}{c|c} \overline{A}_o & \overline{B}_o \\ \hline \overline{C}_o & \overline{D}_0 \end{array}\right] = \left[\begin{array}{c|c} \overline{A} + \overline{B}_u\overline{F} + \overline{L}\overline{C}_y & -\overline{L} \\ \hline \overline{F} & 0 \end{array}\right], \ \hat{\overline{x}} = \begin{bmatrix} \hat{x} \\ \hat{w} \end{bmatrix}, \ \Theta_o = \begin{bmatrix} \Pi \\ I \end{bmatrix},$$

$$\overline{L} = [L_A^T \quad L_S^T]^T , \text{ and } \overline{F} = [F \quad \Gamma - F\Pi] .$$

As stated earlier, $Q := LFT(\hat{J}, K)$ is unstable because $\overline{A} + \overline{B}\overline{F}$ is unstable. However, the unstable dynamics are shown to be unobservable and disconnected.

Lemma 3: Given a control parameter $Q := LFT(\hat{J}, K)$, where K satisfies (**B1**-**B2**) for the plant P in (1) satisfying (**A1-A4**) and $\hat{J}$ is of the form (13) with (Π,Γ) in common with K, there exists a stable realization of $\tilde{Q} \sim Q$, where

$$\tilde{Q} : \left\{ \begin{bmatrix} \dot{\tilde{q}} \\ \hline \tilde{u} \end{bmatrix} = \left[\begin{array}{c|c} \tilde{A}_q & \tilde{B}_q \\ \hline \tilde{C}_q & \tilde{D}_q \end{array}\right]\begin{bmatrix} \tilde{q} \\ \hline \tilde{y} \end{bmatrix} \right. \tag{14}$$

and

$$\left[\begin{array}{c|c} \tilde{A}_q & \tilde{B}_q \\ \hline \tilde{C}_q & \tilde{D}_q \end{array}\right] = \left[\begin{array}{cc|c} A + B_u D_K C_y & B_u C_K & -L_A + B_u D_K + \Pi L_S \\ B_K C_y & A_K & B_K + \Theta_K L_S \\ \hline -F + D_K C_y & C_K & D_K \end{array}\right].$$

Proof: It can be shown there exists a (Π,Θ_K) satisfying (9) since K is output regulating and internally stabilizing. Now, consider a similarity transform on Q involving (Π,Θ_K) with the following resulting alternate realization:

$$\begin{bmatrix} \dot{q}_1-\Pi\dot{q}_2 \\ \dot{q}_2 \\ \dot{q}_3-\Theta_K\dot{q}_2 \\ \hline \tilde{u} \end{bmatrix} = \left[\begin{array}{ccc|c} \tilde{A}_{q,11} & M_{Q1} & \tilde{A}_{q,12} & \tilde{B}_{q,11} \\ 0 & S & 0 & L_S\bar{C}_y \\ \tilde{A}_{q,21} & M_{Q2} & \tilde{A}_{q,22} & \tilde{B}_{q,21} \\ \hline \tilde{C}_{q,11} & M_{Q3} & \tilde{C}_{q,12} & \tilde{D}_q \end{array}\right] \begin{bmatrix} q_1-\Pi q_2 \\ q_2 \\ q_3-\Theta_K q_2 \\ \hline \tilde{y} \end{bmatrix}$$

where,

$$\begin{aligned} M_{Q1} &:= \tilde{A}_{q,11}\Pi+\tilde{A}_{q,12}\Theta_K+(B_w+B_uD_KD_{yw})-\Pi S \\ M_{Q2} &:= \tilde{A}_{q,21}\Pi+\tilde{A}_{q,22}\Theta_K+B_KD_{yw}-\Theta_K S \\ M_{Q3} &:= \tilde{C}_{q,11}\Pi+\tilde{C}_{q,12}\Theta_K-(\Gamma-F\Pi)+D_KD_{yw} \end{aligned} \quad . \quad (15)$$

The matrix M_{Q1} and M_{Q2} are equal to zero since they are exactly the equations of (9). Moreover given that J has a (Π,Γ) in common with K, M_{Q3} can be shown to be zero by applying (11) and canceling terms. These terms in (15) being zero causes the state q_2 to be unobservable and disconnected. Thus, Q can now be truncated into (14) and still maintain the same output.

The dynamics of $\tilde{Q}$ are stable because $\tilde{A}_q$ is equivalent to the closed loop A-matrix of $T_{ew}(K)$ for which K is given as internally stabilizing. Thus, $\tilde{Q}$ is admissible. □

Having a mapping to a stable Q we can now state the following theorem for an input-output equivalent parameterized controller.

Theorem 1: For every controller K with common (Π,Γ) satisfying (**B1-B2**) for the plant P in (1) satisfying assumptions (**A1-A4**), there exists a mapping to a stable $\tilde{Q}$ such that $K\sim K(\tilde{Q})$.

Proof: From Lemma 1 we can find that a controller $K\sim LFT(J,Q)$ for $Q:=LFT(\hat{J},K)$, where J and $\hat{J}$ are defined in (6) and (8), respectively. From Lemma 3, we can find a stable $\tilde{Q}\sim Q$, where $\tilde{Q}$ is defined as (14). Input-output equivalence allows Q to be replaced with $\tilde{Q}$ and have $K\sim LFT(J,\tilde{Q})$. □

B. Internal Stability for any Stable Q

From Lemma 2, a controller satisfies output regulation (**B2**) if it meets two criteria: achieving internal stability and having (Π,Θ) that satisfies (9). In this subsection we consider the criterion of internal stability for a parameterized controller.

Lemma 4: Given a stable parameter Q and a J of the form (12), the $K(Q) := LFT(J,Q)$ achieves *internal stability* for the plant P in (1) satisfying assumptions (**A1-A4**).

Proof: Achieving internal stability require that the closed loop dynamics $T_{ew}(K(Q))$ are stable when the disturbance input is $w = 0$. With $w = 0$ and simple matrix transformations, we get the following dynamics:

$$\begin{bmatrix} \dot{x} \\ \dot{q} \\ \dot{\hat{\bar{x}}}-\dot{\bar{x}} \end{bmatrix} = \begin{bmatrix} A+B_uF & B_u\bar{C}_o & \bar{B}_u\bar{F} \\ 0 & A_q & -B_q\bar{C}_y \\ 0 & 0 & \bar{A}+\bar{L}\bar{C} \end{bmatrix} \begin{bmatrix} x \\ q \\ \hat{\bar{x}}-\bar{x} \end{bmatrix}.$$

From the upper triangular matrix, it is clear the above closed loop system is exponentially stable if $A+B_uF$, $\bar{A}+\bar{L}\bar{C}_y$ and A_q are stable. We know $A+B_uF$ and $\bar{A}+\bar{L}\bar{C}_y$ to be stable from the construction of J, and Q is given as stable. Thus, $K(Q)$ achieves internal stability. □

C. Output Regulation for any Stable Q

In this subsection, we first present the existence of (Π,Θ_K) satisfying (9) for any parameterized controller $K(Q)$ with a stable Q. Then, we prove for any stable Q, there exists a parameterized controller $K(Q)$ that satisfies (**B1-B2**) with regulating strategy (Π,Γ). Last, we present a summarizing algorithm of the developed parameterization.

Lemma 5: Given a parameterized controller $K(Q)$ defined as $LFT(J,Q)$, where J has the form (12) and Q has the form (7), there exists (Π,Θ_K) satisfying (9) for P *and* $K(Q)$ where,

$$\Theta_K = [\Theta_o \quad 0]^T \qquad (16)$$

and $\Theta_K \in \mathbb{R}^{(n_J+n_q)\times n_w}$.

Proof: Consider the regulating observer based controller $K_0=LFT(J,Q=0)$; and note that (Π,Θ_o) satisfies (9) for P and K_o. Then, consider a pair (Π,Θ_K) in (9) for P and $K(Q)$ where Π is the same but Θ_K has the form $\Theta_K := [\Theta_o \quad \Theta_q]$. For $Q\neq 0$, Θ_q must be zero for (9) to be satisfied. Thus, Θ_K is equal to (16). □

Having internal stability and (Π,Θ_K) satisfying (9) for any $K(Q)$ with stable Q we can now state the follow theorem guaranteeing $K(Q)$ satisfies output regulation.

Theorem 2: Given any stable controller parameter Q, there exists an output regulating parameterized controller $K(Q) := LFT(J, Q)$ for the plant P in (1) satisfying (**A1-A4**).

Proof: Given P in (1) satisfying (**A1-A4**), we can find a central controller J from (12). Having J, Lemma 5 states that there exists a (Π,Θ_K) satisfying (9) form any parameterized controller where Θ_K is defined in (16). Moreover, Lemma 4 guarantees that any stable Q results in an internally stable $K(Q) := LFT(J, Q)$. With these two results, we can invoke Lemma 2 to prove that $K(Q)$ is output regulating. □

The framework for parameterization of output regulating controllers with common (Π,Γ) can be summarized in the following algorithm.

***Algorithm 1** (Construction of a controller $K(Q)$):* Given P satisfying assumptions (**A1-A4**) and K satisfying criteria (**B1-B2**), the following steps lead to an input-output equivalent parameterized controller:

1. Construct $\bar{P}$ in the form of (4).
2. Find $F,\bar{L}$ such that $A+BF$ and $\bar{A}+\bar{L}\bar{C}$ are Hurwitz.
3. Construct J and $\hat{J}$ in the form of (6) and (8).
4. Construct stable Q in the form of (14).
5. Construct $K(Q) := LFT(J, Q)$.

V. Example

Many approach multi-objective control problems through designing multiple local controllers and switching among them. However, transients from controller switching can cause poor performance and even instability. The following example utilizes the parameterization framework established in section IV to construct a switched controller that maintains regulation during arbitrarily fast switching. These simulated results are compared with the nominal Youla parameter switching framework discussed in [5].

A. Simulation Problem Definition

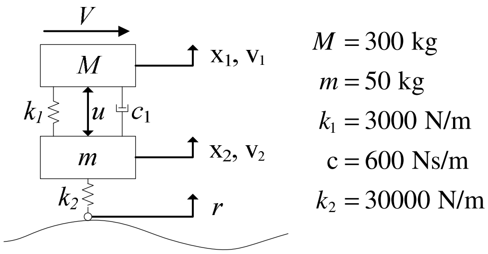


Fig. 3. A simplified representation of an active suspension system for a quarter car model.

Consider the following simplified 1D representation of an active suspension system for a quarter car model in Fig. 3 where the variables are described as follows:

$x_1(t)$, $v_1(t)$, m	position, velocity, mass of car/driver
$x_2(t)$, $v_2(t)$, M	position, velocity, mass of tires/axles
$u(t)$	hydraulic actuator
$r(t)$	roadway height
k_1, c_1	spring, damper of passive shocks
k_2	stiffness of the tires.

The primary performance criterion is the regulation of a constant vertical position for the driver subject to a roadway disturbance characterized by $r(t)$. In this example $r(t)=w_1(t)$ is characterized as an exogenous system representing a sinusoidal-shaped roadway.

The following plant represents the dynamics of in Fig. 3,

$$P_{ew}:\left\{\begin{bmatrix}\dot{x}\\ e\\ y\end{bmatrix}=\left[\begin{array}{c|cc}A & B_w & B_u\\ \hline C_e & 0 & 0\\ C_y & 0 & 0\end{array}\right]\begin{bmatrix}x\\ w\\ u\end{bmatrix}+\begin{bmatrix}0\\ 0\\ v\end{bmatrix}\right. \tag{17}$$

where v is sensor noise,

$$A=\begin{bmatrix}0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ -\frac{k_1}{M} & \frac{k_1}{M} & -\frac{c_1}{M} & \frac{c_1}{M}\\ \frac{k_1}{m} & \frac{-k_1-k_2}{m} & \frac{c_1}{m} & -\frac{c_1}{m}\end{bmatrix},\ B_u=\begin{bmatrix}0\\ 0\\ \frac{1}{M}\\ -\frac{1}{m}\end{bmatrix},\ x=\begin{bmatrix}x_1\\ x_2\\ v_1\\ v_2\end{bmatrix},$$

$$w=\begin{bmatrix}w_1\\ w_2\end{bmatrix},\ B_w=\begin{bmatrix}0 & 0\\ 0 & 0\\ 0 & 0\\ k_2/m & 0\end{bmatrix},\ C_y=\begin{bmatrix}1\\ 0\\ 0\\ 0\end{bmatrix}^T \text{ and } C_e=\begin{bmatrix}1\\ 0\\ 0\\ 0\end{bmatrix}^T.$$

The measurement noise v is white with intensity $\Sigma = 10^{-4}$ $m^2/(Hz)^{1/2}$. The roadway is represented by an exogenous system of the form in (2), where

$$S=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix} \text{ and } w(0)=\begin{bmatrix}0\\ 0.1\end{bmatrix}.$$

An output regulating Youla-like parameterized switched controller, $K_R(Q)$, and Youla parameterized switch controller, $K_Y(Q)$, is designed to maintain a constant driver position with minimal acceleration while subjected to a persistent roadway disturbance $w(t)$ and sensor noise $v(t)$.

B. Local Controller Design

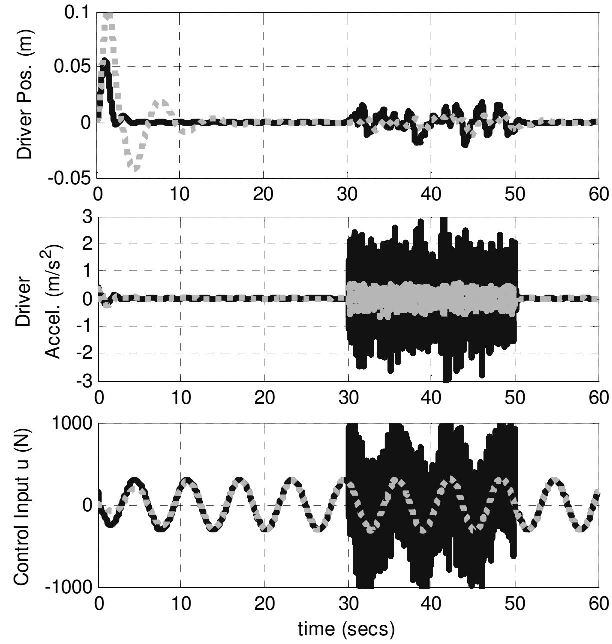


Fig 4. Active suspension simulation comparing controller performance of K_0 (dotted line) vs. K_1 (solid line) under sensor noise at $t \in [30,50]$ seconds and a constant roadway disturbance, $r(t)=0.1\sin(t)$ meters.

Two observer-based measurement controllers K_0 and K_1 were designed (as describe in chapter 2 of [11]) to both achieve internal stabilization and output regulation but separately achieve good noise rejection and good tracking, respectively. The controllers have the following structure:

$$K_i:\left\{\begin{bmatrix}\dot{\hat{\bar{x}}}_i\\ u\end{bmatrix}=\left[\begin{array}{c|c}\bar{A}+\bar{B}_u\bar{F}_i+\bar{L}_i\bar{C}_y & -\bar{L}_i\\ \hline \bar{F}_i & 0\end{array}\right]\begin{bmatrix}\hat{\bar{x}}_i\\ y\end{bmatrix}\right.$$

where, $\hat{\bar{x}}_i$, $\bar{L}_i$, and $\bar{F}_i$ are the state estimates, augmented observer gains, and augmented feedback gains of the i[th] controller, respectively. These two controllers utilize standard LQR methods to find the feedback and observer gains. The feedback gains for the system in (17) assuming no disturbances were calculated by minimizing the cost function

$$J_{reg}:=\int_0^\infty\left(x(\tau)^2+\rho u^2(\tau)\right)d\tau \tag{18}$$

where ρ was chosen to be 10^{-1} and 10^2 for K_0 and K_1 respectively. This feedback gain was then translated to an augmented feedback gain for the augmented plant in (4) using the following output regulating state-feedback controller law $u=\bar{F}\bar{x}$, where

$$\bar{F}=[F \quad (\Gamma-F\Pi)] \text{ and } \bar{x}=[x^T \quad w^T]^T.$$

Next, the augmented observer gains were calculated by also performing LQR but on the dual problem of the augmented plant in (4) using a similar cost function as in (18) but with ρ chosen to be 10^{-4} and 10^1 for K_1 and K_0 respectively. The simulated results of K_1 and K_0 in Fig. 4 motivates controller switching for better performance.

C. Parameterized Controller Design

The controller K_1 gains were also used to construct the central controller J in the form of (6). Then, the controllers K_0 and K_1 were parameterized to the form of (14) to get regulating control parameters Q_0 and Q_1 respectively, where, $K(Q_0) \sim K_0$ and $K(Q_1) \sim K_1$.

For comparison K_0 and K_1 were also parameterized in the nominal Youla parameterization framework described in [5], [15] and [17] to construct a parameterized controller.

D. Simulation Results

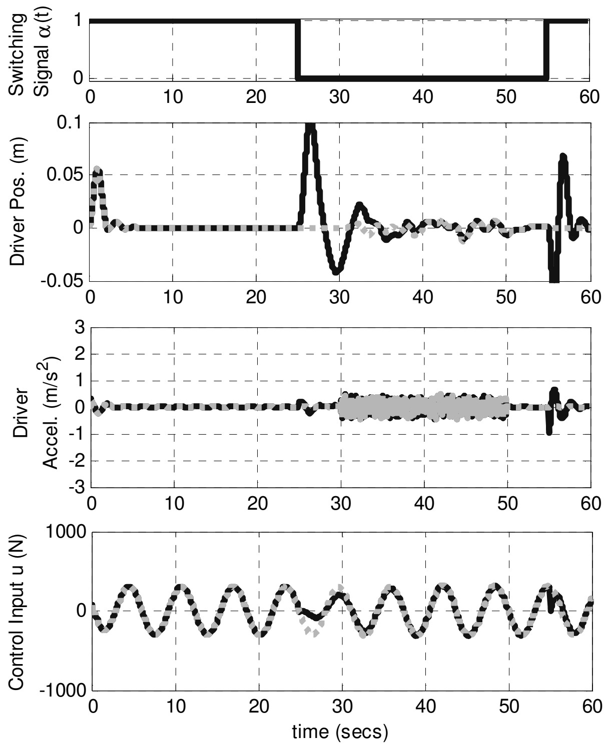


Fig 5. Active suspension simulation comparing Youla parameterized (solid line) vs output regulating Youla-like parameterized (dotted line) switch controllers under sensor noise at $t \in [30,50]$ seconds and a persistent roadway disturbance, $r(t)=0.1 sin(t)$ meters.

In Fig. 4 we present the results of K_0 and K_1 subjected to a persistent road disturbance $w(t)$ and to sensor noise for t=[30,50]. Controller K_1 is very responsive but performs poorly under large sensor noise in terms of minimizing driver vertical acceleration. Controller K_0 is less responsive but performs very well under large sensor noise. These differences motivate development for a switched controller with better performance.

In Fig. 5 we apply the same conditions as in Fig. 4 and compare the performance of $K_Y(Q)$ and $K_R(Q)$. Also, for all switches of both controllers we perform a control parameter reset as done in [5]. Controller $K_R(Q)$ maintains regulation during both switches in Fig. 5 whereas $K_Y(Q)$ does not; it is because the central controller in $K_R(Q)$ is inherently output regulating but not in $K_Y(Q)$. This result gives credence to the proposed parameterization framework over the traditional Youla parameterization framework for parameterization output regulating controllers.

VI. Conclusion

This paper presents a state-space based parameterization framework of output regulating controllers. A set of parameterization criteria is established define when a mapping from controller to a controller parameter to an input-output equivalent controller exists. The advantage of the proposed parameterization is demonstrated on a simulated vehicle active suspension system that shows how switched controllers constructed from the proposed parameterization can significantly reduce switching transients in comparison to traditional Youla parameterization. Future research involves developing a parameterization framework for output regulating controllers with H_2 and/or H_{inf} performance guarantees.

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