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Constrained Consensus via Logarithmic Barrier functions

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Abstract— **In this paper, we consider distributed algorithms for consensus of multiple agents in presence of convex state constraints on individual agent state. Each agent's state is assumed to be constrained in a distinct compact convex set. We show that following the proposed distributed protocol, the agents are guaranteed to reach an agreement on a state that lies at the intersection of individual convex constraint sets. This is accomplished by introducing and sharing auxiliary variables in the network. The auxiliary variable utilizes a logarithmic barrier function to form a convex potential that is augmented to the consensus protocol. The consensus algorithm is then interpreted as a gradient-descent algorithm which operates with the desire to reach consensus while avoiding violation of the constraint sets. This modified consensus algorithm is applicable when each agent is required to satisfy its own constraints while synchronizing with others, e.g., attitude synchronization in presence of attitude constraints. An example is given for two different network topologies to evaluate the effectiveness and the convergence rate of the proposed algorithm.**

I. INTRODUCTION

One of the prime conceptual models for networked dynamic systems is the agreement protocol- in fact, this model is so well-studied that it has multiple names, among them, consensus algorithm, Laplacian dynamics, and the heat kernel [1], [3], [4], [6]. Its applications include modeling heat distribution in a medium, random walks and electrical networks, flocking, attitude alignment, rendezvous, distributed averaging, and even language development. In the agreement protocol, the first order dynamics of each node in the network- abstracted in terms of a graph- is driven by the sum of the differences that it measures with respect to its neighbors. The system of nodes is then allowed to evolve from a particular initialization. The linearity and homogeneity of the node evolution admits a compact representation in terms of an unforced linear time-invariant dynamical system, where the evolution matrix is specified by the negative of underlying graph Laplacian. This on the other hand, allows one to examine system theoretic issues, such as stability and convergence rate, in terms of the spectra of the graph Laplacian. More recently, the dynamics of the agreement protocol in switching, random, noisy, and directed networks, have also been considered as they pertain directly to cooperation and control of multiple networked vehicle systems. Moreover, the basic setup has been extended to nonlinear models including unicycle models [7] and attitude dynamics [9], [10]. We note that the agreement protocol is inherently an "uncontrolled" and "unconstrained" dynamical systems, with a trajectory that is determined by initial conditions and the interconnection topology. In the meantime, most of the previous works in this area have not considered the problems where the agent values are *constrained*. Examples for pertinent situations where the presence of constraints in individual agent's configuration space is important include motion planning and attitude alignment problems in presence of agent's restricted regions or attitude zones. Constrained consensus problem has been examined in a handful of recent research works. In this venue, Moore *et al.* [13] has examined the constrained consensus when a subset of state variables are constrained. In Nedic *et al.* [11], the authors have proposed a framework for the constrained consensus by utilizing a projection algorithm and presented its convergence properties. The projection algorithm [12] provides means by which one can determine the state's closeness to a set defined by linear constraints, which in turn, can be parameterized using measures on an appropriate function space.

In this paper, we present a constrained consensus algorithm framework where the agents values are constrained to be in convex sets and each agent is only aware of its own constrained set. By running proposed modified consensus algorithm, each agent's value evolves to reach consensus asymptotically. A distinct feature of our algorithm is the introduction of an auxiliary variable for each agent that is shared with others over the network. This auxiliary variable is associated with a convex function representing the constraint set and current agent's state. We then proceed to show that by updating the auxiliary variable and agent's state via a consensus-type algorithm, each agent's value converges to the point that lies at the intersection of the compact convex constraint sets for all agents.

The rest of the paper is organized as follows. §II contains the notation, relevant mathematical background and an brief overview of graph theory. In §III, the problem formulation and the continuous constrained consensus algorithm are introduced. We then provide convergence analysis for state evolution of the agents as generated by the proposed algorithm. In order to evaluate the effectiveness of the algorithm, two simulations are examined in §IV. Conclusions and potential future extensions of this work are detailed in §V.

II. NOTATION AND GRAPH THEORY

Here, we provide a brief background on graph theory along with notation and terminology.

Throughout the paper, a vector is a column array and denoted in bold by $\boldsymbol{x}$; its ith component is denoted by x_i. For a stacked vector $\boldsymbol{x} = [\ \boldsymbol{x}_1^T \ \cdots \ \boldsymbol{x}_n^T]^T$, its subcomponent are denoted as $x_{i,l}$, indicating the lth element of ith vector in $\boldsymbol{x}$. In addition, I_m denotes an $m \times m$ identity matrix. The

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Kronecker product is denoted by $\otimes$. Throughout the paper, we let $[\cdot]_{\otimes}$ denote $[\cdot] \otimes \mathbf{I}_m$ for brevity. We use $\|\cdot\|$ to denote the standard Euclidean norm. Namely, $\|\boldsymbol{x}\| = \sqrt{\boldsymbol{x}^T \boldsymbol{x}}$. The communication topology of a network among nodes/agents are represented using a directed graph $\mathcal{G}$, e.g., in such a graph i can transmit information to j when the link $i \to j$ is present in the graph. An undirected graph $\mathcal{G}$ is represented as a pair $\mathcal{G} = (V,\, E)$, where V is a finite nonempty set of nodes as $V = \{v_1, v_2, \ldots, v_n\}$ and E is referred as the set of edges of $\mathcal{G}$ and denoted $E = \{e_1, e_2, \ldots e_q\}$. An element of E, e.g., e_j, consists of pairs of distinct nodes, where node v_i and v_j are connected if $\{v_i,\, v_j\} \in E$. Analogously, the neighbors of ith nodes are denoted by

$$\mathcal{N}_i = \{v_j \in V \,|\, \{v_i,\, v_j\} \in E\}\,.$$

The graph $\mathcal{G}$ is connected if for every pair of distinct nodes in V, there is a path that has them as its end nodes. The graph $\mathcal{G}$ can be represented in terms of matrices. Under the assumption that labels have been associated with the edges in a graph, arbitrarily oriented, the $n \times q$ incidence matrix $D(\mathcal{G})$ is defined as

$$D = [d_{ij}], \text{ where } d_{ij} = \begin{cases} -1 & \text{if } v_i \text{ is the tail of } e_j \\ 1 & \text{if } v_i \text{ is the head of } e_j \\ 0 & \text{otherwise.} \end{cases}$$

The advantage of using an incidence matrix over an adjacency representation is that it holds the orientation (vector) of the connection between two nodes. The constrained consensus algorithm proposed in this paper is equally applicable to weighted graphs. Another matrix representation of a graph $\mathcal{G}$, used in this paper, is the graph Laplacian, $L(\mathcal{G})$. The symmetric Laplacian matrix which is defined as $L(\mathcal{G}) = DD^T$ holds the connection information among pairs of nodes. This matrix is a positive semi-definite matrix and has eigenvalues that can be ordered as

$$\lambda_1(\mathcal{G}) \leq \lambda_2(\mathcal{G}) \leq \cdots \leq \lambda_n(\mathcal{G}) \tag{1}$$

where $\lambda_1(\mathcal{G}) = 0$. The graph $\mathcal{G}$ is connected if and only if $\lambda_2(\mathcal{G}) > 0$ [6]. The weighted graph Laplacian associated with the weighted graph $\mathcal{G} = (V,\, E,\, w)$ can be formed as

$$L_w(\mathcal{G}) = \frac{1}{2}\, DWD^T, \tag{2}$$

where W is a diagonal matrix whose elements consist of the numeric weights $w(e_j)$ corresponding to an edge e_j. It is well-known that $\lambda_2(\mathcal{G}) > 0$ still holds valid for weighted connected graphs as long as the weights are positive.

III. Problem Statement

For the case of undirected graphs, consider the sum of squares of state variables [1] as

$$\boldsymbol{x}^T L_{\otimes} \boldsymbol{x} = \frac{1}{2} \sum_{i,j \in E} a_{ij} \left\| \boldsymbol{x}_j - \boldsymbol{x}_i \right\|^2, \tag{3}$$

where $\boldsymbol{x}$ denotes the stack of vectors $\boldsymbol{x}_i \in \mathrm{R}^m$, a_{ij} denotes a weight acting on the respective edge, and $L_{\otimes}$ denotes $L \otimes \mathrm{I}_m$. Without of loss of generality, we will assume that $m = 1$ in subsequent sections; we denote $L(\mathcal{G})$ by L. Then, defining the quadratic potential function as

$$J(\boldsymbol{x}) = \frac{1}{2} \boldsymbol{x}^T L \boldsymbol{x}, \tag{4}$$

the canonical consensus algorithm can be derived by taking a negative gradient of Eq. (4) as

$$\dot{x}_i = -\nabla J = \sum_{i,j \in E} a_{ij}(x_j - x_i). \tag{5}$$

It is recognized that $J(\boldsymbol{x})$ is convex in $\boldsymbol{x}$ and has a unique global minimum for $\boldsymbol{x} = \alpha \mathbf{1}$ for some α when the graph is connected. This potential function can also be seen as representing the amount of disagreement among the agents' states. Geometrically speaking, the algorithm provides a weighted directional derivative for the agent's state as a function of its relative states with its neighbors. Thus, it is intuitive that this algorithm leads to a consensus on the average of the agent's initial state $(x(0)^T \mathbf{1}/n)\ \mathbf{1}$ [2].

Now, we introduce a continuous constrained consensus algorithm in an analogous manner. First, under undirected links, we assume that we have n nodes and the domain of the ith node's state $\boldsymbol{x}_i$ is restricted to *compact convex* sets. It is further assumed that the interior of these sets can be represented by p convex functions $f_k : \mathbf{R}^m \to \mathbf{R}$ defined as

$$\mathbf{Dom}\, \boldsymbol{x}_i \doteq \{\boldsymbol{x}_i \in \mathrm{R}^m \,|\, f_1(\boldsymbol{x}_i) < 0, \ldots, f_p(\boldsymbol{x}_i) < 0\}\,. \tag{6}$$

Additionally, we define a shift parameter $\beta_{i,k}$ which is computed from a lower bound of the kth convex function of ith node as:

$$L_{i,k} = \inf\, (f_k(\boldsymbol{x}_i))\,,$$

where $\inf(.)$ denotes the infimum operation. Then, the positive scalar $\beta_{i,k}$ is chosen as

$$\beta_{i,k} \geq \frac{\|L_{i,k}\|^3}{\|f_k(\boldsymbol{x}_i = \mathbf{0})\|^2}. \tag{7}$$

The shift parameter $\beta_{i,k}$ is always attainable in Dom $\boldsymbol{x}_i$ since the sets are bounded and closed. Now, we propose the potential function $J(\boldsymbol{x})$ given by

$$\begin{aligned} J(\boldsymbol{x}) &= \sum_i \sum_{i \leftarrow j} \frac{1}{2} \|\boldsymbol{x}_j - \boldsymbol{x}_i\|^2 \left[\sum_k - \log\left(-\frac{f_k(\boldsymbol{x}_i)}{\beta_{i,k}} \right) \right] \\ &= \frac{1}{2} \boldsymbol{x}^T [DWD^T] \boldsymbol{x} \\ &= \boldsymbol{x}^T L_w \boldsymbol{x}, \end{aligned} \tag{8}$$

where $\dot{\boldsymbol{x}} = [\ \dot{\boldsymbol{x}}_1^T \ \ \cdots \ \ \dot{\boldsymbol{x}}_n^T\ ]^T$, W denotes a $q \times q$ diagonal matrix where q designates the number of edges in the graph $\mathcal{G}$ and D denotes the incidence matrix representing this graph.

We note that each diagonal element in L_w is nonnegative.

The weighting matrix W has the form

$$W = \begin{bmatrix} \sum_k -\log\left(-\frac{f_{1,k}(\boldsymbol{x}_1)}{\beta_{1,k}}\right) & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \sum_k -\log\left(-\frac{f_{n,k}(\boldsymbol{x}_n)}{\beta_{n,k}}\right) \end{bmatrix}. \tag{9}$$

If we now consider a gradient flow with $J(\boldsymbol{x})$ as the potential, we arrive at a distributed protocol that is a generalization of the consensus algorithm over the network, namely,

$$\dot{\boldsymbol{x}} = -\nabla J \tag{10}$$

$$= -[DWD^T]\boldsymbol{x} - \frac{1}{2}\begin{bmatrix} \boldsymbol{x}^T D \frac{\partial}{\partial x_{1,1}}(W) D^T \boldsymbol{x} \\ \vdots \\ \boldsymbol{x}^T D \frac{\partial}{\partial x_{n,m}}(W) D^T \boldsymbol{x} \end{bmatrix}, \tag{11}$$

where $\boldsymbol{x} = [\ \boldsymbol{x}_1^T \ \cdots \ \boldsymbol{x}_n^T]^T$ and its component vector $\boldsymbol{x}_i = [\ \boldsymbol{x}_{i,1} \ \cdots \ \boldsymbol{x}_{i,m}\ ]^T$. We adopt the notation

$$\frac{\partial}{\partial \boldsymbol{x}_{n,m}} W$$

that denotes the partial differentiation of the diagonal components of the matrix W with respect to $x_{n,m}$ for notational brevity. Thus we have,

$$\dot{\boldsymbol{x}}_i = \sum_{i \leftarrow j} \underbrace{(g_{j,k} + g_{i,k})[\boldsymbol{x}_j - \boldsymbol{x}_i]}_{\text{attractive}} + \frac{1}{2}\underbrace{\|\boldsymbol{x}_j - \boldsymbol{x}_i\|^2 [-\nabla g_i]}_{\text{repulsive}} \tag{12}$$

where $g_{j,k}$ refers to the auxiliary variable from jth agent and computed as

$$g_{j,k} = \sum_{k=1}^{p} -\log\left(-\frac{f_{j,k}(\boldsymbol{x}_j)}{\beta_{j,k}}\right).$$

Note that ith agent does not need to know the corresponding convex function for the jth agent, e.g., $f_{j,k}(\boldsymbol{x}_j)$, since only $\boldsymbol{x}_j$ and $g_{j,k}$ are transmitted to ith agent.

This weighted non-nlinear consensus algorithm, Eq. (12) has repulsive and attractive components. As illustrated in Fig. 1, the repulsive component provides "interior pointing" vector acting normal to its domain's boundary, while the attractive component guides the agents states toward consensus.

This also can be viewed as an agreement protocol on a state dependent network. In fact, Eq. (12) reduces to

$$\dot{\boldsymbol{x}}_i = \sum_{i \leftarrow j} p_{i,j}(\boldsymbol{x}_j - \boldsymbol{x}_i) \tag{13}$$

$$= -DW'D^T\boldsymbol{x} \tag{14}$$

where $p_{i,j} = f(\boldsymbol{g}, \boldsymbol{x}) = (g_{j,k} + g_{i,k}) - \frac{1}{2}\nabla g_i(\boldsymbol{x}_j - \boldsymbol{x}_i)$.

Proposition 1: The potential function J has a global minimum when all agents' states reach consensus

$$x_1 = x_2 = \cdots = x_n, \tag{15}$$

while each state variable x_i is restricted to stay in its own constrained domain, $\mathbf{Dom}\,\boldsymbol{x}_i$.

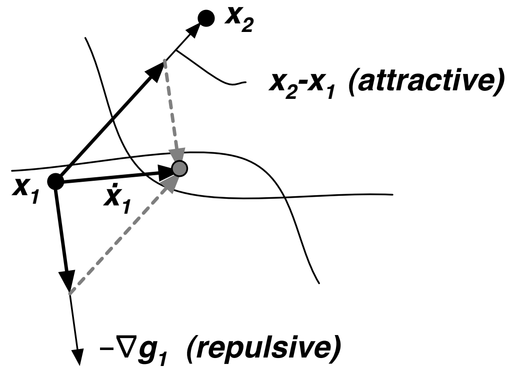

Fig. 1. Geometric illustration of the vector $\dot{x}_1$ from constrained consensus algorithm in presence of 1 neighbor

Proof: The proof is based on the fact that the potential function J can made to be convex in $\boldsymbol{x}$ by having appropriate values for $\beta_{i,k}$. First, given the *positive definite* matrix W, the potential J is always semi-positive when the network is connected as

$$J = \frac{1}{2}\boldsymbol{x}^T[DWD^T]\boldsymbol{x} \geq 0, \tag{16}$$

whose minimum is contained in the set

$$\{x \in \boldsymbol{R}^n \mid J(t) = 0\} = \mathbf{span}\,\{\mathbf{1}\} \tag{17}$$

which in turn, is exactly the null space of L_w. This subspace corresponds to the agreement subspace in Eq. (15). Moreover, its Hessian is calculated as

$$\begin{aligned}\nabla^2 J = {} & [DWD^T] \\ & + \begin{bmatrix} D\frac{\partial}{\partial x_{1,1}} WD^T\boldsymbol{x} & \cdots & D\frac{\partial}{\partial x_{n,m}} WD^T\boldsymbol{x} \end{bmatrix} \\ & + \begin{bmatrix} \boldsymbol{x}^T D\frac{\partial}{\partial x_{1,1}} WD^T \\ \vdots \\ \boldsymbol{x}^T D\frac{\partial}{\partial x_{n,m}} WD^T \end{bmatrix} + \mathrm{Diag}\left(\frac{1}{2}\boldsymbol{x}^T D\frac{\partial^2}{\partial \boldsymbol{x}_i^2} WD^T\boldsymbol{x}\right).\end{aligned} \tag{18}$$

Consequently,

$$\boldsymbol{x}^T\nabla^2 J\boldsymbol{x} = \sum_i \sum_{i \leftarrow j} \|x_j - x_i\|^2 \left[g_i + 2\nabla g_i^T \boldsymbol{x}_i + \frac{1}{2}\|\boldsymbol{x}_i\|^2 \nabla^2 g_i\right] \tag{19}$$

where

$$g_i = \sum_{k=1}^{p} -\log\left(-\frac{f_{i,k}(\boldsymbol{x}_i)}{\beta_{i,k}}\right).$$

As the Hessian $\nabla^2 g_i$ is non-negative ($f_i(x_i)$ is convex) we conclude that g_i is convex on the set

$$\{\boldsymbol{x}_i \in \mathrm{R}^m \mid f_1(\boldsymbol{x}_i) < 0, \ldots, f_p(\boldsymbol{x}_i) < 0\}$$

by the composition rule. Meanwhile, all convex functions,

$$g : \mathbf{R}^m \to \mathbf{R}$$

satisfy the following first-order convexity condition:

$$g(0) \geq g(x) - x^T \nabla g(x)$$

where $x,\ 0 \in \text{Dom } g$. This inequality condition provides a lower bound on the first two terms in the bracket in Eq. (19) as

$$\begin{aligned} g_i + 2\nabla g_i^T \boldsymbol{x}_i &\geq 3g_i - 2g_i(\boldsymbol{0}) \\ &\geq \sum_{k=1}^{p} -3\log\left(-f_{i,k}(x_i)\right) + 2\log\left(-f_{i,k}(\boldsymbol{0})\right) \\ &\quad + \log \beta_{i,k} \end{aligned} \tag{20}$$

In order for the right hand side of Eq. (19) to stay non-negative, we manipulate the value $\beta_{i,k}$ accordingly. Thereby, a reasonable choice of $\beta_{i,k}$ for the right hand side is

$$\frac{(-f_{i,k}(\boldsymbol{0}))^2 \beta_{i,k}}{(-f_{i,k}(\boldsymbol{x}_i))^3} \geq 1, \quad \forall\ i \text{ and } k$$

and the lower bound of $\beta_{i.k}$ is found as

$$\beta_{i,k} \geq \frac{(-f_{i,k}(\boldsymbol{x}_i))^3}{(-f_{i,k}(\boldsymbol{0}))^2} \geq \frac{(-\inf[f_{i,k}(\boldsymbol{x}_i)])^3}{(f_{i,k}(\boldsymbol{0}))^2}.$$

Therefore, the permissible choices of $\beta_{i,j}$ are given as

$$\beta_{i,k} \geq \frac{\|L_{i,k}\|^3}{\|f_k(\boldsymbol{x}_i = \boldsymbol{0})\|^2} \tag{21}$$

where

$$L_{i,k} = \inf(f_{i,k}(\boldsymbol{x}_i)).$$

With the value of $\beta_{i,j}$ chosen as above, Eq. (19) remains positive for all $\boldsymbol{x} \in \mathrm{R}^{nm}$. Hence the Hessian $\nabla^2 V$ is positive definite as

$$\boldsymbol{x}^T \nabla^2 J \boldsymbol{x} > 0 \quad \text{for all nonzero } \boldsymbol{x}. \tag{22}$$

Note that we can select an arbitrary large positive value for $\beta_{i,k}$ to have this Hessian positive, but it will cause the gradient of $-log$ function to approach infinity.

According to Eq. (8), when the graph is connected, the potential J retains a positive value away from consensus and at consensus,

$$x_1 = x_2 = \cdots = x_n, \tag{23}$$

it has a unique minimum of zero. Since J is strictly convex, the time evolution of the function $J(\boldsymbol{x})$ along the trajectory generated by the proposed consensus algorithm, is monotonically decreasing for all x_i as

$$\dot{J} = -\|\nabla J\|^2 \leq 0. \tag{24}$$

The aforementioned conditions now qualifies J as a *strong Lyapunov function*. Moreover, $J(\boldsymbol{x}) \to \infty$ as $\|\boldsymbol{x}\| \to \infty$. Therefore, by Lyapunov's second stability theorem [14], the Lyapunov function J is a certificate for globally asymptotical stability of the agreement subspace. ■

Proposition 2: Any connected graph $\mathcal{G} = (V,\ E)$ can be partitioned into a tree graph and edges that complete its cycles. In particular, any graph Laplacian matrix associated with a connected graph can be written as the sum

$$L(\mathcal{G}) = L(\mathcal{G}_{tree}) + L(\mathcal{G}_{edge})$$

where $\mathcal{G}_{tree} = (V,\ E_1)$, $\mathcal{G}_{edge} = (V,\ E_2)$, and $E(\mathcal{G}) = E_1 \cup E_2$.

Proof: See [2] ■

Proposition 3: The constrained agreement protocol, Eq. (12) converges to the agreement set with a rate of convergence that is at least that of λ_2 for its spanning tree.

Proof: The convergence rate of the proposed consensus algorithm is governed by Eq. (24). According to Prop. 2, Eq. (11) can be written as

$$\begin{aligned} \dot{\boldsymbol{x}} &= -\nabla J \\ &= -\frac{1}{2}\boldsymbol{L}_t\boldsymbol{x} - \frac{1}{4}\begin{bmatrix} \boldsymbol{x}^T d\boldsymbol{L}_{t_{1,1}}\boldsymbol{x} \\ \vdots \\ \boldsymbol{x}^T d\boldsymbol{L}_{t_{n,m}}\boldsymbol{x} \end{bmatrix} - \frac{1}{2}\boldsymbol{L}_e\boldsymbol{x} - \frac{1}{4}\begin{bmatrix} \boldsymbol{x}^T d\boldsymbol{L}_{e_{1,1}}\boldsymbol{x} \\ \vdots \\ \boldsymbol{x}^T d\boldsymbol{L}_{e_{n,m}}\boldsymbol{x} \end{bmatrix} \\ &= -(\nabla J_t + \nabla J_e), \end{aligned}$$

where $\boldsymbol{L}_w = \boldsymbol{L}_t + \boldsymbol{L}_e$, and $d\boldsymbol{L}_{w_{n,m}} = d\boldsymbol{L}_{t_{n,m}} + d\boldsymbol{L}_{e_{n,m}}$. The fact that ∇J_t and ∇J_e share the sign convention and $\lambda_2(\boldsymbol{L}_w) \geq \lambda_2(\boldsymbol{L}_t) > 0$ leads to

$$\|\nabla J\| \geq \|\nabla J_t\|.$$

Thus completing the proof. ■

IV. Example

In this section, we present simulation results for two applications of constrained consensus algorithms.

A. Example 1 - Unicycles

Consider that five homogeneous unicycles moving at constant speed v and subject to steering controls θ_i for changing their respective orientations. The dynamics of the unicycles are given as

$$\dot{r}_i = e^{\boldsymbol{j}\theta_i} \tag{25}$$

$$\dot{\theta}_i = u_i, \quad i = 1, 2, ..., 5. \tag{26}$$

When the control law u_i depends only on their relative orientations, i.e., $u_i = \theta_k - \theta_i$, the state vector evolves on *SO(2)*. If the state variables, however, are constrained to given subsets, e.g., $-\frac{4}{5}\pi \leq \theta_i \leq \frac{1}{2}\pi$, the proposed consensus algorithm is applicable for reaching an agreement set that lies at the intersection of the constrained sets. Constrained orientations can be represented by quadratic forms with convex domains, as shown in Table I. One can recognize that the consensus value will be dominated by 4th unicycle due to its narrowest domain among all unicycles. The initial conditions are randomly selected but close to their boundaries in order to reproduce the worst case scenario. The undirected communication topology is initially given a complete graph and later 3 edges among 5 unicycles are randomly eliminated while keeping connected for the convergence rate comparison as shown in Fig. 2. The second smallest eigenvalues of the graph Laplacian $L(\mathcal{G}_1)$ and $L(\mathcal{G}_2)$ are given as 3.17 and 10, respectively. Note that a consensus takes place quite far off the average because of

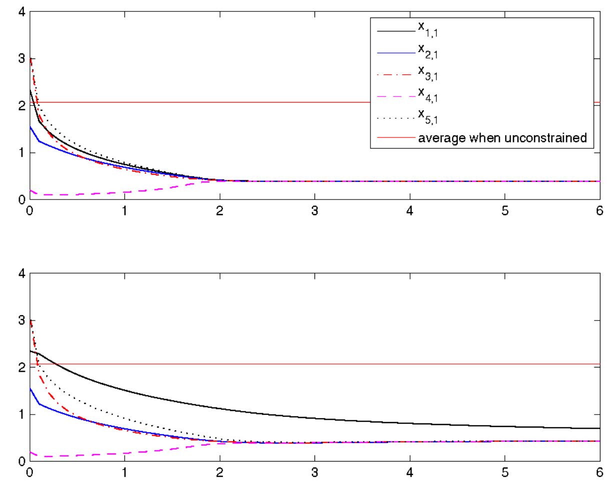

Fig. 2. The convergence rate comparison between complete and simpler graph

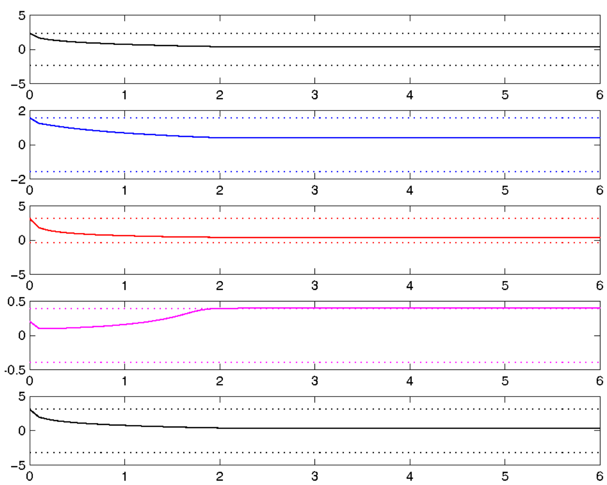

Fig. 3. Each unicycle's orientation over time with bounds

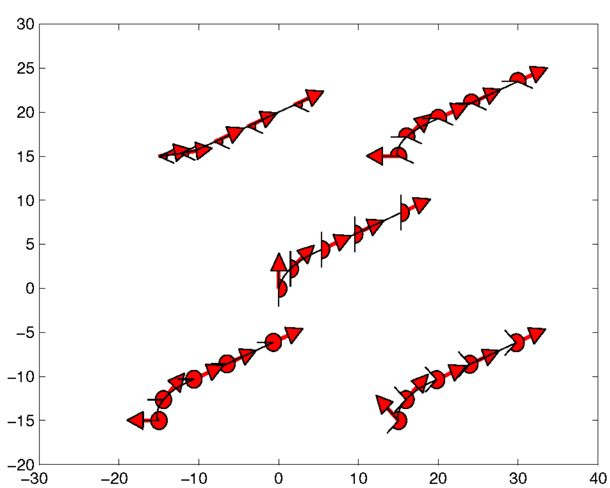

Fig. 4. Unicycle trajectories when completely connected

$\mathbf{Dom}\,\boldsymbol{\theta}_1 =$	$\{\boldsymbol{\theta}_1 \in \mathrm{S}^1 \,\vert\, f_1 = (\theta_1 - \frac{3}{4}\pi)^2 + (\theta_1 + \frac{3}{4}\pi)^2 < 0\}$
$\mathbf{Dom}\,\boldsymbol{\theta}_2 =$	$\{\boldsymbol{\theta}_2 \in \mathrm{S}^1 \,\vert\, f_2 = (\theta_2 - \frac{1}{2})^2 + (\theta_2 + \frac{1}{2}\pi)^2 < 0\}$
$\mathbf{Dom}\,\boldsymbol{\theta}_3 =$	$\{\boldsymbol{\theta}_3 \in \mathrm{S}^1 \,\vert\, f_3 = (\theta_3 - \pi)^2 + (\theta_3 + \frac{1}{8}\pi)^2 < 0\}$
$\mathbf{Dom}\,\boldsymbol{\theta}_4 =$	$\{\boldsymbol{\theta}_4 \in \mathrm{S}^1 \,\vert\, f_4 = (\theta_4 - \frac{1}{8}\pi)^2 + (\theta_2 + \frac{1}{8}\pi)^2 < 0\}$
$\mathbf{Dom}\,\boldsymbol{\theta}_5 =$	$\{\boldsymbol{\theta}_5 \in \mathrm{S}^1 \,\vert\, f_5 = (\theta_5 - \pi)^2 + (\theta_2 + \pi)^2 < 0\}$

TABLE I

SIMULATION PARAMETERS FOR EXAMPLE 1

β_1	β_2	β_3	β_4	β_5
5.56	2.47	20.01	0.16	9.87

TABLE II

THE SHIFT PARAMETER β_i FOR THE SIMULATION EXAMPLE 1

the 4th unicycle's the narrowest domain. This property can be utilized as a forced consensus. Fig. 3 shows the trajectory of each unicycle's orientation with bounds. Note that they stay well in their upper and lower bounds. The shifting parameters β_i to calculate the auxiliary variables are given in Table II.

B. *Example 2 - Single integrators*

Another practical application for the proposed algorithm is the initialization of unmanned aerial vehicles (UAV) formation flying. In this simulation, the UAV is represented via its kinematic model as a single integrator. Furthermore, it has been assumed that there are 4 UAVs involved in the formation flying. Each UAV is taking off from the corresponding ground station, whose remote control coverage is described by a convex set defined as the domain of a known convex function. Two undirected communication topologies, given as graphs $\mathcal{G}_1$ and $\mathcal{G}_2$, are shown in Fig. 5. The second smallest eigenvalues of the graph Laplacian $L(\mathcal{G}_1)$ and $L(\mathcal{G}_2)$ are 4 and 8, respectively. The remote control coverages are shown in Table III and $\beta_{i,k}$ are chosen in order to satisfy the condition, Eq. (21) as in Table IV. The simulation results

$\mathbf{Dom}\,\boldsymbol{x}_1 =$	$\{\boldsymbol{x}_1 \in \mathrm{R}^2 \,\vert\, f_1 = (x_{1,1} - 3)^2 + (x_{1,2} - 1)^6 - 30 < 0\}$
$\mathbf{Dom}\,\boldsymbol{x}_2 =$	$\{\boldsymbol{x}_2 \in \mathrm{R}^2 \,\vert\, f_2 = 3x_{2,1}^2 + 2x_{2,2}^2 - 4x_{2,1}x_{2,2} - 5 < 0\}$
$\mathbf{Dom}\,\boldsymbol{x}_3 =$	$\{\boldsymbol{x}_3 \in \mathrm{R}^2 \,\vert\, f_{3,1} = x_{3,2} + 0.3x_{3,1} - 7 < 0$
	$f_{3,2} = -6x_{3,2} + x_{3,1} - 2 < 0$
	$f_{3,3} = 2x_{3,2} - 2 - x_{3,1} < 0\}$
$\mathbf{Dom}\,\boldsymbol{x}_4 =$	$\{\boldsymbol{x}_4 \in \mathrm{R}^2 \,\vert\, f_4 = x_{4,1}^2 + (x_{4,2} + 3)^2 - 25 < 0\}$

TABLE III

THE REMOTE CONTROL COVERAGES - SIMULATION PARAMETERS

are illustrated in Figs. 6-8. Fig. 6 depicts the trajectory of each UAV as well as the initial position ($\times$ mark) and the

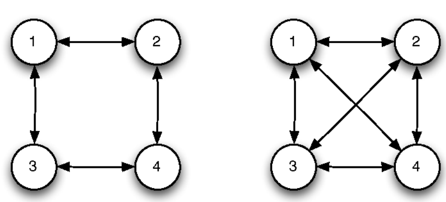

Fig. 5. Two Communication topologies for $\mathcal{G}_1$ and $\mathcal{G}_2$ for 4 UAVs with undirected communication links.

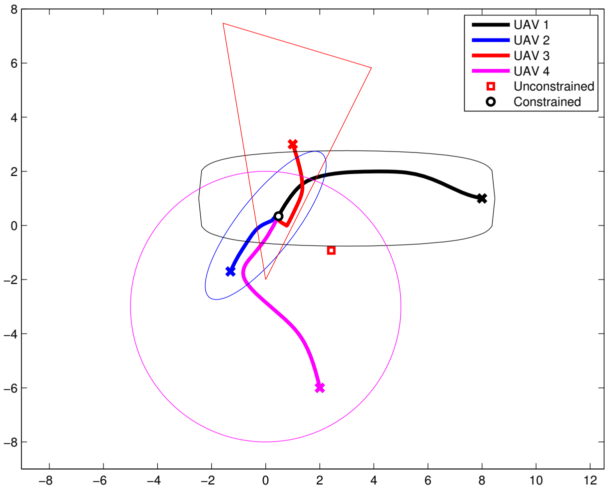

Fig. 6. UAV trajectories on 2D space

β_1	β_2	$\beta_{3,1}$	$\beta_{3,2}$	$\beta_{3,3}$	β_4
67.5	5	13.68	167.48	167.48	61.04

TABLE IV

THE SHIFT PARAMETER $\beta_{i,k}$ FOR THE SIMULATION EXAMPLE 2

final position (○ mark) in black. Note that the unfilled square denotes the final position when unconstrained. Fig. 7 and Fig. 8 show, respectively, the state evolution histories for $\mathcal{G}_1$ and $\mathcal{G}_2$. Note in both cases consensus is asymptotically reached, but the convergence rate for $\mathcal{G}_2$ is faster. The final state attained by the agents depends on the initial conditions and state dependent weights acting on the edges of the communication graph and the shape of the respective convex sets defining each agent's constrained set.

V. CONCLUSION AND FUTURE WORK

In this paper, we have considered the continuous constrained consensus algorithm with the aid of the logarithmic barrier function. The proposed algorithm is applicable when each agent's state is restricted to stay in the interior of a compact convex set. By choosing an appropriate auxiliary variable, namely $\beta_{i,k}$, satisfying required conditions needed for appropriate convexification, we have shown that the updated states for all agents asymptotically approach consensus lying at the intersection of all constraint sets. This algorithm is motivated by our research on spacecraft attitude synchronization over a network in presence of multiple attitude constrained zones.

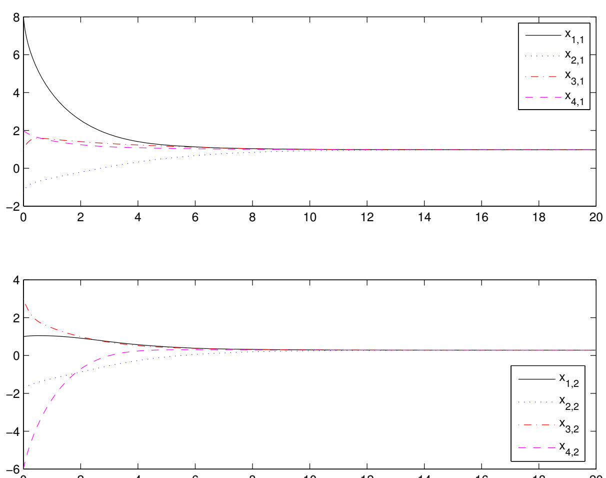

Fig. 7. State evolution over time for $\mathcal{G}_1$

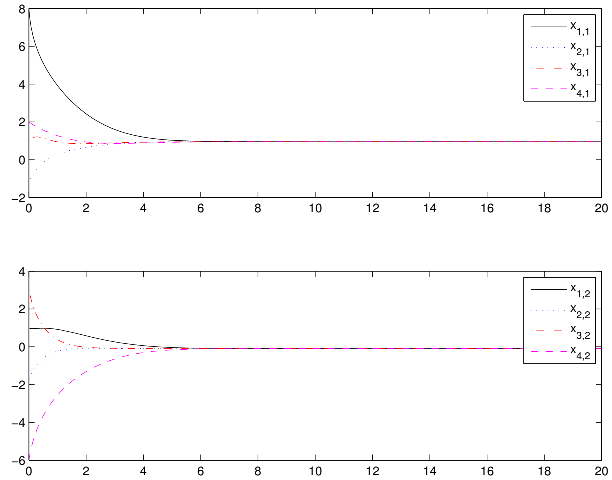

Fig. 8. State evolution over time for $\mathcal{G}_2$

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